RECOMMENDED LITERATURE

1. Rudnyeva G.V. Elements of linear algebra and analytic geometry. Textbook for students of technical specialties. – Kharkiv: NTU KhPI, 2008. – 168 p.

2. Higher mathematics. Problems solving and variants of typical calculations. Vol. I. Educational textbook for students of physical engineering, energomachinebuilding and economic specialities. *Kurpa L.V.* (Ed.) – Kharkiv: NTU “KhPI”, 2002 – 339 p.

3. Higher mathematics. Analytical geometry and linear algebra. Vol. 1. *Kurpa L.V.* (Ed.) – Kharkiv: NTU “KhPI”, 2006. – 364 p./ Высшая математика. Аналитическая геометрия и линейная алгебра.Под ред. проф. Л.В. Курпа. – Харьков, НТУ "ХПИ", 2006. – 364 с - Русск. и англ.

4. Higher mathematics. Differential and Integral calculus of one-variable functions. Vol. 2. *Kurpa L.V.* (Ed.) – Kharkiv: NTU “KhPI”, 2006. – 542 p./ Высшая математика. Дифференциальное и интегральное исчисление функций одной переменной. Higher mathematics. В 4-х томах. Т. 2/ Под ред. проф. Л.В. Курпа. – Харьков, НТУ "ХПИ", 2006. – 542 с. - Русск. и англ.

5. Kurpa L.V., Shmatko T.V. Differential calculus for one variable functions. – Kharkiv: NTU “KhPI”, 2008. – 200 p./ Л.В. Курпа, Т.В. Шматко. Диференціальне числення функції однієї змінної.– Харьков, НТУ "ХПИ", 2008. – 200 с. – Англ. мовою.

6. Kurpa L.V., Shmatko T.V. Integral calculus for one variable functions. – Kharkiv: NTU “KhPI”, 2007. – 96 p./ Л.В. Курпа, Т.В. Шматко. Інтегральне числення для функцій однієї змінної.– Харьков, НТУ "ХПИ", 2007. – 96 с. – Англ. мовою.

7. Higher mathematics. Differential and Integral calculus of functions of several variables. Vol. 3. *Kurpa L.V.* (Ed.) – Kharkiv: NTU “KhPI”, 2006. – 364 p./ Высшая математика. Дифференциальное и интегральное исчисление функций нескольких переменной. В 4-х томах. Т. 3/ Под ред. проф. Л.В. Курпа. – Харьков, НТУ "ХПИ", 2006. – 364 с. - Русск. и англ.

LECTURE #1: BASIC DEFINITIONS OF LINEAR ALGEBRA

1.1 Concept of Matrix

Solutions of many important problems in engineering, economic, etc. lead to the solving of the systems of linear algebraic equations. The subject **LINEAR ALGEBRA** *deals with methods for solving system of linear algebraic equations.*

In general case the system of *m* linear algebraic equations with *n* unknown variables has the following form:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2 \\ \ldots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n = b_m \end{cases} \quad (1)$$

where x_i with subscripts $i = 1,2,\ldots,n$ are unknown variables, a_{ij} $(i = \overline{1,m},\ j = \overline{1,n})$ are real coefficients, $b_i\ (i = \overline{1,m})$ are real numbers called free members.

The coefficients of a_{ij} $(i = \overline{1,m},\ j = \overline{1,n})$ can be considered as a rectangular array of numbers or matrix. Therefore,

Definition. The matrix of the size *m* by *n* is a rectangular array of numbers with *m* rows and *n* columns. We use the following notations to sign the matrices: either capital letters of Latin alphabet or small letters in round or square brackets with indices, i.e.

$$A = \begin{pmatrix} a_{11}\ a_{12}\ a_{13}\ \ldots\ a_{1n} \\ \ldots \\ a_{m1}\ a_{m2}\ a_{m3}\ \ldots\ a_{mn} \end{pmatrix}$$

or

$$A = \left(a_{ij}\right)_{m,n} = [a_{ij}]_{m,n}$$

$i = \overline{1\ldots m},\ j = \overline{1\ldots n}$.

The individual elements a_{ij} are called *entries* or *components of matrix A*. The first subscript is the number of the *i*-th row and the second one is the number of the *j*-th column where the entry a_{ij} is situated. Here we have to intend *m* is a total number of rows, while *n* is a total number of columns in the matrix *A*.

Investigations of such arrays corresponding to the systems of linear equations helps to find solutions of formulated above systems (1).

Definition. If m=n then the matrix A is called the *square matrix*. In other words, A is a square matrix if it has the same number of rows and columns.

$$A=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nj} & \dots & a_{nn} \end{pmatrix}$$

Definition. If m=1 then the matrix is called the *row matrix*.

$$A = (a_{11}\, a_{12} \dots a_{1n});$$

Definition. If n=1 then the matrix is called the *column matrix*.

$$B=\begin{pmatrix} b_{11} \\ b_{12} \\ \dots \\ b_{m1} \end{pmatrix};$$

Definition. The matrix with all its entries equal to zero is called a *null matrix* or *zero matrix*. It is denoted by O.

$$O_{n\times m}=\begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix};$$

In fact, there is a zero matrix for every size!

Let us consider various matrices only.

Definition. The set of the elements $a_{11}, a_{22}, \dots, a_{nn}$ of a square matrix A is called the *main* (or *leading*) *diagonal* of the matrix, and these elements are called the elements of the main diagonal.

Example. Let us consider the matrix $A=\begin{pmatrix} 1 & 3 & -1 \\ 4 & 0 & 5 \\ 7 & 5 & 2 \end{pmatrix}$. The elements (1,0,2) are elements of the main diagonal.

Definition. The set of the elements $a_{1n}, a_{2n-1}, \dots, a_{n1}$ of a square matrix *A* is called the *secondary diagonal* of the matrix, while these elements are called the elements of the secondary diagonal.

The matrix elements (−1,0,7) from the previous example are the elements of the secondary diagonal.

Definition. A square matrix *A* with zero elements everywhere except in the leading diagonal is called a *diagonal matrix*.

$$A=\begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}.$$

Definition. The identity (unit) matrix is a diagonal matrix with all its diagonal elements equal to one. The identity matrix is always a square matrix, and it has the property that there are ones down the main diagonal and zeroes elsewhere.

Usual notations for such matrices are by the letters *I* or *E*.

$$I \equiv E = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

Here are some identity matrices of various sizes.

$$[1], \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The first is the 1×1 identity matrix, the second is the 2×2 identity matrix, and so on. By extension, you can likely see what the $n \times n$ identity matrix would be. When it is necessary to distinguish which size of identity matrix is being discussed, we will use the notation I_n for the $n \times n$ identity matrix.

The identity matrix is so important that there is a special symbol to denote the ij^{th} entry of the identity matrix. This symbol is given by $I_{ij} = \delta_{ij}$ where δ_{ij} is the ***Kronecker symbol*** defined by

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

Definition. If all the elements of a square matrix A located below (above) main diagonal are equal to zero then this matrix is called the *upper* (*lower*) *triangular matrix*.

$$A_{n\times n}=\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{nn} \end{pmatrix}.$$

Example. Suppose we have the following matrices

$$A=\begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 8 \end{pmatrix},\ B=\begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 5 & 3 & 8 \end{pmatrix},\ C=\begin{pmatrix} 9 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 8 \end{pmatrix},\ D=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Here matrix A is upper triangular, B is lower triangular, C is diagonal and D is nor diagonal, nor unit since D is not square.

Definition. If all the elements of a rectangular $r\times n$ matrix A are located as follows

$$A_{r\times k}=\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1r} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2r} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3r} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{rr} & \dots & a_{rn} \end{pmatrix},$$

where $a_{ij}\neq 0$, $i=1,2,\dots,r;\ r\leq k$, one can say that the matrix is in *echelon form* (*row echelon form*).

Specifically, a matrix is in row echelon form if

1. If there are any zero rows, they must be at the bottom of the matrix.
2. The first nonzero entry from the left of a nonzero row is a 1, which is also called the leading one of that row.
3. The leading one for each nonzero row appears to the right and below any leading ones in the previous rows.
4. For a column with a leading one, the other entries in that column are zero.

A similar definition can be made for reduced column echelon form and column echelon form.

Definition. Two matrices A and B of identical sizes are called *equal* if their corresponding elements are equal, i.e.

$$A=B \Leftrightarrow a_{ij}=b_{ij} \quad i=\overline{1,m},\ j=\overline{1,n}.$$

(a) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ because they are different sizes. Also,

(b) $\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ because, although they are the same size, their corresponding entries are not identical.

In the following section, we explore linear operations of matrices.

1.2 Operations on Matrices

Since the matrices are mathematical objects It is naturally to introduce some algebraic operations on them such as, for example, the addition, subtraction, multiplication and division.

Definition. The algebraic sum of matrices A and B of identical sizes is called the matrix $C=A+B$ of the same size as A and B with the elements defined by the following formula

$$c_{ij}=a_{ij}+b_{ij},\ i=\overline{1,m},\ j=\overline{1,n}.$$

Example. Let $A=\begin{pmatrix} 1 & 0 & 2 \\ 3 & 4 & 5 \end{pmatrix}$, $B=\begin{pmatrix} 3 & 1 & -2 \\ -1 & 0 & 2 \end{pmatrix}$. Then

$$A+B=\begin{pmatrix} 1 & 0 & 2 \\ 3 & 4 & 5 \end{pmatrix}+\begin{pmatrix} 3 & 1 & -2 \\ -1 & 0 & 2 \end{pmatrix}=\begin{pmatrix} 1+3 & 0+1 & 2-2 \\ 3-1 & 4+0 & 5+2 \end{pmatrix}=\begin{pmatrix} 4 & 1 & 0 \\ 2 & 4 & 7 \end{pmatrix}.$$

Note 1. Cannot be added two matrices of different sizes.

Note 2. Addition of matrices obeys very much the same properties as normal addition with numbers. Note that when we write for example $A+B$ then we assume that both matrices are of equal size so that the operation is indeed possible.

Basic properties of addition of matrices:

1. *Commutative Law of Addition:* $A+B=B+A$ (commutability)
2. *Associative Law of Addition*: $(A+B)+C=A+(B+C)$ (associability)

3. *Existence of an Additive Identity*: "There exists a zero matrix 0 such that $A + 0 = A$"
4. *Existence of an Additive Inverse*: “"There exists a matrix $-A$ such that $A + (-A) = 0$ ”

We call the zero matrix in [3] the *additive identity*. Similarly, we call the matrix $-A$ in [4] the *additive inverse*. $-A$ is defined to equal $(-1)A = [-a_{ij}]$. In other words, every entry of A is multiplied by -1 .

Definition. The result of multiplying a matrix $A = (a_{ij})_{m,n}$ by a real number k (a scalar) is a matrix $B = (b_{ij})_{m,n}$ whose elements b_{ij} are $b_{ij} = ka_{ij}$, $i = \overline{1,m}$, $j = \overline{1,n}$, i.e. k times the components of A.

Note. The subtraction can be defined then in the following way

$$C=A-B=A+(-1)B.$$

Examples:

If $A = (3 \quad -1 \quad 2)$ then $3A = 3(3 \quad -1 \quad 2) = (3\cdot 3 \quad 3\cdot(-1) \quad 3\cdot 2) = (9 \quad -3 \quad 6)$.

If $A = \begin{bmatrix} 2 & 0 \\ 1 & -4 \end{bmatrix}$, then $7A = 7\begin{bmatrix} 2 & 0 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} 7(2) & 7(0) \\ 7(1) & 7(-4) \end{bmatrix} = \begin{bmatrix} 14 & 0 \\ 7 & -28 \end{bmatrix}$

If $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$, then $5A = \begin{pmatrix} 15 & 10 \\ 10 & 15 \end{pmatrix}$.

Corollary: A common factor of all elements of the matrix can be taken out the matrix.

Example:

$$\begin{pmatrix} 30 & 10 & 8 \\ 20 & 16 & 4 \end{pmatrix} = 2\begin{pmatrix} 15 & 5 & 4 \\ 10 & 8 & 2 \end{pmatrix}.$$

Similarly to addition of matrices, there are several properties of scalar multiplication which hold.

Basic properties of scalar multiplication of a matrix:

1. *Associative Law for Scalar Multiplication*: $(ks)A=k(sA)=s(kA)$
2. *Distributive Law over Scalar Addition*: $(k+s)A=kA+sA$
3. *Distributive Law over Matrix Addition*: $k(A+B)=kA+kB$

4. *Rule for Multiplication by 1:* $1A = A$

The Vector Form of a System of Linear Equations

Now we can demonstrate how scalar multiplication relates to linear systems of equations.

First, a linear combination is a sum consisting of vectors (column matrices) multiplied by scalars. For example,

$$\begin{bmatrix} 50 \\ 122 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 4 \end{bmatrix} + 8\begin{bmatrix} 2 \\ 5 \end{bmatrix} + 9\begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

is a linear combination of three vectors.

Suppose we have a system of equations given by

$$\begin{gathered} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{gathered}$$

We can express this system in vector form which is as follows:

$$x_1\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Notice that each vector used here is one column from the corresponding matrix of the system. There is one vector for each variable in the system, along with the constant vector.

The *operation of matrix multiplication* or the product of two matrices *AB* can be formed not for any two matrices, only for such two matrices *A* and *B* when the number of columns of matrix *A* is equal to the number of rows of matrix *B.*

Consider a product AB where A has size $m \times n$ and B has size $n \times p$. Then, the product in terms of size of matrices is given by

these must match!

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$$(m \times \quad n)(n \times p \quad) = m \times p$$

Note the two outside numbers give the size of the product. One of the most important rules regarding matrix multiplication is the following. If the two middle numbers don't match, you can't multiply the matrices!

Definition. Product of the matrix *A* of the size *m* by *n* and the matrix *B* of the size *n* by *p* is the matrix *C*=*AB* of the size *m* by *p* with elements defined as

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + a_{i3} b_{3j} + \ldots + a_{in} b_{nj}, \; i = \overline{1, m}, \; j = \overline{1, p}.$$

Note. To calculate the entry c_{ij} of the matrix *C*=*AB* we use all entries of the first matrix from the row number *i* and all entries of the second matrix from the column number *j*. That is why the rule of matrix multiplication can be called the rule "row by column" and formulated in the following way:

In order to get the element of the matrix *C*=*AB* situated in the i^{th} row and j^{th} column it is necessary to multiply all entries of the matrix *A* from i^{th} row by the corresponding entries of the matrix *B* from j^{th} column and then summarize the obtained products.

Example. Let $A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 4 & 5 \end{pmatrix}$, $B = \begin{pmatrix} -2 & -1 \\ 0 & 2 \\ 1 & 0 \end{pmatrix}$.

Then $AB = \begin{pmatrix} 1\cdot(-2) + 0\cdot 0 + 2\cdot 1 & 1\cdot(-1) + 0\cdot 2 + 2\cdot 0 \\ 3\cdot(-2) + 4\cdot 0 + 5\cdot 1 & 3\cdot(-1) + 4\cdot 2 + 5\cdot 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 5 \end{pmatrix}$.

Compare the products AB and BA, for matrices $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

The first product, AB is

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$

The second product, BA is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

Therefore,

$$AB \neq BA$$

This example illustrates that you cannot assume $AB = BA$ even when multiplication is defined in both orders. If for some matrices A and B it is true that $AB = BA$, then we say that A and B ***commute***. This is one important property of matrix multiplication.

The following are other important properties of matrix multiplication. Notice that these properties hold only when the size of matrices are such that the products are defined.

Basic properties of the matrix multiplication:

1. $k(AB)=(kA)B=A(kB)$ (distributivity)
2. $(A+B)C=AC+BC$, $C(A+B)=CA+CB$ and $A(kB+nC)=k(AB)+k(AC)$
3. $(AB)C=A(BC)$ (associability)
4. *Multiplication by the Identity Matrix* $AE=A$, $EA=A$
5. In general case $AB \neq BA$. Moreover, both AB and BA do not always exist at the same time.

Definition. The matrices A and B that satisfy the equality $AB=BA$ are called the *commutative matrices*.

Definition. The matrix A to the power n is the matrix $C = \underbrace{A\cdot A\cdot A\cdot A\cdot\ldots\cdot A}_{n \text{ times}}$.

Example. Let us find the n-th power of the matrix A, where

$$A=\begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}.$$

First, we are going to find several first powers of A to understand the rule.

$$A^2=\begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}=$$

$$=\begin{pmatrix} 0\cdot0+2\cdot0+0\cdot0 & 0\cdot2+2\cdot0+0\cdot0 & 0\cdot0+2\cdot3+0\cdot0 \\ 0\cdot0+0\cdot0+3\cdot0 & 0\cdot2+0\cdot0+3\cdot0 & 0\cdot0+0\cdot3+3\cdot0 \\ 0\cdot0+0\cdot0+0\cdot0 & 0\cdot2+0\cdot0+0\cdot0 & 0\cdot0+0\cdot3+0\cdot0 \end{pmatrix}=\begin{pmatrix} 0 & 0 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

$$A^3=\begin{pmatrix} 0 & 0 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}=$$

$$=\begin{pmatrix} 0\cdot0+2\cdot0+6\cdot0 & 0\cdot2+0\cdot0+6\cdot0 & 0\cdot0+0\cdot3+6\cdot0 \\ 0\cdot0+0\cdot0+0\cdot0 & 0\cdot2+0\cdot0+0\cdot0 & 0\cdot0+0\cdot3+0\cdot0 \\ 0\cdot0+0\cdot0+0\cdot0 & 0\cdot2+0\cdot0+0\cdot0 & 0\cdot0+0\cdot3+0\cdot0 \end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since for any n such that $n>3$ $A^n = A^{n-3}A^3$ and A^3 is zero matrix then $A^n = 0$ for $n>3$.

Another important operation on matrices is that of taking the *transpose*. For a matrix A, we denote the transpose of A by A^T. Before formally defining the transpose, we explore this operation on the following matrix:

$$\begin{bmatrix} 1 & 4 \\ 3 & 1 \\ 2 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 1 & 6 \end{bmatrix}$$

Definition. The matrix A^T obtained from the given matrix A by interchanging its rows with its columns is called the *transposed matrix*. In other words, we switch the row and column location of each entry:

$$A_{mxn} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{pmatrix} \quad A^T_{n\times m} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{m1} \\ a_{21} & a_{22} & \dots & a_{m2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{mm} \end{pmatrix}$$

Example. Let $A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 4 & 5 \end{pmatrix}$, $A^T = \begin{pmatrix} 1 & 3 \\ 0 & 4 \\ 2 & 5 \end{pmatrix}$

Properties of the Transpose of a Matrix

1. $(A^T)^T = A$
2. $(AB)^T = B^T A^T$
3. $(kA + nB)^T = kA^T + nB^T$

Definition. The matrix A is called the *symmetric matrix* if $A=A^T$, i.e.

$$a_{ij} = a_{ji} \quad i = \overline{1,m},\ j = \overline{1,n}.$$

Example:

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 5 & -3 \\ 3 & -3 & 7 \end{bmatrix}, A^T = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 5 & -3 \\ 3 & -3 & 7 \end{bmatrix}$$

So, A is symmetric.

Definition. The matrix A is called *skew-symmetric matrix* if $A = -A^T$, i.e.

$$a_{ij} = -a_{ji} \quad i = \overline{1,m},\ j = \overline{1,n}.$$

Example:

$$A = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & 2 \\ -3 & -2 & 0 \end{bmatrix}, A^T = \begin{bmatrix} 0 & -1 & -3 \\ 1 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix}$$

So, A is skew-symmetric.

The Matrix Form of a System of Linear Equations

Using the above operation, we can also write a system of linear equations in matrix form. In this form, we express the system as a matrix multiplied by a vector. Consider the following definition.

Suppose we have a system of equations given by

$$\begin{gathered} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{gathered}$$

Then we can express this system in matrix form as follows.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Note. The expression $AX = B$ is also known as the *Matrix Form* of the corresponding system of linear equations. The matrix A is simply the coefficient matrix of the system, the vector X is the column vector constructed from the variables of the system, and finally the vector B is the column vector constructed from the constants of the system. It is important to note that any system of linear equations can be written in this form.

Notice that if we write a *homogeneous system of equations* in matrix form, it would have the form $AX = 0$, for the zero vector 0.