Lecture #2: Definition of Determinants.

Any square matrix A of the n^{th} order (or $n \times n$ size) can be associated with a number called as a *determinant* of the n^{th} order and denoted by either $\det A$ or $|A|$.

2.1 The Determinants of the Second and Third Orders

Definition. The determinant of the second order $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ is a number corresponding to the square two-by-two matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and calculated by the following formula:

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

It is a difference of two terms, where the first one is a product of entries of the main diagonal and the second one is a product of entries of the secondary diagonal of the matrix.

Examples:

a) $\begin{vmatrix} 1 & 2 \\ 5 & 3 \end{vmatrix} = 1 \cdot 3 - 2 \cdot 5 = 3 - 10 = -7$.

b) $\begin{vmatrix} 1 & -5 \\ 7 & 8 \end{vmatrix} = 1 \times 8 - (-5) \times 7 = 8 + 35 = 43;$

c) $\begin{vmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix} = \cos^2\alpha - \sin^2\alpha = \cos 2\alpha.$

Definition. The determinant of the third order $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ is a number corresponding to the square third-by-third matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and calculated by the following formula:

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

To remember this formula, we can use the following "***rule of triangles***":

entries of the matrix A at the vertices of each triangle are multipliers of the triples in the formula. We have to take such products with the sign “+” if the corresponding triangle has a side parallel to the main diagonal of the matrix and with the sign “–” if it has a side parallel to the secondary diagonal of the matrix A, as shown in Fig. 1.

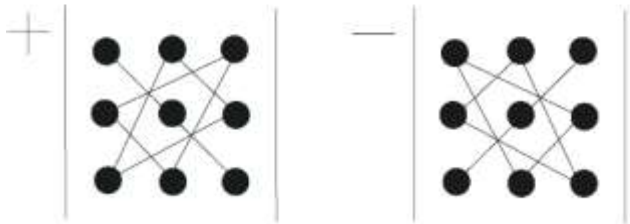


Figure 1. The “**Rule by Triangles**” to calculate the determinant of the third order.

***Examples*:**

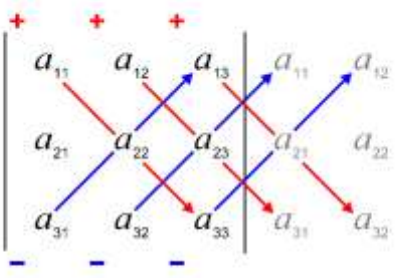
$$\begin{vmatrix} 1 & 2 & -1 \\ 2 & 3 & 5 \\ 1 & -1 & 0 \end{vmatrix} = \underbrace{1\cdot 3\cdot 0}_{entries\,of\ the\,main\,diagonal} + \overbrace{2\cdot 5\cdot 1}^{entries\,at\,the\,vertices} + \underbrace{(-1)\cdot 2\cdot(-1)}_{entries\,at\,the\,vertices}$$

$$- \underbrace{(-1)\cdot 3\cdot 1}_{entries\,of\ the\ secondary\,diagonal} - \overbrace{2\cdot 2\cdot 0}^{entries\,at\,the\,vertices} - \underbrace{1\cdot 5\cdot(-1)}_{entries\,at\,the\,vertices} =$$

$$= 0 + 10 + 2 + 3 - 0 + 5 = 20.$$

The other method for evaluating the third order determinant is the **Rule of Sarrus** (a mnemonic device) named after the French mathematician Pierre Frédéric Sarrus.

Rule of Sarrus: The determinant of the three columns on the left is the sum of the products along the down-right diagonals minus the sum of the products along the up-right diagonals.



$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

Figure 2. The “Rule of Sarrus” to calculate the determinant of the third order.

2.2 The Determinant of the n^{th} Order. Preliminary Idea

Let us look at the formulae more details than one can conclude that the determinant of <u>the second</u> order is the sum of <u>two factorial</u> terms where each one is the product of <u>two</u> elements of matrix taken by one from each row and each column with sign “+” or “–”.The determinant of <u>the third</u> order is the sum of <u>three factorial</u> terms where each one is the product of <u>three</u> elements of matrix taken by one from each row and each column with sign “+” or “–”.

Therefore, it is natural to assume that the determinant of <u>the n^{th}</u> order is the sum of <u>n factorial</u> terms where each one is the product of <u>n</u> elements of matrix taken by one from each row and each column with sign “+” or “–”. Herewith,

1) if the order of elements in each product is changed in the order of increasing the first subscript a formula to calculate the determinant of the n^{th} order is the following:

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = \sum (-1)^{[j_1 \;\; j_2 \;\; \cdots \;\; j_n]} a_{1j_1} a_{2j_2} a_{3j_3} \dots a_{nj_n}, \tag{1}$$

2) if the order of elements in each product is changed in the order of increasing the second subscript a formula to calculate the determinant of the n^{th} order is the following:

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = \sum (-1)^{[i_1 \;\; i_2 \;\; \cdots \;\; i_n]} a_{i_1 1} a_{i_2 2} a_{i_3 3} \dots a_{i_n n} \tag{2}$$

To define these formulae several new definitions should be introduced first.

Definition. Permutation of numbers $1,2,3,\dots n$ (or any different integer numbers $a_1, a_2, a_3, \dots a_n$) is any arrangement of them in some order.

Example. Let us consider several sequences of numbers given as

(1 2 3 4), (1 3 4 2), (2 3 1 4), and so on

They are different permutations of integers 1,2,3,4.

Note. Since any n numbers may be numbered by numbers $1,2,3,\ldots n$ it is enough to investigate the properties of permutations of these integers $1,2,3,\ldots n$.

How many such permutations do exist for a selected number of integers?
The number of all possible permutations is equal to 1*2*3*…*n=n!

Example, number of all permutations of numbers 1,2,3,4 is 4!=24.

Definition. Two numbers i and j in the permutation are said are creating *an inversion* if $i > j$, and i is followed by j, i.e. the bigger number stands before the smaller one.

A number of all inversions in the permutation $(a_1\ a_2\ a_3 \ldots a_n)$ we denote as $N(a_1\ a_2\ a_3 \ldots a_n)$ or $[a_1\ a_2\ a_3 \ldots a_n]$.

Example. Let us calculate number of inversions in the permutation (3 1 4 2).

- number 3 forms 2 inversions with numbers 1 and 2 (they are less then 3 but situated after 3);
- number 1 forms 0 inversions with numbers 4 and 2 since 1 is less then 4 and 2;
- number 4 forms 1 inversion with number 2; number 2 forms 0 inversions since it is the last number.

So, a number of all inversions is [3 1 4 2]=2+0+1+0=3.

Definition. The permutation is called *odd* if the number of all its inversions is odd number. And this permutation is called *even* if the number of all its inversions is even number.

Example. By means of the definitions introduced, we can now define the determinant of the third order.

Let us keep the *first subscripts* the same and consider all possible permutations of the *second subscripts,* i.e. we are going to use the second given above formula (2) to define the third-order determinant. All the results we put into the following table:

Permutation $(j_1\, j_2\, j_3)$	Number of inversions $[j_1\, j_2\, j_3]$	Sign of $(-1)^{[j_1 j_2 j_3]}$	Product $a_{1j_1}a_{2j_2}a_{3j_3}$
(1 2 3)	0+0+0=0	+	$a_{11}a_{22}a_{33}$
(1 3 2)	0+1+0=1	–	$a_{11}a_{23}a_{32}$
(2 1 3)	1+0+0=1	–	$a_{12}a_{21}a_{33}$
(2 3 1)	1+1+0=2	+	$a_{12}a_{23}a_{31}$
(3 1 2)	2+0+0=2	+	$a_{13}a_{21}a_{32}$
(3 2 1)	2+1+0=3	–	$a_{13}a_{22}a_{31}$

Thus, if the order of elements in each product is changed in the order of increasing the second subscript a formula to calculate the determinant of the 3rd order is

$$\det A = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}.$$

So, we obtained the same expression as the given early.

2.3 Properties of the Determinant of the n^{th} Order.

We will see later that using the following properties can greatly assist in finding determinants. This section will use the theorems as motivation to provide various examples of the usefulness of the properties.

Property 1. Determinant of the Transpose: Let A be a matrix where A^T is the transpose of A. Then, $\det A = \det A^T$.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}.$$

Proof. $\det A = \sum (-1)^{[j_1 \;\; j_2 \;\; \cdots \;\; j_n]} a_{1j_1}a_{2j_2}a_{3j_3}\ldots a_{nj_n}$. Let us denote elements of the transposed matrix A^T as a'_{ij}, $i = \overline{1,n}$, $j = \overline{1,n}$. Then $a'_{ij} = a_{ji}$.

By definition:

$$\det A^T = \sum (-1)^{[j_1 \;\; j_2 \;\; \cdots \;\; j_n]} a'_{j_1 1}a'_{j_2 2}a'_{j_3 3}\ldots a'_{j_n n} =$$

$$= \sum (-1)^{[j_1 \;\; j_2 \;\; \cdots \;\; j_n]} a_{1j_1}a_{2j_2}a_{3j_3}\ldots a_{nj_n}.$$

We obtained the same expression as for $\det A$. ***The property is proven.***

Note. All properties that are valid for the rows will be valid for the columns.

Property 2. If A is an $n \times n$ matrix such that one of its rows (columns) consists of zeros, then the determinant of the matrix is equal to zero.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{vmatrix} = 0.$$

Proof. Let the i^{th} row be zero-row. Then

$$\det A = \sum (-1)^{[j_1 \quad j_2 \quad \cdots \quad j_n]} a_{1j_1} \ldots a_{ij_i} \ldots a_{nj_n} =$$

$$= \sum (-1)^{[j_1 \quad j_2 \quad \cdots \quad j_n]} a_{1j_1} \ldots 0 \ldots a_{nj_n} = \sum 0 = 0.$$

The property is proven.

Property 3. Switching Rows (Columns): Let A be an $n \times n$ matrix and let B be a matrix which results from switching two rows (columns) of A. Then $det(B) = -det(A)$.

In other words, if the determinant Δ' is obtained from Δ by interchanging any two rows (columns), the sign of the determinant changes by the opposite one, $\Delta' = -\Delta$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \\ a_{32} & a_{31} & a_{33} \end{vmatrix}.$$

Property 4. The determinant with two identical rows (columns) is equal to 0.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a & b & c \\ a & b & c \end{vmatrix} = \begin{vmatrix} a & a & a_{13} \\ b & b & a_{23} \\ c & c & a_{33} \end{vmatrix} = 0.$$

Proof. If we interchange these two identical rows in the determinant Δ then new determinant Δ' will be at the same time equal to Δ and equal to $-\Delta$ by property 3. So $\Delta' = \Delta = -\Delta \Rightarrow 2\Delta = 0 \Rightarrow \Delta = 0$. ***The property is proven.***

Property 5. Multiplying a Row by a Scalar: Let A be an $n \times n$ matrix and let B be a matrix which results from multiplying some row of A by a scalar k. Then $det(B) = kdet(A)$

Ii other words, if the determinants Δ and Δ' differ only in that the elements of some one row of Δ' are equal to k times corresponding elements of Δ then $\Delta' = k\Delta$, i.e. you can take the common factor k of the row (column) elements out the determinant.

$$\begin{vmatrix} a_{11} & a_{12} & ka_{13} \\ a_{21} & a_{22} & ka_{23} \\ a_{31} & a_{32} & ka_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

Proof. Let the i^{th} row in Δ' have the elements $ka_{i1}, ka_{i2}, ka_{i3}, \ldots ka_{in}$. Then

$$\Delta' = \sum (-1)^{[j_1 \quad j_2 \quad \cdots \quad j_n]} a_{1j_1} \ldots a_{i-1 j_{i-1}} (ka_{ij_i}) a_{i+1 j_{i+1}} \ldots a_{nj_n} =$$

$$= k \sum (-1)^{[j_1 \quad j_2 \quad \cdots \quad j_n]} a_{1j_1} \ldots a_{i-1 j_{i-1}} a_{ij_i} a_{i-1 j_{i-1}} \ldots a_{nj_n} = k\Delta.$$ ***The property is proven.***

Example.

$$\begin{vmatrix} 19 & -76 \\ 13 & 2 \end{vmatrix} = 19 \begin{vmatrix} 1 & -4 \\ 13 & 2 \end{vmatrix} = 19 \cdot 2 \begin{vmatrix} 1 & -2 \\ 13 & 1 \end{vmatrix} = 38(1 \cdot 1 - 13 \cdot (-2)) = 38 \cdot 27 = 1026.$$

Notice that this theorem is true when we multiply one row of the matrix by k. If we were to multiply two rows of A by k to obtain B, we would have $\det(B) = k^2 \det(A)$. Suppose we were to multiply all n rows of A by k to obtain the matrix B, so that $B = kA$. Then, $\det(B) = k^n \det(A)$. This gives the next theorem.

Theorem. Scalar Multiplication: Let A and B be $n \times n$ matrices and k a scalar, such that $B = kA$. Then $\det(B) = k^n \det(A)$.

Property 6. The determinant with two proportional rows (columns) is equal to zero.

Proof. Suppose the i^{th} and the j^{th} rows are proportional. Then

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & & & \\ ka_{i1} & ka_{i2} & \cdots & ka_{in} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \begin{matrix} \\ \\ (i^{th}\ row) \\ \\ (j^{th}\ row) \\ \\ \\ \end{matrix} = [\text{by property 5}] =$$

$$= k \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \begin{matrix} \\ \\ (i^{th}\ row) \\ \\ (j^{th}\ row) \\ \\ \\ \end{matrix} = [\text{by property 4}] = 0.$$

The property is proven.

Property 7. Suppose each element of the i^{th} row in the determinant Δ is equal to sum $a_{ij} = c_{ij} + b_{ij}\ (j = \overline{1,n})$. Then $\Delta = \Delta' + \Delta''$, where Δ' is obtained from Δ by replacing its elements from i^{th} row a_{ij} by c_{ij} and Δ'' is obtained from Δ by replacing its elements from i^{th} row a_{ij} by b_{ij}.

$$\begin{vmatrix} a_{11} & a_{12}+b_{12} & a_{13} \\ a_{21} & a_{22}+b_{22} & a_{23} \\ a_{31} & a_{32}+b_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & b_{12} & a_{13} \\ a_{21} & b_{22} & a_{23} \\ a_{31} & b_{32} & a_{33} \end{vmatrix}.$$

Proof.

$$\Delta = \sum (-1)^{[j_1\ j_2\ \dots\ j_n]} a_{1j_1} \dots a_{i-1\,j_{i-1}} (c_{ij_i} + b_{ij_i}) a_{i+1\,j_{i+1}} \dots a_{nj_n} =$$

$$= \sum (-1)^{[j_1\ j_2\ \dots\ j_n]} (a_{1j_1} \dots a_{i-1\,j_{i-1}} c_{ij_i} a_{i+1\,j_{i+1}} \dots a_{nj_n} + a_{1j_1} \dots a_{i-1\,j_{i-1}} b_{ij_i} a_{i+1\,j_{i+1}} \dots a_{nj_n}) =$$

$$= \sum (-1)^{[j_1\ j_2\ \dots\ j_n]} a_{1j_1} \dots a_{i-1\,j_{i-1}} c_{ij_i} a_{i+1\,j_{i+1}} \dots a_{nj_n} +$$

$$+ \sum (-1)^{[j_1\ j_2\ \dots\ j_n]} a_{1j_1} \dots a_{i-1\,j_{i-1}} b_{ij_i} a_{i+1\,j_{i+1}} \dots a_{nj_n} = \Delta' + \Delta''.$$

The property is proven.

Property 8. Adding a Multiple of a Row to Another Row: Let A be an $n \times n$ matrix and let B be a matrix which results from adding a multiple of a row to another row. Then $det(A) = det(B)$, i.e. the value of the determinant is unchanged if we will add to the elements of one row of this determinant the corresponding elements of another row multiplied by a real number.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11}+ka_{13} & a_{12} & a_{13} \\ a_{21}+ka_{23} & a_{22} & a_{23} \\ a_{31}+ka_{33} & a_{32} & a_{33} \end{vmatrix}.$$

Here we can define a new concept known as a *linear combination* of row or columns.

Definition. A linear combination of row (or columns) of any matrix is called a sum of the products where each of them is a multiplication any row (column) by any real number. As a result, a new row (column) is obtained.

$$a_{ij} = \sum_{s=1}^{M} k_s a_{sj}\ (j = \overline{1,n}) \text{ or } b_{ij} = \sum_{s=1}^{M} k_s b_{sj}\ (j = \overline{1,n})$$

Property 9. If $a_{is} = \sum_{\substack{j=1 \\ j \neq i}}^{n} k_j a_{js}\ (s = \overline{1,n})$ in the determinant then it is equal to zero, i.e.

if one of the row (column) in the determinant is a linear combination of other rows (columns) then this determinant is equal to zero.

Proof. Without loss of generality we can assume that

$$a_{1s} = \sum_{j=2}^{n} k_j a_{js} \ (s = \overline{1,n}) \text{. Then}$$

$$\begin{vmatrix} \sum_{j=2}^{n} k_j a_{j1} & \sum_{j=2}^{n} k_j a_{j2} & \cdots & \sum_{j=2}^{n} k_j a_{jn} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & & a_{nn} \end{vmatrix} = [by\ pr.5,7] = \sum_{j=2}^{n} k_j \begin{vmatrix} a_{j1} & a_{j2} & \cdots & a_{jn} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & & a_{nn} \end{vmatrix} =$$

$= [\text{by pr.4}] = \sum_{j=2}^{n} k_j 0 = 0$. ***The property is proven.***

Example. In the example to property 7 $a_{11} = 1 \cdot a_{12} + 1 \cdot a_{13}$, $a_{21} = 1 \cdot a_{22} + 1 \cdot a_{23}$, $a_{31} = 1 \cdot a_{32} + 1 \cdot a_{33}$, i.e. the first column is linear combination (in this case just sum) of the others columns. By property 9 this determinant is equal to zero.

Property 10. Determinant of a Product: Let A and B be square matrices of the n^{th} order. Then

$$\det AB = \det A \det B \text{.}$$

Without proof.

Property 11. Determinant of the Inverse: Let A be an $n \times n$ matrix. Then A is invertible if and only if $det(A) \neq 0$. If this is true, it follows that

$$det(A^{-1}) = \frac{1}{det(A)}$$

Without proof.