Higher ODEs

1. Reduction of Order

The differential equation of the *n*th order in the general case has the form:

$$F(x; y(x), y'(x), y''(x), \dots, y^{(n)}(x)) = 0,$$

where F is a continuous function of the specified arguments: an unknown function of one real or complex variable x, its derivatives.

A second order differential equation is written in general form as

$$F(x, y, y', y'') = 0,$$

where F is a function of the given arguments.

If the differential equation can be resolved for the highest derivative it can be represented in the following explicit form:

$$y^{(n)}(x) = f\left(x; y(x), y'(x), y''(x), \dots, y^{(n-1)}(x)\right)$$

For the *2*nd order differential equation, the following explicit form for the second derivative y'' may exist:

$$y'' = f(x, y, y').$$

In special cases the function f in the right side may contain only some variables. Such incomplete equations include 5 different types considered below.

Those special cases of the function f for the *2*nd order differential equation include the situation when in the right side may contain only one or two variables. Such incomplete equations of the 3 different types are as follows:

1) $y'' = f(x)$,

2.1) $y'' = f(x, y')$, 2.2) $y'' = f(y')$,

3.1) $y'' = f(y, y')$, 3.2) $y'' = f(y)$

With the help of certain substitutions, these equations can be transformed into first order equations.

In the general case of a second order differential equation, its order can be reduced if this equation has a certain symmetry. Below we discuss two types of such equations (cases 4 and 5):

4) The function $F(x, y, y', y'')$ is a homogeneous function of the arguments y, y', y'';

5) The function $F(x, y, y', y'')$ is an exact derivative of the first order function

$\Phi(x, y, y')$.

Consider examples for the various cases of order reduction of the second order and higher order differential equations.

Case 1. Equation of type $y'' = f(x)$

For an equation of type $y'' = f(x)$ its order can be reduced by introducing a new function $p(x)$ such that $y' = p(x)$. As a result, we obtain the first order differential equation

$$p' = f(x).$$

Solving it, we find the function $p(x)$. Then we solve the second equation

$$y' = p(x).$$

and obtain the general solution of the original equation.

Example 1. Solve the differential equation $y'' = \sin x + \cos x$.

This example relates to the Case 1. Consider the function $y' = p(x)$. Then $y'' = p'$. Consequently,

$$p' = \sin x + \cos x.$$

Integrating, we find the function $p(x)$:

$$\frac{dp}{dx} = \sin x + \cos x, \Rightarrow dp = (\sin x + \cos x)dx, \Rightarrow \int dp = \int (\sin x + \cos x)dx, \Rightarrow p = -\cos x + \sin x + C_1.$$

Given that $y' = p$ we integrate one more equation of the 1st order:

$$y' = -\cos x + \sin x + C_1, \Rightarrow \int dy = \int (-\cos x + \sin x + C_1)dx, \Rightarrow$$

$$y = -\sin x - \cos x + C_1 x + C_2.$$

The latter formula gives the general solution of the original differential equation.

Example 2. Find the general solution of the differential equation $y''' = x^2 - 1$.

We use the consecutive *n* times integration of the given right hand part of the differential equation. Then the general solution of the equation is represented as

$$y(x) = \frac{x^5}{60} - \frac{x^3}{6} + C_1 x^2 + C_2 x + C_3.$$

Here C_1, C_2, C_3 are arbitrary numbers.

Example 3. Find a particular solution of the equation $y^{IV} = \sin x + 1$ with the initial conditions $x_0 = 0, y_0 = 1, y_0' = y_0'' = y_0''' = 0$.

We first construct the general solution, successively integrating the given equation:

$$y''' = -\cos x + x + C_1,$$

$$y'' = -\sin x + \frac{x^2}{2} + C_1 x + C_2,$$

$$y' = \cos x + \frac{x^3}{6} + \frac{C_1 x^2}{2} + C_2 x + C_3,$$

$$y = \sin x + \frac{x^4}{24} + \frac{C_1 x^3}{6} + \frac{C_2 x^2}{2} + C_3 x + C_4.$$

Substituting the initial values, we determine the coefficients C_1–C_4 from the system of equations:

$$\left\{\begin{matrix} 0 = -1 + C_1 \\ 0 = C_2 \\ 0 = 1 + C_3 \\ 1 = C_4 \end{matrix}\right., \Rightarrow \left\{\begin{matrix} C_1 = 0 \\ C_2 = 0 \\ C_3 = -1 \\ C_4 = 1 \end{matrix}\right..$$

Hence, the particular solution satisfying the initial conditions has the form:

$$y(x) = \sin x + \frac{x^4}{24} + \frac{x^3}{6} - x + 1.$$

Example 4. Find the general solution of the differential equation $(y'')^2 - (y'')^3 = x$.

This equation can be solved by the parametric method. We put $y'' = t$. Then,

$$x = t^2 - t^3.$$

Given that $d(y') = y''dx$, we find the derivative y' expressed in terms of the parameter t:

$$d(y') = y''dx = t(2t - 3t^2)dt = (2t^2 - 3t^3)dt, \Rightarrow$$

$$y' = \int (2t^2 - 3t^3)dt = \frac{2t^3}{3} - \frac{3t^4}{4} + C_1.$$

Similarly, we perform one more integration:

$$dy = y'dx = (\frac{2t^3}{3} - \frac{3t^4}{4} + C_1) \cdot (2t - 3t^2)dt$$
$$= (\frac{4t^3}{3} - \frac{3t^4}{2} + 2C_1 t - t^5 + \frac{9t^6}{4} - 3C_1 t^2)dt,$$

$$\Rightarrow y = \int (\frac{4t^3}{3} - \frac{3t^4}{2} + 2C_1 t - t^5 + \frac{9t^6}{4} - 3C_1 t^2)dt$$

$$= \frac{t^4}{3} - \frac{3t^5}{10} + C_1 t^2 - \frac{t^6}{6} + \frac{9t^7}{28} - C_1 t^3 + C_2$$

$$= \frac{9t^7}{28} - \frac{t^6}{6} - \frac{3t^5}{10} + \frac{t^4}{3} - C_1 t^3 + C_1 t^2 + C_2.$$

Thus, the general solution is represented in parametric form as

$$\begin{cases} x = t^2 - t^3 \\ y = \frac{9t^7}{28} - \frac{t^6}{6} - \frac{3t^5}{10} + \frac{t^4}{3} - C_1 t^3 + C_1 t^2 + C_2 \end{cases}$$

where C_1, C_2 are arbitrary constants.

Case 2.1 Equation of type $y'' = f(x, y')$

Here we use the substitution $y' = p(x)$ where $p(x)$ is a new unknown function. As a result, we obtain the first order equation:

$$p' = \frac{dp}{dx} = f(x, p).$$

By integrating, we find the function $p(x)$. Next, we solve one more equation of the 1st order

$$y' = p(x)$$

and find the general solution $y(x)$.

Example 5. Solve the differential equation $\sqrt{x}y'' = (y')^2$.

This equation does not explicitly include the variable y, i.e. it corresponds to the type 4 in our classification. We introduce the new variable $y' = p(x)$. The original equation is transformed into the first order equation:

$$\sqrt{x}p' = p^2,$$

which is solved by separation of variables:

$$\sqrt{x}\frac{dp}{dx} = p^2, \Rightarrow \frac{dp}{p^2} = \frac{dx}{\sqrt{x}}, \Rightarrow \int \frac{dp}{p^2} = \int \frac{dx}{\sqrt{x}}, \Rightarrow -\frac{1}{p} = 2\sqrt{x} + C_1, \Rightarrow$$

$$p = y' = \frac{-1}{2\sqrt{x} + C_1}.$$

Integrating the resulting equation once more yields the function $y(x)$

$$\frac{dy}{dx} = \frac{-1}{2\sqrt{x} + C_1}, \Rightarrow dy = -\frac{dx}{2\sqrt{x} + C_1}, \Rightarrow y = -\int \frac{dx}{2\sqrt{x} + C_1}.$$

To compute the last integral we make the substitution: $x = t^2, dx = 2tdt$. As a result, we have

$$y = -\int \frac{dx}{2\sqrt{x} + C_1} = -\int \frac{2tdt}{2t + C_1} = -\int \frac{2t + C_1 - C_1}{2t + C_1} dt = -\int (1 - \frac{C_1}{2t + C_1}) dt$$

$$= -t + C_1 \int \frac{dt}{2t + C_1} = -t + \frac{C_1}{2} \int \frac{d(2t + C_1)}{2t + C_1}$$

$$= -t + \frac{C_1}{2} \ln|2t + C_1| + C_2.$$

Returning to the variable x, we finally obtain

$$y = -\sqrt{x} + \frac{C_1}{2} \ln|2\sqrt{x} + C_1| + C_2.$$

Case 2.2 Equation of type $y'' = f(y')$

In this case, to reduce the order we introduce the function $y' = p(x)$ and obtain the equation

$$y'' = p' = \frac{dp}{dx} = f(p),$$

which is a first order equation with separable variables p and x. Integrating, we find the function $p(x)$, and then the function $y(x)$.

Example 6. Solve the differential equation $y'' = \sqrt{1 - (y')^2}$.

This equation does not contain the function y and the independent variable x (Case 3). Therefore, we set $y' = p(x)$. Then this equation takes the form

$$y'' = p' = \sqrt{1 - p^2}.$$

The resulting first-order equation for the function $p(x)$ is a separable equation and can be easily integrated:

$$\frac{dp}{dx} = \sqrt{1 - p^2}, \Rightarrow \frac{dp}{\sqrt{1 - p^2}} = dx, \Rightarrow \int \frac{dp}{\sqrt{1 - p^2}} = \int dx, \Rightarrow$$

$$\arcsin p = x + C_1, \Rightarrow p = \sin(x + C_1).$$

Replacing p by y', we obtain

$$y' = \sin(x + C_1).$$

Integrating again, we find the general solution of the original differential equation:

$$\frac{dy}{dx} = \sin(x + C_1), \Rightarrow dy = \sin(x + C_1)\,dx, \Rightarrow \int dy = \sin\int (x + C_1)\,dx, \Rightarrow$$

$$y = -\cos(x + C_1) + C_2, \Rightarrow y = C_2 - \cos(x + C_1).$$

Example 7. Find the general solution of the differential equation $y''' = \sqrt{1 - (y'')^2}$.

This equation is of type 2. We introduce the new variable $z = y''$. This leads to the first-order equation:

$$z' = \sqrt{1 - z^2}.$$

Integrating, we find:

$$\frac{dz}{dx} = \sqrt{1 - z^2}, \frac{dz}{\sqrt{1 - z^2}} = dx, \int \frac{dz}{\sqrt{1 - z^2}} = \int dx,$$

$$\arcsin z = x + C_1, \Rightarrow z = \sin(x + C_1).$$

In fact, we have transformed the initial equation to an equation of type 1. The general solution $y(x)$ is most easily obtained by double integration of the expression for z:

$$y'' = \sin(x + C_1),$$

$$y' = -\cos(x + C_1) + C_2,$$

$$y = -\sin(x + C_1) + C_2 x + C_3,$$

where C_1, C_2, C_3 are arbitrary constants.

Case 3.1 Equation of type $y'' = f(y)$

The right-hand side of the equation depends only on the variable y. We introduce a new function $p(y)$, setting $y' = p(y)$. Then we can write

$$y'' = \frac{d}{dx}(y') = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = \frac{dp}{dy}p,$$

so the equation becomes:

$$\frac{dp}{dy}p = f(y).$$

Solving it, we find the function $p(y)$. Then we find the solution of the equation $y' = p(y)$ that is, the function $y(x)$.

Example 8. Solve the differential equation $y'' = \frac{1}{4\sqrt{y}}$.

This is an equation of type 3, where the right-hand side depends only on the variable y. We introduce the parameter $p(y) = y'$. Then the equation can be written as

$$y'' = \frac{dp}{dy}p = \frac{1}{4\sqrt{y}}.$$

We obtain the equation of the 1st order for the function $p(y)$ with separable variables. Integrating gives:

$$\frac{dp}{dy}p = \frac{1}{4\sqrt{y}}, \Rightarrow 2pdp = \frac{dy}{2\sqrt{y}}, \Rightarrow \int 2pdp = \int \frac{dy}{2\sqrt{y}}, \Rightarrow p^2 = \sqrt{y} + C_1,$$

where C_1 is a constant of integration.

Taking the square root of both sides, we find the function $p(y)$

$$p = \pm\sqrt{\sqrt{y} + C_1}.$$

Now recall that $y' = p$ and solve another equation of the 1st order:

$$y' = \pm\sqrt{\sqrt{y} + C_1}, \Rightarrow \frac{dy}{dx} = \pm\sqrt{\sqrt{y} + C_1}.$$

Separate the variables and integrate:

$$\frac{dy}{\sqrt{\sqrt{y} + C_1}} = \pm dx, \Rightarrow \int \frac{dy}{\sqrt{\sqrt{y} + C_1}} = \pm\int dx.$$

To calculate the integral on the left-hand side, make the replacement:

$$\sqrt{y} + C_1 = z, \Rightarrow dz = \frac{dy}{2\sqrt{y}}, \Rightarrow dy = 2\sqrt{y}dz = 2(z - C_1)dz.$$

Then the left-hand integral is equal to

$$\int \frac{dy}{\sqrt{\sqrt{y} + C_1}} = \int \frac{2(z - C_1)dz}{\sqrt{z}} = 2\int \left(\frac{z}{\sqrt{z}} - \frac{C_1}{\sqrt{z}}\right)dz = 2\int \left(z^{\frac{1}{2}} - C_1 z^{-\frac{1}{2}}\right)dz$$

$$= 2\left(\frac{z^{\frac{3}{2}}}{\frac{3}{2}} - C_1\frac{z^{\frac{1}{2}}}{\frac{1}{2}}\right) = \frac{4}{3}z^{\frac{3}{2}} - 4C_1 z^{\frac{1}{2}} = \frac{4}{3}\sqrt{(\sqrt{y} + C_1)^3} - 4C_1\sqrt{\sqrt{y} + C_1}.$$

As a result, we obtain the following algebraic equation:

$$\frac{4}{3}\sqrt{(\sqrt{y} + C_1)^3} - 4C_1\sqrt{\sqrt{y} + C_1} = C_2 \pm x,$$

where C_1, C_2 are constants of integration

The last expression is the general solution of the differential equation in implicit form.

Case 3.2 Equation of type $y'' = f(y, y')$

To solve this equation, we introduce a new function $p(y)$, setting $y' = p(y)$, similar to case 3.1. Differentiating this expression with respect to x leads to the equation

$$y'' = \frac{d(y')}{dx} = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = \frac{dp}{dy}p.$$

As a result, our original equation is written as an equation of the 1st order

$$p\frac{dp}{dy} = f(y, p).$$

Solving it, we find the function $p(y)$. Then we solve another first order equation

$$y' = p(y)$$

and determine the general solution $y(x)$.

Example 9. Solve the differential equation $y'' = (2y + 3)(y')^2$.

This equation does not explicitly contain the independent variable x, that is refers to the Case 3. Let $y' = p(y)$ Then the equation can be written as

$$p' = (2y + 3)p^2.$$

Separate variables and integrate:

$$\frac{dp}{p^2} = (2y + 3)dy, \Rightarrow \int \frac{dp}{p^2} = \int (2y + 3)dy, \Rightarrow -\frac{1}{p} = y^2 + 3y + C_1, \Rightarrow$$

$$p = y' = \frac{-1}{y^2 + 3y + C_1}.$$

Integrating again, we obtain the final solution in implicit form:

$$(y^2 + 3y + C_1)dy = -dx, \Rightarrow \int (y^2 + 3y + C_1)dy = -\int dx,$$

$$\Rightarrow y^3 + \frac{3y^2}{2} + C_1 y + C_2 = -x, \Rightarrow 2y^3 + 3y^2 + C_1 y + C_2 + 2x = 0,$$

where C_1, C_2 are constants of integration.