3. *Higher Order Linear Nonhomogeneous Differential Equations*

To complete the course, we must also consider the higher order linear nonhomogeneous differential equations. An n-th order nonhomogeneous differential equation of this type can be written as

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = f(x),$$

where the coefficients $a_1(x)$, …, $a_n(x)$ and the right hand side $f(x)$ are continuous functions on some interval $[a, b]$.

With the help of a linear differential operator L this equation can be written in compact form:

$$Ly(x) = f(x),$$

where L includes the operations of differentiation, multiplication by the coefficients $a_i(x)$, and addition.

As it is known, the general solution $y(x)$ of a nonhomogeneous differential equation is the sum of the *general solution* $y_0(x)$ of the corresponding homogeneous equation and a *particular solution* $y_1(x)$ of the nonhomogeneous equation:

$$y(x) = y_0(x) + y_1(x).$$

We focus our attention on constructing solutions of the nonhomogeneous equations. The method of variation of constants also known as the Lagrange method is commonly used for this purpose. With this method, we can obtain the general solution of the nonhomogeneous equation, if the general solution of the homogeneous equation is known.

Method of Variation of Constants

Suppose we want to solve an n-th order nonhomogeneous differential equation. We will assume that the general solution of the associated homogeneous equation is found and expressed by the formula

$$y_0(x) = C_1Y_1(x) + C_2Y_2(x) + \cdots + C_nY_n(x),$$

containing n arbitrary constants C_1, …, C_n.

The idea of this method is to replace the constants C_1, …, C_n with continuously differentiable functions $C_1(x)$, …, $C_n(x)$, which are chosen so that the solution

$$y(x) = C_1(x)Y_1(x) + C_2(x)Y_2(x) + \cdots + C_n(x)Y_n(x) = \sum_{i=1}^{n} C_i(x)Y_i(x)$$

satisfies the nonhomogeneous differential equation.

The first derivatives of the functions $C_i(x)$ are determined from the system of n equations of the form

$$\left\{\begin{array}{c} C_1'(x)Y_1(x) + C_2'(x)Y_2(x) + \cdots + C_n'(x)Y_n(x) = 0 \\ C_1'(x)Y_1'(x) + C_2'(x)Y_2'(x) + \cdots + C_n'(x)Y_n'(x) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ C_1'(x)Y_1^{(n-1)}(x) + C_2'(x)Y_2^{(n-1)}(x) + \cdots + C_n'(x)Y_n^{(n-1)}(x) = f(x) \end{array}\right.$$

Note that the main determinant of this system is the Wronskian $W(x)$ constructed on the basis of the fundamental system of solutions $Y_1, \ldots, Y_n$. As the solutions $Y_1, \ldots, Y_n$ re linearly independent, the Wronskian is not zero.

The unknown derivatives $C_i{}'(x)$ are calculated by Cramer's rule:

$$C_i'(x) = \frac{W_i(x)}{W(x)}, i = 1,2, \ldots, n,$$

where the determinant $W_i(x)$ is obtained from the Wronskian $W(x)$ by replacing the i th column with the column $[0,0, \ldots, f(x)]$ on the right side.

Further, the expressions for $C_i(x)$ can be found by integration:

$$C_i(x) = \int \frac{W_i(x)}{W(x)} dx + A_i, i = 1,2, \ldots, n.$$

Here A_i denote constants of integration.

As a result, the general solution of the nonhomogeneous equation can be written as

$$y(x) = \sum_{i=1}^{n} C_i(x)Y_i(x) = \sum_{i=1}^{n} (\int \frac{W_i(x)}{W(x)} dx + A_i)Y_i(x)$$

$$= \sum_{i=1}^{n} A_iY_i(x) + \sum_{i=1}^{n} (\int \frac{W_i(x)}{W(x)} dx)Y_i(x) = y_0(x) + y_1(x).$$

In the last expression, the first sum corresponds to the general solution $y_0(x)$ of the homogeneous equation (with arbitrary numbers A_i), and the second sum describes a particular solution $y_1(x)$ of the nonhomogeneous equation.

Example 1. Find the general solution of the differential equation $y'' - 2y' + y = \frac{e^t}{t^2}$

The complementary homogeneous equation is

$$y'' - 2y' + y = 0$$

The general solution of this equation takes a form:

$$y_0(x) = c_1 e^x + c_2 x e^x$$

Since $y_1 = e^x$ and $y_2 = xe^x$ we can calculate the derivatives $y_1'(x) = e^x$ and $y_2'(x) = e^x + xe^x$. Then, we have a system of equations

$$C_1'(x)e^x + C_2'(x)xe^x = 0$$

$$C_1'(x)e^x + C_2'(x)(e^x + xe^x) = \frac{e^x}{x^2}.$$

Applying Cramer's rule we have

$$C_1'(x) = \frac{\begin{vmatrix} 0 & xe^x \\ \frac{e^x}{x^2} & e^x + xe^x \end{vmatrix}}{\begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix}} = \frac{0 - xe^x(\frac{e^x}{x^2})}{e^x(e^x + xe^x) - e^x xe^x} = \frac{-\frac{e^{2x}}{x}}{e^{2x}} = -\frac{1}{x}$$

and

$$C_2'(x) = \frac{\begin{vmatrix} e^x & 0 \\ e^x & \frac{e^t}{t^2} \end{vmatrix}}{\begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix}} = \frac{e^x(\frac{e^x}{x^2})}{e^{2x}} = \frac{1}{x^2}$$

Integrating, we get

$$C_1(x) = -\ln|x| + C_1,\ C_2(x) = -\frac{1}{x} + C_2$$

Then we have

$$y(x) = (-\ln|x| + C_1)e^x + \left(-\frac{1}{x} + C_2\right)xe^x = c_1 e^x + c_2 xe^x - e^x \ln|x| - e^x$$

where

$$y_1(x) = -e^x \ln|x| - e^x$$

Example 2. Find the general solution of the differential equation $(x^2 - 2)y''' - 2xy'' - (x^2 - 2)y' + 2xy = 2x - \frac{4}{y}$.

First we find the general solution of the homogeneous equation

$$(x^2 - 2)y''' - 2xy'' - (x^2 - 2)y' + 2xy = 0.$$

We use the symmetry of the equation and introduce the new variable

$$v = y'' - y.$$

Then the equation becomes:

$$(x^2-2)v'-2xv=0.$$

The resulting equation can be easily solved by separation of variables:

$$(x^2-2)\frac{dv}{dx}=2xv, \Rightarrow \frac{dv}{v}=\frac{2xdx}{x^2-2}, \Rightarrow \int \frac{dv}{v}=\int \frac{2xdx}{x^2-2}, \Rightarrow \int \frac{dv}{v}=\int \frac{d(x^2-2)}{x^2-2},$$

$$\Rightarrow \ln|v|=\ln|x^2-2|+\ln B_1 (B_1>0), \Rightarrow \ln|v|=\ln(B_1|x^2-2|),$$

$$\Rightarrow |v|=B_1|x^2-2|, \Rightarrow v=B_2(x^2-2),$$

We now find the function $y(x)$:

$$y''-y=v, \Rightarrow y''-y=B_2(x^2-2).$$

We have obtained a nonhomogeneous equation of order 2. The solution of the corresponding homogeneous equation is given by

$$y''-y=0, \Rightarrow \lambda^2-1=0, \Rightarrow \lambda_{1,2}=\pm 1, \Rightarrow y_0(x)=C_1e^x+C_2e^{-x}.$$

Given that the right side $B_2(x^2-2)$ is a quadratic polynomial, we seek a particular solution in the form

$$y_1=Dx^2+Ex+F.$$

We substitute this function and its derivatives

$$y_1'=2Dx+E, y_1''=2D$$

in our nonhomogeneous equation and find the coefficients D, E, F:

$$2D-(Dx^2+Ex+F)=B_2x^2-2B_2, \Rightarrow 2D-Dx^2-Ex-F=B_2x^2-2B_2.$$

Consequently,

$$\begin{cases} -D=B_2 \\ -E=0 \\ 2D-F=-2B_2 \end{cases}, \Rightarrow \begin{cases} D=-B_2 \\ E=0 \\ F=0 \end{cases}.$$

Thus, the particular solution y_1 is given by

$$y_1=-B_2x^2.$$

Replacing the arbitrary number B_2 with C_3, we finally obtain the general solution of the homogeneous equation:

$$y_0(x)=C_1e^x+C_2e^{-x}+C_3x^2.$$

Here, the functions $Y_1=e^x$, $Y_2=e^{-x}$, $Y_3=x^2$ form a fundamental system of solutions.

Now we find the solution of the nonhomogeneous equation using the method of variation of constants. The general solution is represented as

$$y(x)=C_1(x)e^x+C_2(x)e^{-x}+C_3(x)x^2,$$

where the derivatives of the unknown functions $C_1(x)$, $C_2(x)$, $C_3(x)$ satisfy the system of equations

$$\begin{cases} C_1' e^x + C_2' e^{-x} + C_3' x^2 = 0 \\ C_1' e^x - C_2' e^{-x} + 2C_3' x = 0 \\ C_1' e^x + C_2' e^{-x} + 2C_3' = 2x - \dfrac{4}{x} \end{cases}$$

Calculate the determinants of this system:

$$W = \begin{vmatrix} e^x & e^{-x} & x^2 \\ e^x & -e^{-x} & 2x \\ e^x & e^{-x} & 2 \end{vmatrix} = e^x e^{-x} \begin{vmatrix} 1 & 1 & x^2 \\ 1 & -1 & 2x \\ 1 & 1 & 2 \end{vmatrix}$$

$$= 1 \cdot [1(-2 - 2x) - 1(2 - x^2) + 1(2x + x^2)]$$

$$= -2 - \cancel{2x} - 2 + x^2 + \cancel{2x} + x^2 = 2x^2 - 4;$$

$$W_1 = \begin{vmatrix} 0 & e^{-x} & x^2 \\ 0 & -e^{-x} & 2x \\ 2x - \dfrac{4}{x} & e^{-x} & 2 \end{vmatrix} = (2x - \frac{x}{4}) \cdot (2xe^{-x} + x^2 e^{-x})$$

$$= (2x^2 - 4)(x + 2)e^{-x},$$

$$W_2 = \begin{vmatrix} e^x & 0 & x^2 \\ e^x & 0 & 2x \\ e^x & 2x - \dfrac{4}{x} & 2 \end{vmatrix} = -(2x - \frac{x}{4}) \cdot (2xe^x - x^2 e^x) = (2x^2 - 4)(x - 2)e^x,$$

$$W_3 = \begin{vmatrix} e^x & e^{-x} & 0 \\ e^x & -e^{-x} & 0 \\ e^x & e^{-x} & 2x - \dfrac{4}{x} \end{vmatrix} = e^{-x} e^x \begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 2x - \dfrac{4}{x} \end{vmatrix} = (2x - \frac{x}{4})(-1 - 1)$$

$$= -\frac{2}{x}(2x^2 - 4).$$

Then the derivatives $C_1{}'(x)$, $C_2{}'(x)$, $C_3{}'(x)$ are

$$C_1' = \frac{W_1}{W} = \frac{\cancel{(2x^2 - 4)}(x + 2)e^{-x}}{\cancel{2x^2 - 4}} = (x + 2)e^{-x},$$

$$C_2' = \frac{W_2}{W} = \frac{\cancel{(2x^2 - 4)}(x - 2)e^{x}}{\cancel{2x^2 - 4}} = (x - 2)e^{x},$$

$$C_3{}' = \frac{W_3}{W} = \frac{-\frac{2}{x}\cancel{(2x^2 - 4)}}{\cancel{2x^2 - 4}} = -\frac{2}{x}.$$

Integrating, we find the functions

$$C_1(x) = \int (x+2)e^{-x}dx = \begin{bmatrix} u = x+2 \\ v' = e^{-x} \\ u' = 1 \\ v = -e^{-x} \end{bmatrix} = -(x+2)e^{-x} - \int (-e^{-x})dx$$
$$= -(x+2)e^{-x} + \int e^{-x}dx = -(x+2)e^{-x} - e^{-x} + A_1$$
$$= -(x+3)e^{-x} + A_1,$$

$$C_2(x) = \int (x-2)e^{x}dx = \begin{bmatrix} u = x-2 \\ v' = e^{x} \\ u' = 1 \\ v = e^{x} \end{bmatrix} = (x-2)e^{x} - \int e^{x}dx$$
$$= (x-2)e^{x} - e^{x} + A_2 = (x-3)e^{x} + A_2,$$

$$C_3(x) = \int (-\frac{2}{x})dx = -2\int \frac{dx}{x} = -2\ln|x| + A_3.$$

Now we can write the general solution of the nonhomogeneous equation:

$$y(x) = C_1(x)Y_1(x) + C_2(x)Y_2(x) + C_3(x)Y_3(x)$$
$$= [-(x+3)e^{-x} + A_1]e^{x} + [(x-3)e^{x} + A_2]e^{-x} + [-2\ln|x| + A_3]x^2$$
$$= A_1e^{x} + A_2e^{-x} + A_3x^2 - (x+3) + x - 3 - 2x^2\ln|x|$$
$$= A_1e^{x} + A_2e^{-x} + A_3x^2 - 2x^2\ln|x| - 6.$$

4. *Higher Order Linear Nonhomogeneous Differential Equations with Constant Coefficients*

These equations have the form

$$y^{(n)}(x) + a_1y^{(n-1)}(x) + \cdots + a_{n-1}y'(x) + a_ny(x) = f(x),$$

where a_1, …, a_n are real or complex numbers, and the right-hand side $f(x)$ is a continuous function on some interval $[a, b]$.

Using the linear differential operator $L(D)$ equal to

$$L(D) = D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n,$$

the nonhomogeneous differential equation can be written as

$$L(D)y(x) = f(x).$$

The general solution $y(x)$ of the nonhomogeneous differential equation is q sum of the general solution $y_0(x)$ of the corresponding homogeneous equation and a particular solution $y_1(x)$ of the nonhomogeneous equation:

$$y(x) = y_0(x) + y_1(x).$$

For an arbitrary right side $f(x)$, the general solution of the nonhomogeneous equation can be found using the *method of variation of parameters*. If the right-hand side is the product of a polynomial and exponential functions, it is more convenient to seek a particular solution by the *method of undetermined coefficients*.

Method of Variation of Parameters

We assume that the general solution of the homogeneous differential equation of the nth order is known and given by

$$y_0(x) = C_1Y_1(x) + C_2Y_2(x) + \cdots + C_nY_n(x).$$

According to the method of variation of constants (or Lagrange method), we consider the functions $C_1(x)$, …, $C_n(x)$ instead of the regular numbers C_1, …, C_n. These functions are chosen so that the solution

$$y = C_1(x)Y_1(x) + C_2(x)Y_2(x) + \cdots + C_n(x)Y_n(x)$$

satisfies the original nonhomogeneous equation.

The derivatives of n unknown functions $C_1{}'(x)$, …, $C_n{}'(x)$ are determined from the system of n equations:

$$\begin{cases} C_1'(x)Y_1(x) + C_2'(x)Y_2(x) + \cdots + C_n'(x)Y_n(x) = 0 \\ C_1'(x)Y_1'(x) + C_2'(x)Y_2'(x) + \cdots + C_n'(x)Y_n'(x) = 0 \\ \dots\dots\dots\dots\dots\dots\dots \\ C_1'(x)Y_1^{(n-1)}(x) + C_2'(x)Y_2^{(n-1)}(x) + \cdots + C_n'(x)Y_n^{(n-1)}(x) = f(x) \end{cases}$$

The determinant of this system is the Wronskian of $Y_1(x)$, …, $Y_n(x)$ forming a fundamental system of solutions. By the linear independence of these functions, the determinant is not zero and the system is uniquely solvable. The final expressions for the functions $C_1(x)$, …, $C_n(x)$ can be found by integration.

Method of Undetermined Coefficients

If the right-hand side $f(x)$ of the differential equation is a function of the form

$$P_n(x)e^{\alpha x} \quad \text{or} \quad [P_n(x)\cos\beta x + Q_m(x)\sin\beta x]e^{\alpha x},$$

where $P_n(x)$ and $Q_m(x)$ are polynomials of degree n and m, respectively, then the method of undetermined coefficients may be used to find a particular solution.

In this case, we seek a particular solution in the form corresponding to the structure of the right-hand side of the equation.

Case 1: if the function has the form

$$f(x) = P_n(x)e^{\alpha x},$$

the particular solution is given by

$$y_1(x) = x^sA_n(x)e^{\alpha x},$$

where $A_n(x)$ is a polynomial of the same degree n as $P_n(x)$. The coefficients of the polynomial $A_n(x)$ are determined by direct substitution of the trial solution $y_1(x)$ in the nonhomogeneous differential equation.

In the so-called *resonance case*, when the number of α in the exponential function coincides with a root of the characteristic equation, an additional factor x^s, where s is the multiplicity of the root, appears in the particular solution. In the non-resonance case, we set $s = 0$.

Case 2: if the function has the form

$$f(x) = [P_n(x)\cos\beta x + Q_m(x)\sin\beta x]e^{\alpha x}.$$

Here the particular solution has a similar structure and can be written as

$$y_1(x) = x^s[A_n(x)\cos\beta x + B_n(x)\sin\beta x]e^{\alpha x},$$

where $A_n(x)$, $B_n(x)$ are polynomials of degree n (for $n \geq m$), and the degree s in the additional factor x^s is equal to the multiplicity of the complex root $\alpha \pm \beta i$ in the resonance case (i.e. when the numbers α and β coincide with the complex root of the characteristic equation), and accordingly, $s = 0$ in the non-resonance case.

Superposition Principle

The superposition principle is stated as follows. Let the right-hand side $f(x)$ be the sum of two functions:

$$f(x) = f_1(x) + f_2(x).$$

Suppose that $y_1(x)$ is a solution of the equation

$$L(D)y(x) = f_1(x),$$

and the function $y_2(x)$ is, accordingly, a solution of the second equation

$$L(D)y(x) = f_2(x).$$

Then the sum of the functions

$$y(x) = y_1(x) + y_2(x)$$

will be a solution of the linear nonhomogeneous equation

$$L(D)y(x) = f(x) = f_1(x) + f_2(x).$$

Example 1. Find the general solution of the differential equation $y''' + 3y'' - 10y' = x - 3$.

First we find the general solution of the homogeneous equation

$$y''' + 3y'' - 10y' = 0.$$

Calculate the roots of the characteristic equation:

$$\lambda^3 + 3\lambda^2 - 10\lambda = 0, \Rightarrow \lambda(\lambda^2 + 3\lambda - 10) = 0, \Rightarrow \lambda(\lambda - 2)(\lambda + 5) = 0.$$

Hence,

$$\lambda_1 = 0, \lambda_2 = 2, \lambda_3 = -5.$$

So the general solution of the homogeneous equation is given by

$$y_0(x) = C_1 + C_2 e^{2x} + C_3 e^{-5x},$$

where C_1, C_2, C_3 are arbitrary numbers.

The right side of the equation contains only a polynomial. However, if we take into account that $e^0 = 1$,, we see that in fact we have the resonance case (in disguised form) as one of the roots of the characteristic equation is also zero: $\lambda_1 = 0$. Therefore, we will seek a particular solution in the form

$$y_1(x) = x(Ax + B) = Ax^2 + Bx.$$

Substitute the derivatives

$$y_1' = 2Ax + B, y_1'' = 2A, y_1''' = 0.$$

into the nonhomogeneous equation and determine the coefficients A, B

$$0 + 3 \cdot 2A - 10(2Ax + B) = x - 3, \Rightarrow 6A - 20Ax - 10B = x - 3,$$

$$\Rightarrow \begin{cases} -20A = 1 \\ 6A - 10B = -3 \end{cases}, \Rightarrow \begin{cases} A = -\frac{1}{20} \\ B = \frac{27}{100} \end{cases}, \Rightarrow \begin{cases} A = -\frac{5}{100} \\ B = \frac{27}{100} \end{cases}.$$

The particular solution y_1 is written as

$$y_1(x) = x\left(-\frac{5}{100}x + \frac{27}{100}\right) = \frac{x}{100}(27 - 5x).$$

Thus, the general solution of nonhomogeneous differential equation is given by

$$y(x) = y_0(x) + y_1(x) = C_1 + C_2 e^{2x} + C_3 e^{-5x} + \frac{x}{100}(27 - 5x).$$

Example 2. Find the general solution of the differential equation $y''' - y' = \sin 3x$.

We construct the general solution of the homogeneous equation

$$y''' - y' = 0.$$

The roots of the characteristic equation are

$$\lambda^3 - \lambda = 0, \Rightarrow \lambda(\lambda^2 - 1) = 0, \Rightarrow \lambda(\lambda - 1)(\lambda + 1) = 0, \Rightarrow \lambda_1 = 0, \lambda_2 = 1, \lambda_3 = -1.$$

Consequently, the general solution of the homogeneous equation can be written as

$$y_0(x) = C_1 + C_2e^x + C_3e^{-x},$$

where C_1, C_2, C_3 are arbitrary numbers.

Based on the structure of the right-hand side, we seek a particular solution in the form of trial function

$$y_1(x) = A\sin 3x + B\cos 3x.$$

The derivatives of this function are as follows:

$$y_1' = 3A\cos 3x - 3B\sin 3x,$$

$$y_1'' = -9A\sin 3x - 9B\cos 3x,$$

$$y_1''' = -27A\cos 3x + 27B\sin 3x.$$

Substituting these derivatives into the equation, we obtain

$$-27A\cos 3x + 27B\sin 3x - 3A\cos 3x + 3B\sin 3x = \sin 3x,$$

$$\Rightarrow -30A\cos 3x + 30B\sin 3x = \sin 3x, \Rightarrow \begin{cases} -30A = 0 \\ 30B = 1 \end{cases}, \Rightarrow \begin{cases} A = 0 \\ B = \frac{1}{30} \end{cases}.$$

Thus, a particular solution can be written as

$$y_1(x) = \frac{1}{30}\cos 3x.$$

Accordingly, the general solution of the nonhomogeneous equation is described by

$$y(x) = y_0(x) + y_1(x) = C_1 + C_2e^x + C_3e^{-x} + \frac{1}{30}\cos 3x.$$

Example 3. Find the general solution of the differential equation $y^{IV} - y = 2\cos x$.

We first consider the homogeneous equation

$$y^{IV} - y = 0$$

and construct its general solution. The characteristic equation

$$\lambda^4 - 1 = 0$$

has the following roots:

$$(\lambda^2 - 1)(\lambda^2 + 1) = 0, \Rightarrow (\lambda - 1)(\lambda + 1)(\lambda^2 + 1) = 0, \Rightarrow$$

$$\lambda_1 = 1, \lambda_2 = -1, \lambda_{3,4} = \pm i.$$

Consequently, the general solution of the homogeneous equation has the form:

$$y_0(x) = C_1e^x + C_2e^{-x} + C_3\cos x + C_4\sin x,$$

where $C_1, \dots, C_4$ are arbitrary numbers.

Now we find a particular solution of the nonhomogeneous equation. Here we have the resonance case, since the expression in the right side corresponds to one of the roots of the characteristic equation. Hence, we seek a particular solution in the form

$$y_1(x) = x(A\cos x + B\sin x).$$

The derivatives of this function are

$$y_1' = A\cos x + B\sin x + x(-A\sin x + B\cos x),$$

$$y_1'' = -A\sin x + B\cos x + (-A\sin x + B\cos x) + x(-A\cos x - B\sin x)$$
$$= -2A\sin x + 2B\cos x - x(A\cos x + B\sin x),$$

$$y_1''' = -2A\cos x - 2B\sin x - (A\cos x + B\sin x) - x(-A\sin x + B\cos x)$$
$$= -3A\cos x - 3B\sin x + x(A\sin x - B\cos x),$$

$$y^{IV} = 3A\sin x - 3B\cos x + (A\sin x - B\cos x) + x(A\cos x + B\sin x)$$
$$= 4A\sin x - 4B\cos x + x(A\cos x + B\sin x).$$

Substitute the derivatives in the nonhomogeneous equation and determine the coefficients A, B

$$4A\sin x - 4B\cos x + \cancel{x(A\cos x + B\sin x)} - \cancel{x(A\cos x + B\sin x)} = 2\cos x,$$

$$\Rightarrow \begin{cases} 4A = 0 \\ -4B = 2 \end{cases}, \Rightarrow \begin{cases} A = 0 \\ B = -\frac{1}{2} \end{cases}.$$

Thus, a particular solution is expressed as

$$y_1(x) = -\frac{x}{2}\sin x.$$

Then the general solution of the original nonhomogeneous equation can be written as

$$y(x) = y_0(x) + y_1(x) = C_1 e^x + C_2 e^{-x} + C_3\cos x + C_4\sin x - \frac{x}{2}\sin x.$$

Example 4. Solve the equation $y^{IV} + y''' - 3y'' - 5y' - 2y = e^{2x} - e^{-x}$.

First we find the general solution of the homogeneous equation

$$y^{IV} + y''' - 3y'' - 5y' - 2y = 0.$$

Write the characteristic equation and find its roots:

$$\lambda^4 + \lambda^3 - 3\lambda^2 - 5\lambda - 2 = 0, \Rightarrow \lambda^4 - 2\lambda^3 + 3\lambda^3 - 6\lambda^2 + 3\lambda^2 - 6\lambda + \lambda - 2 = 0,$$
$$\Rightarrow \lambda^3(\lambda - 2) + 3\lambda^2(\lambda - 2) + 3\lambda(\lambda - 2) + \lambda - 2 = 0,$$
$$\Rightarrow (\lambda^3 + 3\lambda^2 + 3\lambda + 1) \cdot (\lambda - 2) = 0, \Rightarrow (\lambda + 1)^3(\lambda - 2) = 0.$$

It is seen that the equation has two roots:

$$\lambda_1 = -1, \lambda_2 = 2,$$

and the multiplicity of the first root is 3.

Then the general solution of the homogeneous equation can be written as

$$y_0(x) = (C_1 + C_2x + C_3x^2)e^{-x} + C_4e^{2x},$$

where C_1, ..., C_4 are as usual arbitrary numbers.

We now construct a particular solution of the nonhomogeneous equation. Using the superposition principle, it is convenient to consider two nonhomogeneous equations of the form

$$y^{IV} + y''' - 3y'' - 5y' - 2y = e^{2x};$$

$$y^{IV} + y''' - 3y'' - 5y' - 2y = -e^{-x}.$$

The sum of the right sides of these equations corresponds to the right side of the original nonhomogeneous equation.

Note that we have the resonance cases in both equations. In the first equation the number 2 in the exponential function coincides with the root $\lambda_2 = 2$ of multiplicity 2. In the second equation the number −1 in the exponential function coincides with another root $\lambda_1 = -1$, the multiplicity of which is equal to 3. With this in mind, we seek particular solutions in the forms

$$y_1 = Axe^{2x}, y_2 = Bx^3e^{-x}.$$

The derivatives for the trial solution y_1 have the form

$$y_1' = A(e^{2x} + 2xe^{2x}) = A(2x + 1)e^{2x},$$

$$y_1'' = A[2e^{2x} + (4x + 2)e^{2x}] = A(4x + 4)e^{2x},$$

$$y_1''' = A[4e^{2x} + (8x + 8)e^{2x}] = A(8x + 12)e^{2x},$$

$$y_1^{IV} = A[8e^{2x} + (16x + 24)e^{2x}] = A(16x + 32)e^{2x}.$$

Substituting this into the first equation, we find the coefficient A:

$$A(16x + 32)e^{2x} + A(8x + 12)e^{2x} - 3A(4x + 4)e^{2x} - 5A(2x + 1)e^{2x} - 2Axe^{2x} = e^{2x},$$

$$\Rightarrow A(\cancel{16x} + \cancel{8x} - \cancel{12x} - \cancel{10x} - \cancel{2x})e^{2x} + A(32 + \cancel{12} - \cancel{12} - 5)e^{2x} = e^{2x},$$

$$\Rightarrow 27A = 1, \Rightarrow A = \frac{1}{27}.$$

Therefore, the particular solution y_1 is given by

$$y_1(x) = \frac{x}{27}e^{2x}.$$

Similarly, we find the particular solution y_2 The derivatives of the trial function y_2 are

$$y_2' = B(3x^2e^{-x} - x^3e^{-x}) = B(-x^3 + 3x^2)e^{-x},$$
$$y_2'' = B[(-3x^2 + 6x)e^{-x} - (-x^3 + 3x^2)e^{-x}] = B(x^3 - 6x^2 + 6x)e^{-x},$$
$$y_2''' = B[(3x^2 - 12x + 6)e^{-x} - (x^3 - 6x^2 + 6x)e^{-x}]$$
$$= B(-x^3 + 9x^2 - 18x + 6)e^{-x},$$
$$y_2^{IV} = B[(-3x^2 + 18x - 18)e^{-x} - (-x^3 + 9x^2 - 18x + 6)e^{-x}]$$
$$= B(x^3 - 12x^2 + 36x - 24)e^{-x}.$$

Substituting these derivatives into the second equation, we calculate the coefficient B

$$B(x^3 - 12x^2 + 36x - 24)e^{-x} + B(-x^3 + 9x^2 - 18x + 6)e^{-x} - 3B(x^3 - 6x^2 + 6x)e^{-x} - 5B(-x^3 + 3x^2)e^{-x} - 2Bx^3e^{-x} = -e^{-x},$$
$$\Rightarrow B(\cancel{x^3} - \cancel{x^3} - \cancel{3x^3} + \cancel{5x^3} - \cancel{2x^3})e^{-x} + B(-\cancel{12x^2} + \cancel{9x^2} + \cancel{18x^2} - \cancel{15x^2})e^{-x} + B(\cancel{36x} - \cancel{18x} - \cancel{18x})e^{-x} + B(-24 + 6)e^{-x} = -e^{-x},$$
$$\Rightarrow -18B = -1, \Rightarrow B = \frac{1}{18}.$$

We obtain the solution y_2 as follows:

$$y_2(x) = \frac{x^3}{18}e^{-x}.$$

In accordance with the principle of superposition, a particular solution of the original nonhomogeneous equation is represented as

$$y_{\mathrm{p}} = y_1(x) + y_2(x) = \frac{x}{27}e^{2x} + \frac{x^3}{18}e^{-x}.$$

Finally, the general solution is given by

$$y(x) = (C_1 + C_2x + C_3x^2)e^{-x} + C_4e^{2x} + \frac{x}{27}e^{2x} + \frac{x^3}{18}e^{-x}$$
$$= (C_1 + C_2x + C_3x^2 + \frac{x^3}{18})e^{-x} + (C_4 + \frac{x}{27})e^{2x}.$$

Example 5. Solve the equation $y''' + y' = \frac{1}{\cos x}$

First we solve the corresponding homogeneous equation

$$y''' + y' = 0.$$

The roots of its characteristic equation are:

$$\lambda^3 + \lambda = 0, \Rightarrow \lambda(\lambda^2 + 1) = 0, \Rightarrow \lambda_1 = 0, \lambda_{2,3} = \pm i.$$

Consequently, the general solution of the homogeneous equation has the form:

$$y_0(x) = C_1 + C_2\cos x + C_3\sin x,$$

where C_1, C_2, C_3 are arbitrary numbers.

According to the method of variation of constants, we will consider the functions $C_1(x)$, $C_2(x)$, $C_3(x)$ instead of the numbers C_1, C_2, C_3 to construct the general solution of the nonhomogeneous equation. These functions will satisfy the nonhomogeneous equation, provided

$$\begin{cases} C_1'Y_1 + C_2'Y_2 + C_3'Y_3 = 0 \\ C_1'Y_1' + C_2'Y_2' + C_3'Y_3' = 0 \\ C_1'Y_1'' + C_2'Y_2'' + C_3'Y_3'' = \dfrac{1}{\cos x} \end{cases}$$

Here the functions Y_1, Y_2, Y_3 are the fundamental system of solutions. They were found in the solution of the homogeneous equation:

$$Y_1 = 1, Y_2 = \cos x, Y_3 = \sin x.$$

Then the system of equations takes the form:

$$\begin{cases} C_1' \cdot 1 + C_2' \cos x + C_3' \sin x = 0 \\ C_1' \cdot 0 + C_2'(-\sin x) + C_3' \cos x = 0 \\ C_1' \cdot 0 + C_2'(-\cos x) + C_3'(-\sin x) = \dfrac{1}{\cos x} \end{cases}, \Rightarrow \begin{cases} C_1' + C_2' \cos x + C_3' \sin x = 0 \\ -C_2' \sin x + C_3' \cos x = 0 \\ -C_2' \cos x - C_3' \sin x = \dfrac{1}{\cos x} \end{cases}$$

The main determinant (Wronskian) is

$$W = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix} = 1 \cdot \begin{vmatrix} -\sin x & \cos x \\ -\cos x & -\sin x \end{vmatrix} = \sin^2 x + \cos^2 x = 1.$$

We find expressions for the derivatives $C_1{}'(x)$, $C_2{}'(x)$, $C_3{}'(x)$ calculating the other three determinants:

$$\Delta_1 = \begin{vmatrix} 0 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ \dfrac{1}{\cos x} & -\cos x & -\sin x \end{vmatrix} = \frac{1}{\cos x} \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \frac{1}{\cos x}(\cos^2 x + \sin^2 x)$$

$$= \frac{1}{\cos x},$$

$$\Delta_2 = \begin{vmatrix} 1 & 0 & \sin x \\ 0 & 0 & \cos x \\ 0 & \dfrac{1}{\cos x} & -\sin x \end{vmatrix} = 1 \cdot \begin{vmatrix} 0 & \cos x \\ \dfrac{1}{\cos x} & -\sin x \end{vmatrix} = -\frac{1}{\cos x} \cdot \cos x = -1,$$

$$\Delta_3 = \begin{vmatrix} 1 & \cos x & 0 \\ 0 & -\sin x & 0 \\ 0 & -\cos x & \dfrac{1}{\cos x} \end{vmatrix} = 1 \cdot \begin{vmatrix} -\sin x & 0 \\ -\cos x & \dfrac{1}{\cos x} \end{vmatrix} = -\sin x \cdot \frac{1}{\cos x} = -\tan x.$$

Consequently, the derivatives $C_1{}'(x)$, $C_2{}'(x)$, $C_3{}'(x)$ are given by

$$C_1' = \frac{\Delta_1}{W} = \frac{1}{\cos x}, C_2' = \frac{\Delta_2}{W} = -1, C_3' = \frac{\Delta_3}{W} = -\tan x.$$

The integrals of these functions are tabulated, so that we can immediately write:

$$C_1(x) = \int \frac{dx}{\cos x} = \ln|\tan(\frac{x}{2} + \frac{\pi}{4})| + A_1,$$

$$C_2(x) = \int (-1)dx = -x + A_2,$$

$$C_3(x) = \int (-\tan x)dx = \ln|\cos x| + A_3,$$

where A_1, A_2, A_3 are constants of integration.

Substituting this into the general solution, we find the answer in the following form:

$$\begin{aligned} y(x) &= C_1(x) + C_2(x)\cos x + C_3(x)\sin x \\ &= \ln|\tan(\frac{x}{2} + \frac{\pi}{4})| + A_1 + (-x + A_2)\cos x + (\ln|\cos x| + A_3)\sin x \\ &= A_1 + A_2\cos x + A_3\sin x + \ln|\tan(\frac{x}{2} + \frac{\pi}{4})| - x\cos x \\ &+ \sin x\ln|\cos x|. \end{aligned}$$