5. Higher Order Ordinary Differential Equations

5.1 Prelude to Higher Order Differential Equations

In the previous classes we looked at first order differential equations. We turn now to ODEs of order two and higher. In the first three sections we examine some of the underlying theory of higher order differential equations. Then, just as we did in the last chapter we will look at some special cases of higher order differential equations that we can solve. Unlike the previous chapter however, we are going to have to be even more restrictive as to the kinds of differential equations that we'll look at. This will be required in order for us to actually be able to solve them.

Definition. The differential equation of the *n*th order in the general case has the form:

$$F(x; y(x), y'(x), y''(x), \dots, y^{(n)}(x)) = 0, \tag{1}$$

where F is a continuous function of the specified arguments: an unknown function of one real or complex variable x, its derivatives.

Note: Recall that the order of a differential equation is the highest derivative that appears in the equation.

In some cases, it is possible to solve the *n*th order differential equation (1) for the highest derivative such that

$$y^{(n)}(x) = f\big(x; y(x), y'(x), y''(x), \dots, y^{(n-1)}(x)\big) \tag{2}$$

Definition. A solution to a differential equation is a function $y = f(x)$ that satisfies the differential equation when $f(x)$ and its derivatives are substituted into the equation.

If we try to solve the differential equation, and if everything goes well, then you will end up with a formula for the general solution[1]:

$$y(x) = y(x; C_1, C_2, \dots, C_n) \tag{3}$$

which contains a number of constants $C_1, C_2, \dots, C_n$ (the number of them corresponds to the order of differential equation).

Theorem (The Existence and Uniqueness Theorem) Suppose x_0 is a given "initial point" $x = x_0$, and suppose a_0, a_1, ... , a_{n-1} are given constants. Then there

[1] If we can show that there actually aren't any other solutions than the ones we found (i.e. your formula captures all solutions to the differential equation), then that solution is the general solution.

is exactly one solution to the differential equation (2) which satisfies the initial conditions:

$$y(x_0) = a_0, y'(x_0) = a_1, y''(x_0) = a_2, \dots , y^{(n-1)}(x_0) = a_{n-1} \qquad (4)$$

Note that for an *n*th order equation we can prescribe exactly *n* initial values.

One way to test if a solution is the general solution is to see if it is possible to choose the constants $C_1, C_2, \dots, C_n$ so that the solution satisfies the initial conditions (4). This means that we can compute the derivatives $y'(x), y''(x), \dots, y^{(n-1)}(x)$ of the solution and then check them and the solution itself if we can solve the equations:

$$y(x_0; C_1, C_2, \dots, C_n) = a_0,$$
$$y'(x_0; C_1, C_2, \dots, C_n) = a_1,$$
$$y''(x_0; C_1, C_2, \dots, C_n) = a_2,$$
$$\dots ,$$
$$y^{(n-1)}(x_0; C_1, C_2, \dots, C_n) = a_{n-1}$$

for the constants $C_1, C_2, \dots, C_n$.

Note that we have *n* equations with *n* unknowns here. We'll do this for linear equations below.

5.2 Cases of Reduction of Order

The order of the equation of *n*th order can be reduced if it does not contain some of the arguments, or has a certain symmetry. Below we consider in detail some cases of reducing the order with respect to the differential equations of arbitrary order *n*.

Case 1. Equation of Type $F(x, y^{(n)}) = 0$

If the differential equation does not contain the original function and its derivatives, one can say the differential equation can be solved in quadratures, i.e. its general solution is expressed through one or more integrals.

This equation can be transformed into an explicit form for the derivative $y^{(n)}$, i.e. expressed as

$$y^{(n)}(x) = f(x)$$

The original function $y(x)$ can be found by *n*-fold integration, i.e. We integrate this equation *n* times consecutively in the range from x_0 to x. As a result, we obtain the following expressions for the derivatives and the function $y(x)$.

Example 1. Find the general solution of the differential equation $y''' = e^{2x}$ which satisfies the initial conditions: $y(0) = 1, y'(0) = -1, y''(0) = 0$

Since the right hand side of the third order differential equation is a function of the independent variable x only, the original function $y(x)$ can be found by 3-fold integration as follows:

$$y''(x) = \int e^{2x}\,dx = \frac{1}{2}e^{2x} + C_1$$

Then,

$$y'(x) = \frac{1}{2}\int e^{2x}\,dx + \int C_1\,dx = \frac{1}{4}e^{2x} + C_1 x + C_2$$

Finally, the general solution is presented in the form:

$$y(x) = \frac{1}{4}\int e^{2x}\,dx + \int C_1 x\,dx + \int C_2\,dx = \frac{1}{8}e^{2x} + C_1\frac{x^2}{2} + C_2 x + C_3$$

Given the initial conditions we can find the constants as follows:

$y''(0) = \frac{1}{2}e^{2\cdot 0} + C_1 = 0 \Rightarrow C_1 = -\frac{1}{2}$

$y'(0) = \frac{1}{4}e^{2\cdot 0} + C_1 \cdot 0 + C_2 = -1 \Rightarrow C_2 = -1 - \frac{1}{4} = -\frac{5}{4}$

$y(0) = \frac{1}{8}e^{2\cdot 0} + C_1\frac{0^2}{2} + C_2 \cdot 0 + C_3 = 1 \Rightarrow C_3 = 1 - \frac{1}{8} = \frac{7}{8}$

So, a particular solution at the given initial conditions has a form:

$$y(x) = \frac{1}{8}e^{2x} - \frac{x^2}{4} - \frac{5}{4}x + \frac{7}{8}$$

Case 2. Equation of Type $F(x; y^{(k)}(x), y^{(k+1)}(x), \ldots, y^{(n)}(x)) = 0$

If the differential equation does not contain the original function and its $k-1$ first derivatives, then by replacing

$$y^{(k)} = p(x)$$

the order of this equation is reduced by k units. As a result, the original equation takes the form

$$F(x, p, p', \ldots p^{(n-k)}) = 0.$$

From this equation (if possible) we can determine the function $p(x)$. The original function $y(x)$ can be found by k-fold integration of the replacement, i.e. is solved by the method set out in paragraph 1 above.

Note: If the differential equation does not contain only the original function $y(x)$, that is has the form

$$F(x, y', y'', \ldots, y^{(n)}) = 0,$$

then its order can be reduced by one by the substitution $y' = p(x)$.

Example 2. Find the general solution of the differential equation $y''' + \frac{2}{x}y'' = 0$.
This equation does not contain the function y and its first derivative y'. So we make the change of variable

$$y'' = p(x).$$

We obtain the first-order equation with separable variables:

$$p' + \frac{2}{x}p = 0.$$

Integrating, we find the solution:

$$\frac{dp}{dx} = -\frac{2}{x}p, \Rightarrow \frac{dp}{p} = -\frac{2}{x}dx, \Rightarrow \int \frac{dp}{p} = -2\int \frac{dx}{x}, \Rightarrow$$

$$\ln|p| = -2\ln|x| + \ln C_1, \Rightarrow p = \frac{C_1}{x^2}.$$

Returning to the original variable y, we obtain another differential equation:

$$y'' = \frac{C_1}{x^2}.$$

Integrating twice, we find the general solution of the original equation:

$$y' = -\frac{C_1}{x} + C_2, y = -C_1\ln|x| + C_2x + C_3.$$

Example 3. Find a particular solution of the equation $y^{IV} - y''' = 1$ with initial conditions: $x_0 = 0, y_0 = y_0' = y_0'' = y_0''' = 0$.
This equation is of type 2. We introduce an intermediate variable $z = y'''$. As a result we get a linear first order equation:

$$z' - z = 1.$$

Its general solution is given by the function

$$z = C_1e^x - 1.$$

Hence, we have

$$y''' = C_1e^x - 1,$$

that is the equation is converted to type 1. It can be solved by successive integration:

$y'' = C_1e^x - x + C_2, \ \Rightarrow \ y' = C_1e^x - \frac{x^2}{2} + C_2x + C_3, \ \Rightarrow \ y = C_1e^x - \frac{x^3}{6} + \frac{C_2x^2}{2} + C_3x + C_4$.

The coefficients C_i are determined from the initial conditions:

$$\begin{cases} 0 = C_1 - 1 \\ 0 = C_1 + C_2 \\ 0 = C_1 + C_3 \\ 0 = C_1 + C_4 \end{cases}, \Rightarrow \begin{cases} C_1 = 1 \\ C_2 = -1 \\ C_3 = -1 \\ C_4 = -1 \end{cases}.$$

Thus, a particular solution for the given initial conditions is expressed by the formula

$$y(x) = e^x - \frac{x^3}{6} - \frac{x^2}{2} - x - 1.$$

Example 4. Find the general solution of the equation $5(y''')^2 - 3y''y^{IV} = 0$.

This equation does not contain the function y and its first derivative y'. We make the following change:

$$y'' = p(x).$$

As a result, we obtain the second-order equation:

$$5(p')^2 - 3pp'' = 0.$$

Since this equation does not contain the independent variable x, then we put $p' = z(p)$. Hence,

$$p'' = \frac{d}{dx}(p') = \frac{d}{dp}(p')\frac{dp}{dx} = \frac{dz}{dp} \cdot z = zz'.$$

Then the equation can be written as

$$5z^2 - 3pzz' = 0, \Rightarrow z(5z - 3pz') = 0.$$

There are two solutions: One solution to this equation is given by

$$z = 0, \Rightarrow p' = 0, \Rightarrow p = C_1, \Rightarrow y'' = C_1, \Rightarrow y' = C_1x + C_2, \Rightarrow y = C_1x^2 + C_2x + C_3.$$

It is evident that this solution describes a set of parabolas with arbitrary coefficients C_1, C_2, C_3.

Now we find the second solution of the differential equation:

$$5z - 3pz' = 0, \Rightarrow 3pz' = 5z, \Rightarrow \frac{dz}{z} = \frac{5}{3}\frac{dp}{p}, \Rightarrow \int \frac{dz}{z} = \frac{5}{3}\int \frac{dp}{p}, \Rightarrow$$

$$\ln|z| = \frac{5}{3}\ln|p| + \ln C_4, \Rightarrow z = C_4 p^{\frac{5}{3}}, \Rightarrow p' = C_4 p^{\frac{5}{3}}.$$

The resulting first-order equation is easily integrated:

$$\int p^{-\frac{5}{3}} dp = C_4 \int dx, \Rightarrow -\frac{3p^{-\frac{2}{3}}}{2} = C_4 x + C_5.$$

Renaming the constants C_4, C_5, the solution can be written as

$$p = -(C_4 x + C_5)^{-\frac{3}{2}}.$$

Thus, to determine the second solution, we have the following equation:

$$y'' = -(C_4x + C_5)^{-\frac{3}{2}}.$$

Integrating twice, we find:

$$y' = 2(C_4x + C_5)^{-\frac{1}{2}} + C_6, \Rightarrow y = (C_4x + C_5)^{\frac{1}{2}} + C_6x + C_7, \Rightarrow$$
$$y = \sqrt{C_4x + C_5} + C_6x + C_7.$$

Thus, the general solution of the original equation has two families of functions:

$$y_1 = C_1x^2 + C_2x + C_3,$$
$$y_2 = \sqrt{C_4x + C_5} + C_6x + C_7,$$

where $C_1, C_2, \ldots, C_7$ are arbitrary numbers.

Case 3. Equation of Type $F(y, y', y'', \ldots, y^{(n)}) = 0$

Here the left side does not contain the independent variable x. The order of the equation can be reduced by the substitution $y = p(y)$. The derivatives are defined through the new variables y and p as follows:

$$y' = \frac{dy}{dx} = p,$$

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy},$$

$$y''' = \frac{d^3y}{dx^3} = \frac{d}{dx}(p\frac{dp}{dy}) = \frac{d}{dy}(p\frac{dp}{dy})\frac{dy}{dx} = [p\frac{d^2p}{dy^2} + (\frac{dp}{dy})^2]p = p^2\frac{d^2p}{dy^2} + p(\frac{dp}{dy})^2,$$

…

It is seen that substitution of the derivatives into the original equation gives a new differential equation of the $(n-1)$th order. Solving this equation, we can determine the function $p(y)$ and then find $y(x)$.

Example 5. Find the general solution of the equation $yy'' - (y')^2 - 4yy' = 0$.

This equation does not contain explicitly the variable x. We make the following substitutions:

$$y' = p(y), y'' = p\frac{dp}{dy}$$

As a result, we obtain the first-order equation:

$$yp\frac{dp}{dy} - p^2 - 4yp = 0 \Rightarrow p\left(y\frac{dp}{dy} - p - 4y = 0\right) = 0 \Rightarrow$$

$$\begin{cases} p = 0 \\ y\dfrac{dp}{dy} - p - 4y = 0 \end{cases} \Rightarrow \begin{cases} y' = 0 \\ \dfrac{dp}{dy} = 4 + \dfrac{p}{y}, \quad y \neq 0 \end{cases}$$

There are two solutions: one solution to this equation is given by $y = C_1$, which may include a case of zero-solution. The second equation is a first-order homogeneous equation. Then, substituting $p = u(y)y$ and $\frac{dp}{dy} = \frac{du}{dy}y + u$ into the equation, we obtain

$$\frac{du}{dy}y + u = 4 + \frac{u(y)y}{y} \Rightarrow \frac{du}{dy}y = 4 \Rightarrow du = 4\frac{dy}{y} \Rightarrow$$

$$\int du = 4\int \frac{dy}{y} \Rightarrow u = 4\ln|C_2 y|$$

Then the equation can be rewritten as

$$\frac{p}{y} = 4\ln|C_2 y| \Rightarrow \frac{dy}{dx} = 4y\ln|C_2 y| \Rightarrow \frac{dy}{y\ln|C_2 y|} = 4dx \Rightarrow$$

$$\int \frac{dy}{y\ln|C_2 y|} = 4\int dx \Rightarrow \ln\ln|C_2 y| = 4x + C_3$$

Thus, the general solution of the original equation has two families of functions:

$$y = C_1,$$

$$\ln\ln|C_2 y| = 4x + C_3,$$

where C_1, C_2, C_3 are arbitrary numbers.