2. *Higher Order Linear Homogeneous Differential Equations with Constant Coefficients*

The linear homogeneous differential equation of the *n*th order with constant coefficients can be written as

$$y^{(n)}(x) + a_1 y^{(n-1)}(x) + \cdots + a_{n-1} y'(x) + a_n y(x) = 0,$$

$a_1, a_2, \ldots, a_n$ are constants which may be real or complex.

A solution of the form $y(x) = e^{\lambda x}$, where λ (the lowercase Greek letter lambda) is some constant, can be used.

Hence, if $y(x) = e^{\lambda x}$ than $y'(x) = \lambda e^{\lambda x}$, $y''(x) = \lambda^2 e^{\lambda x}$, …, $y^{(n)}(x) = \lambda^n e^{\lambda x}$.

Substituting these expressions into the equation, we get

$$\lambda^n e^{\lambda x} + a_1 \lambda^{n-1} e^{\lambda x} + \cdots + a_{n-1} \lambda e^{\lambda x} + a_n e^{\lambda x} = 0 \Rightarrow$$

$$e^{\lambda x} \cdot (\lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n) = 0 \Rightarrow$$

Since, $e^{\lambda x}$ does not equal to zero, we get

$$\lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$$

This algebraic equation

$$L(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$$

is called the *characteristic equation* of the differential equation.

According to the fundamental theorem of algebra, a polynomial of degree *n* has exactly *n* roots, counting multiplicity. In this case the roots can be both real and complex (even if all the coefficients of $a_1, a_2, \ldots, a_n$ are real).

Example 1 Solve the following IVP.

$$y^{(3)} - 5y'' - 22y' + 56y = 0, y(0) = 1, y'(0) = -2, y''(0) = -4$$

The characteristic equation is,

$$\lambda^3 - 5\lambda^2 - 22\lambda + 56 = (\lambda + 4)(\lambda - 2)(\lambda - 7) = 0 \Rightarrow \lambda_1 = -4, \lambda_2 = 2, \lambda_3 = 7$$

So, we have three real distinct roots here and so the general solution is,

$$y(x) = C_1 e^{-4x} + C_2 e^{2x} + C_3 e^{7x}$$

Differentiating a couple of times and applying the initial conditions gives the following system of equations that we'll need to solve in order to find the coefficients.

$$\begin{aligned} 1 &= y(0) = c_1 + c_2 + c_3 \\ -2 &= y'(0) = -4c_1 + 2c_2 + 7c_3 \\ -4 &= y''(0) = 16c_1 + 4c_2 + 49c_3 \end{aligned} \Rightarrow \begin{aligned} c_1 &= \frac{14}{33} \\ c_2 &= \frac{13}{15} \\ c_3 &= -\frac{16}{55} \end{aligned}$$

The actual solution is then,

$$y(x) = \frac{14}{33}e^{-4x} + \frac{13}{15}e^{2x} - \frac{16}{55}e^{7x}$$

Also note that we'll not be showing very much work in solving the characteristic polynomial. We are using computational aids here and would encourage you to do the same here. Solving these higher degree polynomials is just too much work and would obscure the point of these examples.

Example 2 Solve the following differential equation

$$2y^{(4)} + 11y^{(3)} + 18y'' + 4y' - 8y = 0$$

The characteristic equation is,

$$2\lambda^4 + 11\lambda^3 + 18\lambda^2 + 4\lambda - 8 = (2\lambda - 1)(\lambda + 2)^3 = 0$$

So, we have two roots here, $\lambda_1 = \frac{1}{2}$ and $\lambda_{2,3,4} = -2$ which is multiplicity of 3. Remember that we'll get three solutions for the second root and after the first one we add x's only the solution until we reach three solutions.

The general solution is then,

$$y(x) = C_1 e^{\frac{1}{2}x} + C_2 e^{-2x} + C_3 x e^{-2x} + C_4 x^2 e^{-2x}$$

Example 3 Solve the following differential equation

$$y^{(5)} + 12y^{(4)} + 104y^{(3)} + 408y'' + 1156y' = 0$$

The characteristic equation is,

$$\lambda^5 + 12\lambda^4 + 104\lambda^3 + 408\lambda^2 + 1156\lambda = \lambda(\lambda^2 + 6\lambda + 34)^2 = 0$$

So, we have one real root $\lambda = 0$ and a pair of complex roots $\lambda = -3 \pm 5i$ each with multiplicity 2. So, the solution for the real root is easy and for the complex roots we'll get a total of 4 solutions, 2 will be the normal solutions and two will be the normal solution each multiplied by x.

The general solution is,

$$\begin{aligned} y(x) = C_1 &+ C_2 e^{-3x}\cos(5x) + C_3 e^{-3x}\sin(5x) + C_4 x e^{-3x}\cos(5x) \\ &+ C_5 x e^{-3x}\sin(5x) \end{aligned}$$

Example 4 Solve the following differential equation

$$y^{(5)} - 15y^{(4)} + 84y^{(3)} - 220y'' + 275y' - 125y = 0$$

The characteristic equation is

$$\lambda^5 - 15\lambda^4 + 84\lambda^3 - 220\lambda^2 + 275\lambda - 125 = (\lambda - 1)(\lambda - 5)^2(\lambda^2 - 4\lambda + 5) = 0$$

In this case we've got one real distinct root, $\lambda = 1$, and double root, $\lambda = 5$, and a pair of complex roots, $\lambda = 2 \pm i$ that only occur once.

The general solution is then,

$$y(x) = C_1 e^x + C_2 e^{5x} + C_3 x e^{5x} + C_4 e^{2x}\cos(x) + C_5 e^{2x}\sin(x)$$

Example 5 Solve the following differential equation

$$y^{(4)} + 16y = 0$$

The characteristic equation is

$$\lambda^4 + 16 = 0$$

So, a really simple characteristic equation. However, in order to find the roots we need to compute the fourth root of -16. The 4 (and yes there are 4!) 4th roots of -16 can be found by evaluating the following,

$$\sqrt[4]{-16} = (-16)^{\frac{1}{4}} = \sqrt[4]{16}\mathbf{e}^{(\frac{\pi}{4}+\frac{\pi k}{2})i} = 2(\cos(\frac{\pi}{4} + \frac{\pi k}{2}) + i\sin(\frac{\pi}{4} + \frac{\pi k}{2}))k = 0,1,2,3$$

Note that each value of k will give a distinct 4th root of -16. Also, note that for the 4th root (and ONLY the 4th root) of any negative number all we need to do is replace the 16 in the above formula with the absolute value of the number in question and this formula will work for those as well.

Here are the 4th roots of -16.

$$k = 0: 2(\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4})) = 2(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i) = \sqrt{2} + i\sqrt{2}$$

$$k = 1: 2(\cos(\frac{3\pi}{4}) + i\sin(\frac{3\pi}{4} + \frac{\pi k}{2})) = 2(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i) = -\sqrt{2} + i\sqrt{2}$$

$$k = 2: 2(\cos(\frac{5\pi}{4}) + i\sin(\frac{5\pi}{4})) = 2(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i) = -\sqrt{2} - i\sqrt{2}$$

$$k = 3: 2(\cos(\frac{7\pi}{4}) + i\sin(\frac{7\pi}{4})) = 2(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i) = \sqrt{2} - i\sqrt{2}$$

So, we have two sets of complex roots: $\lambda = \sqrt{2} \pm i\sqrt{2}$ and $\lambda = \sqrt{2} \pm i\sqrt{2}$.

The general solution is,

$$y(x) = C_1 e^{\sqrt{2}x}\cos(\sqrt{2}x) + C_2 e^{\sqrt{2}x}\sin(\sqrt{2}x) + C_3 e^{-\sqrt{2}x}\cos(\sqrt{2}x) + C_4 e^{-\sqrt{2}x}\sin(\sqrt{2}x)$$