6. Higher Order Linear Differential Equations

An *n*th order differential equation is said to be *linear* if it can be written in the form

$$P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_n(x)y = F(x),$$

where $P_0(x)$, $P_1(x)$,…, $P_n(x)$, $F(x)$ are *Variable Coefficients* which are continuous functions on (a, b) and $P_0(x)$ has no zeros on (a, b).

For simplicity, we can abbreviate the left side of Equation by Ly, that is,

$$Ly = P_0 y^{(n)} + P_1 y^{(n-1)} + \cdots + P_n y$$

Here $L(D)(\dots) = P_0 \cdot D^{(n)}(\dots) + P_1 \cdot D^{(n-1)}(\dots) + \cdots + P_{n-1} \cdot D(\dots) + P_n(\dots)$ is called the *differential polynomial*, where D is a *differential operator*. The simplest differential operator D acting on a function y, "returns" the first derivative of this function: $Dy(x) = y'(x)$.

So, $L(D)$ is a generalization of the operation of differentiation, multiplication by the coefficients $P_i(x)$ and addition acting on a function y. In other words, the operator L is an algebraic polynomial, in which the differential operator plays the role of a variable.

The operator L is linear, and therefore has the following properties:

1. $L[y_1(x) + y_2(x) + \cdots + y_n(x)] = L[y_1(x)] + L[y_2(x)] + \cdots + L[y_n(x)]$,
2. $L[Cy(x)] = CL[y(x)]$

Therefore, we can write a linear nonhomogeneous differential equation in the form:

$$Ly = F(x)$$

For convenience, we may also consider *linear nonhomogeneous differential equations* written in a *normal form*:

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = f(x),$$

where $a_i = P_i/P_0$, $i = 1,2,\dots,n$ and $f(x) = F(x)/P_0$.

Theorem. Suppose $Ly = F$ is normal on (a,b), let x_0 be a point in (a,b), and let k_0 , k_1 , …, k_{n-1} be arbitrary real numbers. Then *the initial value problem*

$$Ly = F, y(x_0) = k_0, y'(x_0) = k_1, \dots, y^{(n-1)}(x_0) = k_{n-1}$$

has a unique solution on (a,b).

6.1 Higher Order Linear Homogeneous Differential Equations

Linear differential equations with a zero right hand part $f(x) = 0$

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0$$

are called as a *linear homogeneous differential equation.*

Since $y \equiv 0$ is obviously a solution of $Ly = 0$, we call it the *trivial solution*. Any other solution is *nontrivial*.

It's easy to show that if y_1,y_2, …, y_n are solutions of $Ly = 0$ on (a,b), then any their linear combination is a solution as well, that is

$$y = C_1y_1 + C_2y_2 + \cdots + C_ny_n$$

where C_1,C_2, …, C_n are constants.

We say that $\{y_1, y_2, \ldots, y_n\}$ is a *fundamental set of solutions* of $Ly = 0$ on (a, b) if every solution of $Ly = 0$ on (a, b) can be written as a linear combination of $\{y_1, y_2, \ldots, y_n\}$.

In this case we say that $y = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n$ is *the general solution* of $Ly = 0$ on (a, b).

It can be shown that if the equation $Ly = 0$ exists on (a, b) then it has *infinitely many* fundamental sets of solutions on (a, b). The next definition will help to identify fundamental sets of solutions of $Ly = 0$.

Definition. We say that $\{y_1, y_2, \ldots, y_n\}$ is *linearly independent* on (a, b) if the only constants $C_1, C_2, \ldots, C_n$ such that

$$C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x) = 0, \quad a < x < b,$$

are $C_1 = C_2 = \ldots = C_n = 0$. If this equality holds for some set of constants $C_1, C_2, \ldots, C_n$ that are not all zero, then $\{y_1, y_2, \ldots, y_n\}$ is *linearly dependent* on (a, b).

Theorem. If $Ly = 0$ exists on (a, b), then a set $\{y_1, y_2, \ldots, y_n\}$ of *n* solutions of $Ly = 0$ on (a, b) is a fundamental set if and only if it is linearly independent on (a, b).

To test functions for linear independence it is convenient to use the Wronskian. It allows testing *n* solutions $\{y_1, y_2, \ldots, y_n\}$ of any *n*th order equation $Ly = 0$ for linear independence on an interval (a, b) on which the equation exists. Thus, if $C_1, C_2, \ldots, C_n$ are constants such that

$$C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x) = 0, \quad a < x < b,$$

then differentiating $n - 1$ times leads to the $n \times n$ system of equations

$$\begin{array}{ccc}
c_1 y_1(x) + c_2 y_2(x) + & \cdots & + c_n y_n(x) = 0 \\
c_1 y_1'(x) + c_2 y_2'(x) + & \cdots & + c_n y_n'(x) = 0 \\
\vdots & \ddots & \vdots \\
c_1 y_1^{(n-1)}(x) + c_2 y_2^{(n-1)}(x) + & \cdots & + c_n y_n^{(n-1)}(x) = 0
\end{array}$$

for $C_1, C_2, \ldots, C_n$. For a fixed x , the determinant of this system is

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{vmatrix}$$

This determinant is called the *Wronskian* of $\{y_1, y_2, \ldots, y_n\}$.

If $W(x) \neq 0$ for some x in (a, b) then the system of equations has only the trivial solution $C_1 = C_2 = \ldots = C_n = 0$, and the Theorem implies that

$$y = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n$$

is the general solution of $Ly = 0$ on (a, b). Otherwise, if $W(x) = 0$ then the functions $\{y_1, y_2, \ldots, y_n\}$ are linearly dependent on the interval (a, b).

Since the fundamental system of solutions uniquely defines a linear homogeneous differential equation. In particular, the fundamental system $\{y_1, y_2, y_3\}$ defines a third-order equation, which can be expressed through determinant as follows:

$$\begin{vmatrix} y_1 & y_2 & y_3 & y \\ y_1' & y_2' & y_3' & y' \\ y_1'' & y_2'' & y_3'' & y'' \\ y_1''' & y_2''' & y_3''' & y''' \end{vmatrix} = 0.$$

Analogously, for the differential equation of the *n*th order one can write:

$$\begin{vmatrix} y_1 & y_2 & \cdots & y_n & y \\ y_1' & y_2' & \cdots & y_n' & y' \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} & y^{(n)} \end{vmatrix} = 0.$$

Example 1. Let $y_1 = x^2$, $y_2 = x^3$ and $y_3 = \frac{1}{x}$ be the solutions of $x^3 y''' - x^2 y'' - 2xy' + 6y = 0$. Calculate the Wronskian of $\{y_1, y_2, y_3\}$?

If $x \neq 0$, then

$$W(x) = \begin{vmatrix} x^2 & x^3 & \frac{1}{x} \\ 2x & 3x^2 & -\frac{1}{x^2} \\ 2 & 6x & \frac{2}{x^3} \end{vmatrix} = 2x^3 \begin{vmatrix} 1 & x & \frac{1}{x^3} \\ 2 & 3x & -\frac{1}{x^3} \\ 1 & 3x & \frac{1}{x^3} \end{vmatrix}$$

Adding the second row of the last determinant to the first and third rows yields

$$W(x) = 2x^3 \begin{vmatrix} 3 & 4x & 0 \\ 2 & 3x & -\frac{1}{x^3} \\ 3 & 6x & 0 \end{vmatrix} = 2x^3 \left(\frac{1}{x^3}\right) \begin{vmatrix} 3 & 4x \\ 3 & 6x \end{vmatrix} = 12x$$

Therefore $W(x) \neq 0$ on $(-\infty, 0)$ and $(0, +\infty)$.

Example 2. Show that the functions x, $\sin x$, $\cos x$ are linearly independent.

We find the Wronskian matrix $W(x)$ for this system of functions:

$$W(x)=\begin{vmatrix} x & \sin x & \cos x \\ 1 & \cos x & -\sin x \\ 0 & -\sin x & -\cos x \end{vmatrix} = x\begin{vmatrix} \cos x & -\sin x \\ -\sin x & -\cos x \end{vmatrix} - 1\cdot\begin{vmatrix} \sin x & \cos x \\ -\sin x & -\cos x \end{vmatrix}$$
$$= x(-\cos^2 x - \sin^2 x) - 1\cdot(-\cancel{\sin x\cos x} + \cancel{\sin x\cos x}) = -x \neq 0.$$

Since the Wronskian is not identically zero, it follows that the given system of functions is linearly independent.

Example 3. Make a differential equation, which is determined by the fundamental system of functions $1, x^2, e^x$.

This equation is written in terms of the determinant as follows:

$$\begin{vmatrix} 1 & x^2 & e^x & y \\ 0 & 2x & e^x & y' \\ 0 & 2 & e^x & y'' \\ 0 & 0 & e^x & y''' \end{vmatrix} = 0, \Rightarrow 1\cdot\begin{vmatrix} 2x & e^x & y' \\ 2 & e^x & y'' \\ 0 & e^x & y''' \end{vmatrix} = 0,$$
$$\Rightarrow 2x(e^x y''' - e^x y'') - 2(e^x y''' - e^x y') = 0,$$
$$\Rightarrow 2xe^x y''' - 2xe^x y''' - 2e^x y''' + 2e^x y' = 0,$$
$$\Rightarrow 2e^x(xy''' - xy'' - y''' + y') = 0,$$
$$\Rightarrow (x-1)y''' - xy'' + y' = 0.$$

Suppose that the functions $y_1, y_2, \dots, y_n$ form a fundamental system of solutions for a differential equations of *n*th order. Suppose that the point x_0 belongs to the interval (a, b). Then the Wronskian is determined *by Liouville's formula*:

$$W(x) = W(x_0)e^{-\int_{x_0}^{x} a_1(t)dt},$$

where a_1 is the coefficient of the derivative $y^{(n-1)}$ in the linear normal differential equation. For the general form of the linear differential equation, Liouville's formula takes the form:

$$W(x) = W(x_0)e^{-\int_{x_0}^{x}\frac{P_1(t)}{P_0(t)}dt}, P_0(t) \neq 0, t \in (a, b).$$

Note: The order of a linear homogeneous equation

$$Ly(x) = y^{(n)} + a_1(x)y^{(n-1)} + \dots + a_{n-1}(x)y' + a_n(x)y = 0$$

can be reduced by one by the substitution $y' = yz$. Unfortunately, usually such a substitution does not simplify the solution, because the new equation in the variable *z* becomes nonlinear.

The next theorem generalizes the features of the solution.

Theorem. Suppose $Ly = 0$ exists on (a, b) and let $y_1, y_2, \dots, y_n$ be *n* solutions of $Ly = 0$ on (a, b). Then the following statements are equivalent; that is, they are either all true or all false:

a. The general solution of $Ly = 0$ on (a, b) is $y = C_1 y_1 + C_2 y_2 + \dots + C_n y_n$.
b. $\{y_1, y_2, \dots, y_n\}$ is a fundamental set of solutions of $Ly = 0$ on (a, b).
c. $\{y_1, y_2, \dots, y_n\}$ is linearly independent on (a, b).
d. The Wronskian of $\{y_1, y_2, \dots, y_n\}$ is nonzero at some point in (a, b).
e. The Wronskian of $\{y_1, y_2, \dots, y_n\}$ is nonzero at all points in (a, b).

6.2 Higher Order Linear Homogeneous Differential Equations with Constant Coefficients

In this section we will be investigating homogeneous higher order linear differential equations with constant coefficients. Those equations allow relatively simple finding a fundamental set of solutions. If we know those solutions, then any linear combination of these solutions give us the general solution.

The linear homogeneous differential equation of the *n*th order with constant coefficients can be written as

$$y^{(n)}(x) + a_1 y^{(n-1)}(x) + \dots + a_{n-1} y'(x) + a_n y(x) = 0,$$

$a_1, a_2, \dots, a_n$ are constants which may be real or complex.

Using the linear differential operator $L(D)$, this equation can be represented as

$$L(D)y(x) = 0,$$

where $L(D) = D^n + a_1 D^{n-1} + \dots + a_{n-1} D + a_n$.

Since all the coefficients are constants, the solutions are probably going to be functions with derivatives that are constant multiples of themselves. We need all the terms to cancel out, and if taking a derivative introduces a term that is not a constant multiple of the original function, it is difficult to see how that term cancels out. *Exponential functions* have derivatives that are constant multiples of the original function, so let’s see what happens when we try a solution of the form $y(x) = e^{\lambda x}$, where λ (the lowercase Greek letter lambda) is some constant.

Hence, if $y(x) = e^{\lambda x}$ than $y'(x) = \lambda e^{\lambda x}$, $y''(x) = \lambda^2 e^{\lambda x}$, …, $y^{(n)}(x) = \lambda^n e^{\lambda x}$. Substituting these expressions into the equation, we get

$$\lambda^n e^{\lambda x} + a_1 \lambda^{n-1} e^{\lambda x} + \dots + a_{n-1} \lambda e^{\lambda x} + a_n e^{\lambda x} = 0 \Rightarrow$$

$$e^{\lambda x} \cdot (\lambda^n + a_1 \lambda^{n-1} + \dots + a_{n-1} \lambda + a_n) = 0 \Rightarrow$$

Since, $e^{\lambda x}$ does not equal to zero, we get

$$\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$$

This algebraic equation

$$L(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$$

is called the *characteristic equation* of the differential equation.

According to the fundamental theorem of algebra, a polynomial of degree *n* has exactly *n* roots, counting multiplicity. In this case the roots can be both real and complex (even if all the coefficients of a_1, a_2, …, a_n are real).

Let us consider in more detail the different cases of the roots of the characteristic equation and the corresponding formulas for the general solution of differential equations.

Case 1. All Roots of the Characteristic Equation are Real and Distinct

We assume that the characteristic equation $L(\lambda) = 0$ has *n* roots $\lambda_1, \lambda_2, \dots, \lambda_n$, which are *Real and Distinct*. In this case the partial solutions are:

$$y_1 = e^{\lambda_1 x}, y_2 = e^{\lambda_2 x}, \dots , y_n = e^{\lambda_n x}$$

Hence, the general solution is written in a simple form:

$$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + \cdots + C_n e^{\lambda_n x},$$

where $C_1, C_2, \dots, C_n$ are constants depending on initial conditions.

Case 2. The Roots of the Characteristic Equation are Real and Multiple

We assume that the characteristic equation $L(\lambda) = 0$ has *m* roots $\lambda_1, \lambda_2, \dots, \lambda_m$, which are *Real and Multiple* (repeated real roots), the multiplicity of which, respectively, is equal to $k_1, k_2, \dots, k_m$. It is clear that the following condition has to hold:

$$k_1 + k_2 + \cdots + k_m = n.$$

In this case, we know $e^{\lambda_1 x}$ is a solution but it is only one solution and we need k_1 linearly independent solutions. Let's try $x^{(i)} e^{\lambda_1 x}$, $i = 1,2, \dots, k_1 - 1$ as the next solutions related to the k_1-times repeated root λ_1. Similarly, for all the other roots.

So, for repeated roots we just add in an x for each of the solutions past the first one until we have a total of k_1 solutions. In general, it needs to compute the Wronskian to verify that these are in fact a set of linearly independent solutions.

Then the general solution of the homogeneous differential equations with constant coefficients has the form

$$y(x) = C_1 e^{\lambda_1 x} + C_2 x e^{\lambda_1 x} + \cdots + C_{k_1} x^{k_1 - 1} e^{\lambda_1 x} + \cdots + C_{n-k_m+1} e^{\lambda_m x} \\ + C_{n-k_m+2} x e^{\lambda_m x} + \cdots + C_n x^{k_m - 1} e^{\lambda_m x}.$$

Case 3. The Roots of the Characteristic Equation are Complex and Distinct

If the coefficients of the differential equation are real numbers, the complex roots of the characteristic equation will be presented in the form of pairs of numbers:

$$\lambda_{1,2} = \alpha \pm i\beta, \lambda_{3,4} = \gamma \pm i\delta, \ldots$$

In this case the complex-valued partial solutions are written as

$y_1 = e^{\lambda_1 x} = e^{(\alpha+i\beta)x} = e^{\alpha x}e^{i\beta x}$, $y_2 = e^{\lambda_2 x} = e^{(\alpha-i\beta)x} = e^{\alpha x}e^{-i\beta x}$, …

In doing so, the Euler's formula tells us that

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Then,

$y_1 = e^{\alpha x}(\cos\beta x + i\sin\beta x)$, $y_2 = e^{\alpha x}(\cos\beta x - i\sin\beta x)$, …

By combining the first two solutions for the complex conjugate roots $\lambda_{1,2}$ we can rewrite their partial solutions in the real-valued forms as follows:

$\tilde{y}_1 = \frac{1}{2}(y_1 + y_2) = e^{\alpha x}\cos\beta x$, $\tilde{y}_2 = \frac{1}{2i}(y_1 - y_2) = e^{\alpha x}\sin\beta x$ and $\tilde{y}_3 = e^{\gamma x}\cos\delta x$, $\tilde{y}_4 = e^{\gamma x}\sin\delta x$, …

As a result, the general solution has a form:

$$y(x) = e^{\alpha x}(C_1\cos\beta x + C_2\sin\beta x) + e^{\gamma x}(C_3\cos\delta x + C_4\sin\delta x) + \cdots$$

Case 4. The Roots of the Characteristic Equation are Complex and Multiple

Now let's suppose that each pair of complex conjugate roots $\alpha \pm i\beta$ has a multiplicity of k (i.e. they occur k times in the list of roots). In this case we can use the work from the repeated roots (case 2) above to get the following set of $2k$ complex-valued particular solutions, $x^{(i)}e^{(\alpha\pm i\beta)x}$, $i = 1,2, \ldots, k-1$ or splitting each one into its real and imaginary parts we can arrive at the following set of $2k$ real-valued solutions:

$$e^{\alpha x}\cos\beta x, e^{\alpha x}\sin\beta x, e^{\alpha x}x\cos\beta x, e^{\alpha x}x\sin\beta x, \ldots, e^{\alpha x}x^{k-1}\cos\beta x, e^{\alpha x}x^{k-1}\sin\beta x.$$

Then the part of the general solution of the differential equation corresponding to a given pair of complex conjugate roots is constructed as follows:

$$y(x) = e^{\alpha x}(C_1\cos\beta x + C_2\sin\beta x) + xe^{\alpha x}(C_3\cos\beta x + C_4\sin\beta x) + \cdots$$
$$+ x^{k-1}e^{\alpha x}(C_{2k-1}\cos\beta x + C_{2k}\sin\beta x).$$

In general, when the characteristic equation has both real and complex roots of arbitrary multiplicity, the general solution is constructed as the sum of the above solutions of the form 1–4.

Before we work a couple of examples here we should point out that the characteristic polynomial is now going to be 2nd and higher order degree polynomials and finding the roots of these by hand is often a very difficult and time consuming process and in many cases if the roots are not rational (i.e. in the form $\frac{p}{q}$) it can be almost impossible to find them all by hand. In practice, for determining all the rational roots of a polynomial use some form of computation aid such as Maple or Mathematica to find all the roots with the Finding Zeroes of Polynomials procedure.

Example 1. Solve the differential equation $y''' + 2y'' - y' - 2y = 0$.

Write the corresponding characteristic equation:

$$\lambda^3 + 2\lambda^2 - \lambda - 2 = 0.$$

Solving it, we find the roots:

$$\lambda^2(\lambda + 2) - (\lambda + 2) = 0, \Rightarrow (\lambda + 2)(\lambda^2 - 1) = 0, \Rightarrow (\lambda + 2)(\lambda - 1)(\lambda + 1) = 0,$$
$$\Rightarrow \lambda_1 = -2, \lambda_2 = 1, \lambda_3 = -1.$$

It is seen that all three roots are real. Therefore, the general solution of the differential equations can be written as

$$y(x) = C_1 e^{-2x} + C_2 e^{x} + C_3 e^{-x},$$

where C_1, C_2, C_3 are arbitrary constants.

Example 2. Solve the differential equation $y''' - 7y'' + 11y' - 5y = 0$.

The corresponding characteristic equation is

$$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0.$$

It is easy to see that one of the roots is the number $\lambda = 1$. Then, factoring the term $(\lambda - 1)$ from the equation, we obtain

$$\lambda^3 - \lambda^2 - 6\lambda^2 + 6\lambda + 5\lambda - 5 = 0, \Rightarrow \lambda^2(\lambda - 1) - 6\lambda(\lambda - 1) + 5(\lambda - 1) = 0,$$
$$\Rightarrow (\lambda - 1) \cdot (\lambda^2 - 6\lambda + 5) = 0, \Rightarrow (\lambda - 1) \cdot (\lambda - 1) \cdot (\lambda - 5) = 0,$$
$$\Rightarrow (\lambda - 1)^2(\lambda - 5) = 0.$$

Thus, the equation has two roots $\lambda_1 = 1$, $\lambda_2 = 5$, the first of which has multiplicity 2. Then the general solution of differential equations can be written as follows:

$$y(x) = (C_1 + C_2 x)e^{x} + C_3 e^{5x},$$

where C_1, C_2, C_3 are arbitrary constants.

Example 3. Solve the differential equation $y^{IV} - y''' + 2y' = 0$.

Write the characteristic equation:

$$\lambda^4 - \lambda^3 + 2\lambda = 0.$$

Factorize the left side and find the roots:

$$\lambda(\lambda^3 - \lambda^2 + 2) = 0.$$

It is easy to see that $\lambda = 0$ is a root. Also, one of the roots of the cubic polynomial is the number $\lambda = -1$. Then, by dividing $\lambda^3 - \lambda^2 + 2$ by $\lambda + 1$ we get

$$\frac{\lambda^3 - \lambda^2 + 2}{\lambda + 1} = \lambda^2 - 2\lambda + 2.$$

As a result, the characteristic equation takes the following form:

$$\lambda(\lambda + 1) \cdot (\lambda^2 - 2\lambda + 2) = 0.$$

We find the roots of the quadratic equation:

$$\lambda^2 - 2\lambda + 2 = 0, \Rightarrow D = 4 - 8 = -4, \Rightarrow \lambda = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i.$$

Thus, the characteristic equation has four distinct roots, two of which are complex:

$$\lambda_1 = 0, \lambda_2 = -1, \lambda_{3,4} = 1 \pm i.$$

The general solution of the differential equation can be represented as

$$y(x) = C_1 + C_2 e^{-x} + e^x(C_3 \cos x + C_4 \sin x),$$

where $C_1, \ldots, C_4$ are arbitrary constants.

Example 4. Solve the differential equation $y^V + 18y''' + 81y' = 0.$

The characteristic equation can be written as

$$\lambda^5 + 18\lambda^3 + 81\lambda = 0.$$

Factorize the left side and calculate the roots:

$$\lambda(\lambda^4 + 18\lambda^2 + 81) = 0, \Rightarrow \lambda(\lambda^2 + 9)^2 = 0.$$

As it can be seen, the equation has the following roots:

$$\lambda_1 = 0, \lambda_{2,3} = \pm 3i,$$

and imaginary roots have multiplicity 2. In accordance with the rules set out above, we write the general solution in the form

$$y(x) = C_1 + (C_2 + C_3 x)\cos 3x + (C_4 + C_5 x)\sin 3x,$$

where $C_1, \ldots, C_5$ are arbitrary constants.

Example 5. Solve the differential equation $y^{IV} - 4y''' + 5y'' - 4y' + 4y = 0.$

Calculate the roots of the characteristic equation

$$\lambda^4 - 4\lambda^3 + 5\lambda^2 - 4\lambda + 4 = 0.$$

Factorize the left side:

$$\lambda^4 - 2\lambda^3 - 2\lambda^3 + 4\lambda^2 + \lambda^2 - 2\lambda - 2\lambda + 4 = 0,$$
$$\Rightarrow (\lambda^4 - 2\lambda^3) - (2\lambda^3 - 4\lambda^2) + (\lambda^2 - 2\lambda) - (2\lambda - 4) = 0,$$
$$\Rightarrow \lambda^3(\lambda - 2) - 2\lambda^2(\lambda - 2) + \lambda(\lambda - 2) - 2(\lambda - 2) = 0,$$
$$\Rightarrow (\lambda - 2) \cdot (\lambda^3 - 2\lambda^2 + \lambda - 2) = 0, \Rightarrow (\lambda - 2) \cdot [\lambda^2(\lambda - 2) + \lambda - 2]$$
$$= 0, \Rightarrow (\lambda - 2) \cdot (\lambda - 2) \cdot (\lambda^2 + 1) = 0, \Rightarrow (\lambda - 2)^2(\lambda^2 + 1) = 0.$$

We see that the roots of the equation are equal

$$\lambda_{1,2} = 2, \lambda_{3,4} = \pm i.$$

The first root is of multiplicity 2. The general solution of the differential equation is given by

$$y(x) = (C_1 + C_2 x)e^{2x} + C_3 \cos x + C_4 \sin x,$$

where $C_1, \dots, C_4$ are arbitrary constants.