COMPLEX NUMBERS

The purpose of this paragraph is to give you a brief overview of complex numbers, notation associated with complex numbers, and some of the basic operations involving complex numbers.

1 Prelude to Complex Numbers

In the real number system, there is no solution to the equation $x^2 = -1$.

The backbone of this new number system (complex numbers) is the number i, also known as the *Imaginary Unit*.

$$i = \sqrt{-1}$$

So,

$$i^2 = -1$$

Hence, $i = \sqrt{-1}$ is defined so that we can deal with square roots of negative numbers as follows,

$$\sqrt{-100} = \sqrt{(100)(-1)} = \sqrt{100}\sqrt{-1} = \sqrt{100}i = 10i$$

By taking multiples of this imaginary unit, we can create infinitely many more new numbers, like $3i$, $\sqrt{5}i$ and $-12i$. These are examples of *imaginary numbers*.

However, we can go even further than that and add real numbers and imaginary numbers, for example $2 + 7i$, $3 - \sqrt{2}i$. These combinations are called *complex numbers*.

Defining complex numbers

So, let's give the definition of a complex number.

Given two real numbers a and b we will define the complex number z as,

$$z = a + bi$$

Note that at this point we've not actually defined just what i is at this point. The number a is called the *real part* of z and the number b is called the *imaginary part* of z and are often denoted as,

$$\mathrm{Re}z = a \text{ and } \mathrm{Im}z = b$$

The table below shows examples of complex numbers, with the real and imaginary parts identified. Some people find it easier to identify the real and imaginary parts if the number is written in standard form.

Complex Number	Standard Form: $a + bi$	Description of parts
$7i - 2$	$-2 + 7i$	The real part is −2, and the

		imaginary part is 7.
$4-3i$	$4+(-3)i$	The real part is 4, and the imaginary part is −3.
$9i$	$0+9i$	The real part is 0, and the imaginary part is 9.
-2	$-2+0i$	The real part is −2, and the imaginary part is 0.

In the last two cases, if the real part is 0 we call the complex number a *pure imaginary number*. Next, a complex number that has a zero imaginary part is in fact *a real number*.

So,

An imaginary number is a complex number $a+bi$, where $a=0$. Similarly, we can say that a real number is a complex number $a+bi$, where $b=0$.

From the first definition, we can conclude that any imaginary number is also a complex number. From the second definition, we can conclude that any real number is also a complex number.

The set of complex numbers is denoted by the symbol ℂ. Thus we have the chain of inclusions:

$$\mathbb{C} \supset \mathbb{R} \supset \mathbb{Q} \supset \mathbb{Z} \supset \mathbb{N}$$

So why do we study complex numbers anyway? Believe it or not, complex numbers have many applications—electrical engineering and quantum mechanics to name a few!

From a purely mathematical standpoint, one cool thing that complex numbers allow us to do is to solve any polynomial equation.

$$x^2+2x+5=0 \Rightarrow x_{1,2}=\frac{-2\pm\sqrt{2^2-4\cdot5}}{2}=\frac{-2\pm\sqrt{-16}}{2}=\frac{-2\pm\sqrt{i^2\cdot16}}{2}=\frac{-2\pm4\cdot i}{2}$$

Hence,

$$x1=-1+2\cdot i \text{ and } x2=-1-2\cdot i$$

To check the solutions, we calculated

$$\begin{aligned}(x-x_1)\cdot(x-x_2)&=(x+1-2\cdot i)\cdot(x+1+2\cdot i)=(x+1)^2-(2\cdot i)^2\\&=x^2+2x+1-(4\cdot i^2)=x^2+2x+1-4\cdot(-1)=x^2+2x+5\end{aligned}$$

As we continue our study of mathematics, we will learn more about these numbers.

2 Complex Number Operations

We next need to define how we do basic operations with complex numbers.

1. Given two complex numbers $z_1 = x_1 + iy_1$and $z_2 = x_2 + iy_2$ we define *addition* and *subtraction* as follows,

$$z = z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

and

$$z = z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i(y_1 - y_2)$$

We just have a bunch of real parts and imaginary parts that we can then add up together or subtract them.

For the sake of exactness, the operation of subtraction is introduced through the definition of an *additive inverse* $-z$ such that $z + (-z) = 0$, i.e. $-z = -a - bi$.

Then, to define the subtraction of two complex numbers, we add a complex number z_1 and an additive inverse of complex number z_2. This is

$$z_1 - z_2 = z_1 + (-z_2)$$

Examples:

1. $(-3 + 5i) + (4 - 8i) = (-3 + 4) + (5 - 8)i = 1 - 3i$
2. $(3 + 2i) + (-1 - 5i) = (3 - 1) + (2 - 5)i = 2 - 3i$
3. $(2 + 3i) + (6 - 3i) = (2 + 6) + (3 - 3)i = 8 - 0i = 8$
4. $(10 - 3i) + (-10 + 3i) = (10 - 10) + (-3 + 3)i = 0 - 0i = 0$

1. $(-5 + 2i) - (3 - 5i) = (-5 - 3) + (2 - (-5))i = -8 + 7i$
2. $(6 + 7i) - (6 - 5i) = (6 - 6) + (7 + 5)i = 12i$
3. $(0{,}3 + 2{,}5i) - (-0{,}75 + 1{,}5i) = (0{,}3 + 0{,}75) + (2{,}5 - 1{,}5)i = 1{,}05 + i$

2. Given two complex numbers $z_1 = x_1 + iy_1$and $z_2 = x_2 + iy_2$ we define multiplication as follows,

$$z = (x_1 + iy_1) \cdot (x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1x_2 + y_1y_2)$$

Example:

$$(1 - 2i) \cdot (3 + 2i) = (1 \cdot 3 - (-2) \cdot 2) + (1 \cdot 2 + (-2) \cdot 3)i = (3 + 4) + (2 - 6)i = 7 - 4i$$

3. Suppose that we have two complex numbers $z_1 = x_1 + iy_1$and $z_2 = x_2 + iy_2$ then the division of these two is defined to be,

$$\frac{z_1}{z_2} = z_1{z_2}^{-1}$$

where ${z_2}^{-1}$ is a *multiplicative inverse*.

A multiplicative inverse for a non-zero complex number $z = a + bi$ is an element denoted by z^{-1} such that $zz^{-1} = 1$. Following this formula, we get

$$z^{-1} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i$$

Hence,

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = (x_1 + iy_1)(x_2 + iy_2)^{-1} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} - i\frac{x_1x_2 - y_1y_2}{x_2^2 + y_2^2}$$

Example:

$$\frac{6 + 3i}{10 + 8i} = (6 + 3i)(10 + 8i)^{-1}$$

where $(10 + 8i)^{-1} = \frac{10}{10^2+8^2} - \frac{8}{10^2+8^2} i = \frac{10-8i}{164}$.

Then,

$$\frac{6 + 3i}{10 + 8i} = (6 + 3i)(10 + 8i)^{-1} = (6 + 3i)\frac{10 - 8i}{164} = \frac{60 - 48i + 30i - 24i^2}{164}$$
$$= \frac{21}{41} - \frac{9}{82} i$$

3. Conjugate and Modulus

There are a couple of other operations that we should take a look at since they tend to show up on occasion. We'll also take a look at quite a few nice facts about these operations.

The first one we'll look at is the *complex conjugate*, (or just the *conjugate*). Given the complex number $z = a + bi$ the complex conjugate is denoted by $\bar{z}$ and is defined to be,

$$\bar{z} = a - bi$$

In other words, we just switch the sign on the imaginary part of the number.

Here are some basic facts about conjugates.

$$\begin{aligned} \bar{\bar{z}} &= z \\ \overline{z_1 \pm z_2} &= \bar{z}_1 \pm \bar{z}_2 \\ \overline{z_1 z_2} &= \bar{z}_1 \bar{z}_2 \\ \overline{\left(\frac{z_1}{z_2}\right)} &= \frac{\bar{z}_1}{\bar{z}_2} \end{aligned}$$

The first one just says that if we conjugate twice we get back to what we started with originally and hopefully this makes some sense. The remaining three just say we can break up sum, differences, products and quotients into the individual pieces and then conjugate.

So, just so we can say that we worked a number example or two let's do a couple of examples illustrating the above facts.

$\overline{\overline{z}}$ for $z = 3 - 15i$: $\overline{z} = 3 + 15i \Rightarrow \overline{\overline{z}} = \overline{3 + 15i} = 3 - 15i = z$

$\overline{z_1 - z_2}$ for $z_1 = 5 + i$ and $z_2 = -8 + 3i$: $z_1 - z_2 = 13 - 2i \Rightarrow \overline{z_1 - z_2} = \overline{13 - 2i} = 13 + 2i$

$\overline{z_1} - \overline{z_2}$ for $z_1 = 5 + i$ and $z_2 = -8 + 3i$: $\overline{z}_1 - \overline{z}_2 = \overline{5 + i} - (\overline{-8 + 3i}) = 5 - i - (-8 - 3i) = 13 + 2i$

There is another nice fact that uses conjugates that we should probably take a look at. However, instead of just giving the fact away let's derive it. We'll start with a complex number $z = a + bi$ and then perform each of the following operations.

$$z + \overline{z} = a + bi + (a - bi) = 2a \qquad z - \overline{z} = a + bi - (a - bi) = 2bi$$

Now, recalling that $\text{Re}z = a$ and $\text{Im}z = b$ we see that we have,

$$\text{Re}z = \frac{z + \overline{z}}{2}, \qquad \text{Im}z = \frac{z - \overline{z}}{2i}$$

The other operation we want to take a look at in this section is the *modulus of a complex number*. Given a complex number $z = a + bi$ the modulus is denoted by $|z|$ and is defined by

$$|z| = \sqrt{a^2 + b^2}$$

Notice that the modulus of a complex number is always a real number and in fact it will never be negative since square roots always return a positive number or zero depending on what is under the radical.

We can get a nice fact about the relationship between the modulus of a complex number and its real and imaginary parts. To see this let's square both sides of Equation and use the fact that $\text{Re}z = a$ and I $\text{Im}z = b$. Doing this we arrive at

$$|z|^2 = a^2 + b^2 = (\text{Re}z)^2 + (\text{Im}z)^2$$

There is a very nice relationship between the modulus of a complex number and its conjugate. Let's start with a complex number $z = a + bi$ and take a look at the following product:

$$z\overline{z} = (a + bi)(a - bi) = a^2 + b^2$$

From this product we can see that

$$z\bar{z} = |z|^2$$

We can also now formalize the process for division from the previous section now that we have the modulus and conjugate notations. In order to get the i out of the denominator of the quotient we really multiplied the numerator and denominator by the conjugate of the denominator. Doing all this gives the following formula for division,

$$\frac{z_1}{z_2} = \frac{z_1\bar{z}_2}{z_2\bar{z}_2} = \frac{z_1\bar{z}_2}{|z_2|^2}$$

Then, given two complex numbers $z_1 = x_1 + iy_1$and $z_2 = x_2 + iy_2$ we define

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{x_1 + iy_1}{x_2 + iy_2} \cdot \frac{x_2 - iy_2}{x_2 - iy_2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} - i\frac{x_1x_2 - y_1y_2}{x_2^2 + y_2^2}$$

Example:

$$\frac{6+3i}{10+8i} = \frac{(6+3i)(10-8i)}{(10+8i)(10-8i)} = \frac{(60+24)-i(60-24)}{10^2+8^2} = \frac{84}{164} - \frac{36}{164}i$$
$$= \frac{21}{41} - \frac{9}{82}i$$

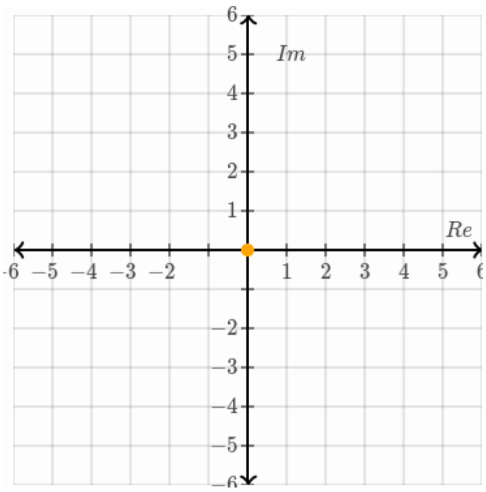
4 Polar & Exponential Form

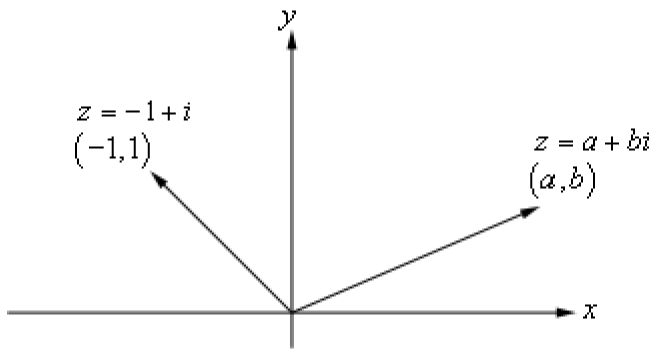
There are some alternate forms of a complex number that are useful at times.

Geometric Interpretation

Before we get into the alternate forms we should first take a very brief look at a natural geometric interpretation of complex numbers since this will lead us into our first alternate form.

Just like we can use the number line to visualize the set of real numbers, we can use *the complex plane* to visualize the set of complex numbers.





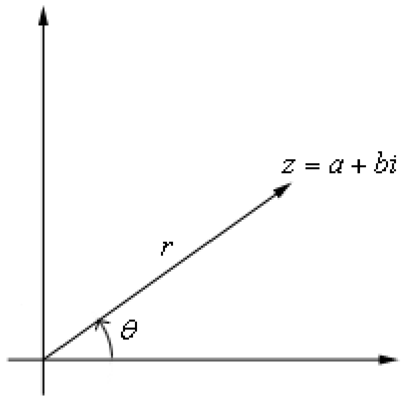
Consider the complex number $z = a + bi$. We can think of this complex number as either the point (a, b) in the standard Cartesian coordinate system or as the vector that starts at the origin and ends at the point (a, b).

The complex plane consists of two number lines that intersect in a right angle at the point (0,0). The horizontal number line (what we know as the *x*-axis on a Cartesian plane) is the real axis. The vertical number line (the *y*-axis on a Cartesian plane) is the imaginary axis.

An example of this is shown in the figure.

Polar Form

Let's now take a look at the first alternate form for a complex number. If we think of the non-zero complex number $z = a + bi$ as the point (a, b) in the *xy*-plane we also know that we can represent this point by the polar coordinates (r, θ), where r is the distance of the point from the origin and θ is the angle, in radians, from the positive x-axis to the ray connecting the origin to the point.



When working with complex numbers we assume that r is positive and that θ can be any of the possible (both positive and negative) angles that end at the ray. Note that this means that there are literally an infinite number of choices for θ.

We excluded $z = 0$ since θ is not defined for the point (0,0). We will therefore only consider the polar form of non-zero complex numbers.

We have the following conversion formulas for converting the polar coordinates (r, θ) into the corresponding Cartesian coordinates of the point, (a, b).

$$a = r\cos\theta\,, \qquad b = r\sin\theta$$

If we substitute these into $z = a + bi$ and factor an r out we arrive at the polar form of the complex number,

$$z = r(\cos\theta + i\sin\theta)$$

Note as well that we also have the following formula from polar coordinates relating r to a and b

$$r = \sqrt{a^2 + b^2}$$

but, the right side is nothing more than the definition of the modulus and we see that,

$$r = |z|$$

So, sometimes the polar form will be written as,

$$z = |z|(\cos\theta + i\sin\theta)$$

The angle θ is called *the argument* of z and is denoted by,

$$\theta = \arg z$$

The argument of z can be any of the infinite possible values of θ each of which can be found by solving

$$\tan\theta = \frac{b}{a}$$

and making sure that θ is in the correct quadrant.

Note as well that any two values of the argument will differ from each other by an integer multiple of 2π. This makes sense when you consider the following.

Exponential Form

Now that we've discussed the polar form of a complex number we can introduce the second alternate form of a complex number. First, we'll need Euler's formula,

$$\mathbf{e}^{i\theta} = \cos\theta + i\sin\theta$$

With Euler's formula we can rewrite the polar form of a complex number into its exponential form as follows.

$$z = r\mathbf{e}^{i\theta}$$

where $\theta = arg z$ and so we can see that, much like the polar form, there are an infinite number of possible exponential forms for a given complex number.

We can also get some formulas for the product or quotient of complex numbers. Given two complex numbers $z_1 = r_1\mathbf{e}^{i\theta_1}$ and $z_2 = r_2\mathbf{e}^{i\theta_2}$ where θ_1 is any value of $\arg z_1$ and θ_2 is any value of $\arg z_2$, we have

$$z_1 z_2 = (r_1\mathbf{e}^{i\theta_1})(r_2\mathbf{e}^{i\theta_2}) = r_1 r_2 \mathbf{e}^{i(\theta_1+\theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1\mathbf{e}^{i\theta_1}}{r_2\mathbf{e}^{i\theta_2}} = \frac{r_1}{r_2}\mathbf{e}^{i(\theta_1-\theta_2)}$$

The polar forms for both of these are,

$$z_1 z_2 \quad = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$$

$$\frac{z_1}{z_2} \quad = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$$

5 Powers and Roots

In this paragraph we're going to take a look at a really nice way of quickly computing integer powers and roots of complex numbers.

We'll start with integer powers of $z = r\mathbf{e}^{i\theta}$ since they are easy enough. If n is an integer then,

$$z^n = (r\mathbf{e}^{i\theta})^n = r^n\mathbf{e}^{in\theta}$$

There really isn't too much to do with powers other than working a quick example.

Example: $(3 + 3i)^5$

Of course, we could just do this by multiplying the number out, but this would be time consuming and prone to mistakes. Instead we can convert to exponential form and then to quickly get the answer.

The exponential form is

$$r = \sqrt{9+9} = 3\sqrt{2}, \qquad \tan\theta = \frac{3}{3} \Rightarrow \mathrm{Arg}z = \frac{\pi}{4}$$

Then,

$$3 + 3i = 3\sqrt{2}\mathbf{e}^{i\frac{\pi}{4}}$$

Now,

$$(3+3i)^5 = (3\sqrt{2})^5\mathbf{e}^{i\frac{5\pi}{4}} = 972\sqrt{2}(\cos(\frac{5\pi}{4}) + i\sin(\frac{5\pi}{4})) = 972\sqrt{2}(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i)$$
$$= -972 - 972i$$

Note that if $r = 1$ then we have,

$$z^n = (\mathbf{e}^{i\theta})^n = \mathbf{e}^{in\theta}$$

and if we take the last two terms and convert to polar form we arrive at a formula that is called *de Moivre's formula*.

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta) n = 0, \pm 1, \pm 2, \ldots$$