Lecture #3: Minors and Algebraic Cofactors

Definition. if A is an $n \times n$ matrix, then the ij −th minor of the matrix A denoted as M_{ij} or $minor(A)_{ij}$ is the determinant of the (n-1)× (n-1) order which results from the determinant $n \times n$ of matrix A by removing the i^{th} row and the j^{th} column, i.e. the row and the column on intersection of which the entry ij −th is situated.

Example. For a matrix A of 3×3, the determinant of the third order is

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

the minors for the entries a_{23} and a_{32} are

$$M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}; \qquad M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}.$$

Note, there is a minor associated with each entry of A .

The ij-th minor of a matrix A is used in another important definition, given next:

Definition. Suppose A is an $n \times n$ matrix. The ij-th algebraic cofactor (or just cofactor), denoted by A_{ij} or $cof(A)_{ij}$ is a number defined to be

$$A_{ij} = (-1)^{i+j} M_{ij} \text{ or } \mathrm{cof}(A)_{ij} = (-1)^{i+j} minor(A)_{ij}$$

Example. For the matrix A with its determinant

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ -3 & 0 & 2 & 0 \end{vmatrix}$$

the algebraic cofactor of the element a_{12} is

$$M_{12} = \begin{vmatrix} 4 & 2 & 1 \\ 0 & 0 & 1 \\ -3 & 2 & 0 \end{vmatrix} = 4\cdot 0\cdot 0 + 2\cdot 1\cdot(-3) + 1\cdot 0\cdot 2 - 1\cdot 0\cdot(-3) - 2\cdot 0\cdot 0 - 4\cdot 1\cdot 2 = -14,$$

$$A_{12} = (-1)^{1+2} M_{12} = (-1)\cdot(-14) = 14.$$

Theorem (Expansion of the determinant by cofactors) The value of the determinant is equal to the sum of products of the elements of any row (column) by their cofactors. The determinant does not depend on the choice of the row (column).

Proof. Let us consider the determinant of the n^{th} order and present all elements of k^{th} row as sum of (n-1) zeros and the element:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ \vdots & & & & \\ a_{k1}+0+\ldots+0 & 0+a_{k2}+0+\ldots+0 & 0+0+a_{k3}+0+\ldots+0 & \ldots & 0+0+\ldots+0+a_{kn} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} \end{vmatrix} =$$ [by property 7]=

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ \vdots & & & & \\ a_{k1} & 0 & 0 & \ldots & 0 \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ \vdots & & & & \\ 0 & a_{k2} & 0 & \ldots & 0 \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} \end{vmatrix} + \ldots + \begin{vmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ \vdots & & & & \\ 0 & 0 & 0 & \ldots & a_{kn} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} \end{vmatrix} =$$

=[by the previous Theorem]= $a_{k1}A_{k1} + a_{k2}A_{k2} + a_{k3}A_{k3} + \ldots + a_{kn}A_{kn}$.

Theorem is proven.

This method of evaluating a determinant by expanding along a row or a column is called **Laplace Expansion** or **Cofactor Expansion**

Example. For a determinant of the third order

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

the determinant can be calculated as one of the following formulas:

$$\Delta = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13};\ \Delta = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31};$$

$$\Delta = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23};\ \Delta = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32};$$

$$\Delta = a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33};\ \Delta = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33}.$$

Note 1. Described in the theorem way to calculate the determinant could be called also as *Expansion of the determinant along the row* or *Expansion of the determinant down the column*.

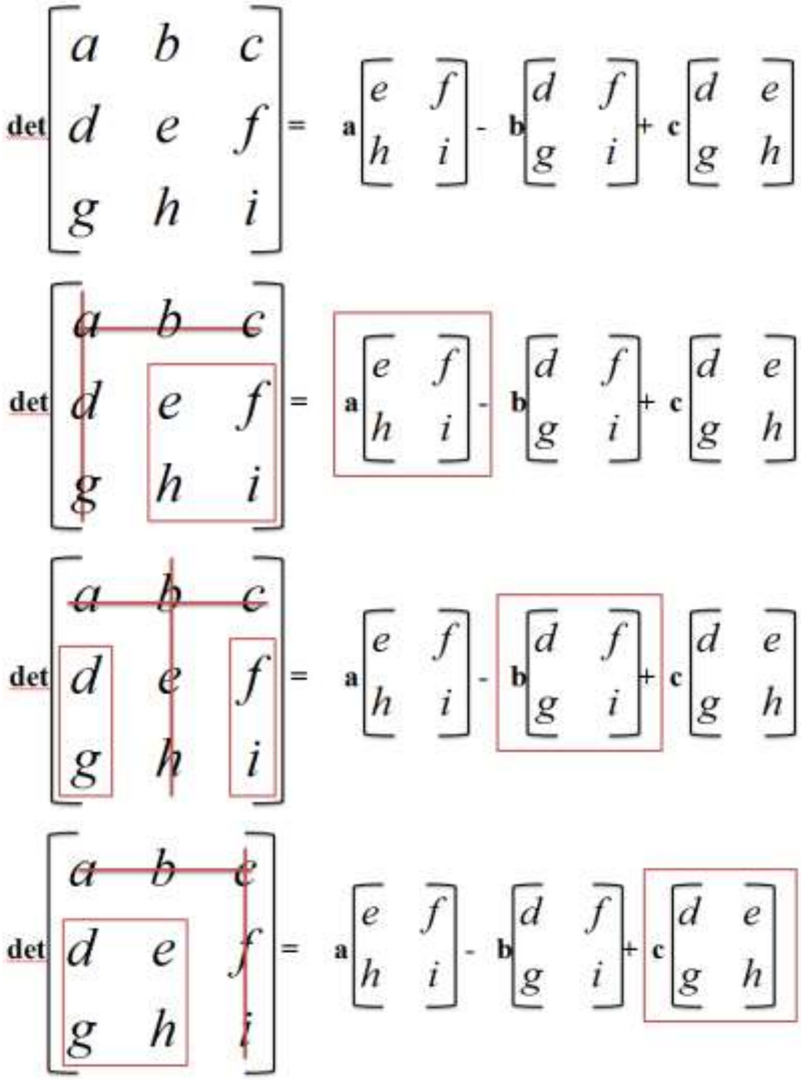
Note 2. It follows from the Theorem that to calculate the determinant of the n^{th} order we should calculate n determinants of the (n-1)$^{\text{th}}$ order:

- along the k-th row: $\begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ \vdots & & & & \\ a_{k1} & a_{k2} & a_{k3} & \dots & a_{kn} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix} = a_{k1}A_{k1} + a_{k2}A_{k2} + a_{k3}A_{k3} + \ldots + a_{kn}A_{kn} = \sum_{i=1}^{n} a_{ki}A_{ki}$;

- along the s-th column:

$$\begin{vmatrix} a_{11} & a_{12} & \ldots a_{1s} & \dots & a_{1n} \\ \vdots & & & & \\ a_{k1} & a_{k2} & \ldots a_{ks} & \dots & a_{kn} \\ \vdots & & & & \\ a_{n1} & a_{n2} & \ldots a_{ns} & \dots & a_{nn} \end{vmatrix} = a_{1s}A_{1s} + a_{2s}A_{2s} + a_{3s}A_{3s} + \ldots + a_{ns}A_{ns} = \sum_{i=1}^{n} a_{is}A_{is}.$$

A Third-Order Determinant is the determinant of a 3×3 matrix



Example: Calculate the determinant:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix},$$

First, we will calculate $det(A)$ by expanding along the first column. Using the Theorem, we take the 1 in the first column and multiply it by its cofactor,

$$1(-1)^{1+1}\begin{vmatrix}3 & 2\\ 2 & 1\end{vmatrix} = (1)(1)(-1) = -1$$

Similar for the other entries in the first column. Finally, we add these numbers together, as given in the following equation:

$$det(A) = 1\overbrace{(-1)^{1+1}\begin{vmatrix}3 & 2\\ 2 & 1\end{vmatrix}}^{\mathrm{cof}(A)_{11}} + 4\overbrace{(-1)^{2+1}\begin{vmatrix}2 & 3\\ 2 & 1\end{vmatrix}}^{\mathrm{cof}(A)_{21}} + 3\overbrace{(-1)^{3+1}\begin{vmatrix}2 & 3\\ 3 & 2\end{vmatrix}}^{\mathrm{cof}(A)_{31}}$$

Calculating each of these, we obtain

$$det(A) = 1(1)(-1) + 4(-1)(-4) + 3(1)(-5) = -1 + 16 + -15 = 0$$

Notice, we can choose to expand along any row or column. Let's try now by expanding along the *second row*. Here, we take the 4 in the second row and multiply it to its cofactor, then add this to the 3 in the second row multiplied by its cofactor, and the 2 in the second row multiplied by its cofactor. The calculation is as follows:

$$det(A) = 4\overbrace{(-1)^{2+1}\begin{vmatrix}2 & 3\\ 2 & 1\end{vmatrix}}^{\mathrm{cof}(A)_{21}} + 3\overbrace{(-1)^{2+2}\begin{vmatrix}1 & 3\\ 3 & 1\end{vmatrix}}^{\mathrm{cof}(A)_{22}} + 2\overbrace{(-1)^{2+3}\begin{vmatrix}1 & 2\\ 3 & 2\end{vmatrix}}^{\mathrm{cof}(A)_{23}}$$

We obtain,

$$det(A) = 4(-1)(-2) + 3(1)(-8) + 2(-1)(-4) = 0$$

Theorem *(***The Determinant is Well Defined**) Expanding the $n \times n$ matrix along any row or column always gives the same answer, which is the determinant.

Definition : The Determinant of an $n \times n$ matrix. Let A be an $n \times n$ matrix where $n \geq 2$ and suppose the determinant of an $(n-1) \times (n-1)$ has been defined. Then

$$det(A) = \sum_{j=1}^{n} a_{ij}\,\mathrm{cof}(A)_{ij} = \sum_{i=1}^{n} a_{ij}\,\mathrm{cof}(A)_{ij}$$

The first formula consists of expanding the determinant along the i-th row and the second expands the determinant along the j-th column.

Theorem (about the sum of products of the elements of any row/column by cofactors of the elements from another row/column) The sum of products of the elements of any row (column) by the cofactors of the corresponding elements of other row (column) is equal to zero.

Note 3. If the determinant has a special form where all elements except the element a_{ij} in some row are equal to zero then this determinant is equal to the product of this element by its algebraic cofactor, i.e.

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ & & & \vdots & & \\ 0 & 0 & \dots & a_{ij} & \dots & 0 \\ & & & \vdots & & \\ a_{n1} & a_{n2} & \dots & a_{nj} & \dots & a_{nn} \end{vmatrix} = a_{ij}(-1)^{i+j} M_{ij} = a_{ij} A_{ij}.$$

Note 4. Since we can choose any row (column), to simplify calculations it will be better to choose row (column) with maximum number of zero elements. Herewith, if the initial determinant does not contain zero elements such row (or column) has to be created by using *property 8* of the determinants to increase the number of zeros in the row (column).

Finding Determinants using Row Operations

In this section, we look at two examples where row operations are used to find the determinant of a large matrix. Recall that when working with large matrices, Laplace Expansion is effective but timely, as there are many steps involved. This section provides useful tools for an alternative method. By first applying row operations, we can obtain a simpler matrix to which we apply Laplace Expansion.

While working through questions such as these, it is useful to record your row operations as you go along. Keep this in mind as you read through the next example.

Examples: Calculate the determinant:

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 0 & 1 \\ -3 & 0 & 2 & 0 \end{vmatrix} =$$

=[Let us add to the elements of the first column the corresponding elements of the forth one multiplied by (-2) and to the elements of the second column the corresponding elements of the forth one multiplied by (-1)]=

$$= \begin{vmatrix} 1-8 & 2-4 & 3 & 4 \\ 4-2 & 3-1 & 2 & 1 \\ 2-2 & 1-1 & 0 & 1 \\ -3-0 & 0-0 & 2 & 0 \end{vmatrix} = \begin{vmatrix} -7 & -2 & 3 & 4 \\ 2 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 \\ -3 & 0 & 2 & 0 \end{vmatrix} =$$

=[Let us expand it along the third row]=

$$= a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} + a_{34}A_{34} = 0 + 0 + 0 + 1 \cdot (-1)^{3+4}\begin{vmatrix} -7 & -2 & 3 \\ 2 & 2 & 2 \\ -3 & 0 & 2 \end{vmatrix} =$$

=[Let us add to the elements of the first row the elements of the second and use property 5]=

$$= -2\begin{vmatrix} -5 & 0 & 5 \\ 1 & 1 & 1 \\ -3 & 0 & 2 \end{vmatrix} = 10\begin{vmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ -3 & 0 & 2 \end{vmatrix} = 10(a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32}) =$$

$$= 10 \cdot 1 \cdot (-1)^{2+2}\begin{vmatrix} 1 & -1 \\ -3 & 2 \end{vmatrix} = 10(2-3) = -10.$$

Find the determinant of the matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 1 & 2 & 3 \\ 4 & 5 & 4 & 3 \\ 2 & 2 & -4 & 5 \end{bmatrix}$$

We will use the properties of determinants outlined above to find $det(A)$. First, add -5 times the first row to the second row. Then add -4 times the first row to the third row, and -2 times the first row to the fourth row. This yields the matrix

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -9 & -13 & -17 \\ 0 & -3 & -8 & -13 \\ 0 & -2 & -10 & -3 \end{bmatrix}$$

Notice that the only row operation we have done so far is adding a multiple of a row to another row. Therefore, by Theorem 8, $det(B) = det(A)$.

At this stage, you could use Laplace Expansion to find $det(B)$. However, we will continue with row operations to find an even simpler matrix to work with. Add -3 times the third row to the second row. By Theorem this does not change the value of the determinant. Then, multiply the fourth row by -3. This results in the matrix

$$C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 11 & 22 \\ 0 & -3 & -8 & -13 \\ 0 & 6 & 30 & 9 \end{bmatrix}$$

Here, $det(C) = -3det(B)$, which means that $\det(B) = \left(-\frac{1}{3}\right)\det(C)$.

Since $det(A) = det(B)$, we now have that $\det(A) = \left(-\frac{1}{3}\right)\det(C)$. Again, you could use Laplace Expansion here to find $\det(C)$. However, we will continue with row operations.

Now replace the add 2 times the third row to the fourth row. This does not change the value of the determinant by Theorem. Finally switch the third and second rows. This causes the determinant to be multiplied by -1. Thus $det(C) = -det(D)$ where

$$D = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -8 & -13 \\ 0 & 0 & 11 & 22 \\ 0 & 0 & 14 & -17 \end{bmatrix}$$

Hence, $\det(A) = \left(-\frac{1}{3}\right)\det(C) = \left(\frac{1}{3}\right)\det(D)$

You could do more row operations or you could note that this can be easily expanded along the first column. Then, expand the resulting 3×3 matrix also along the first column. This results in

$$det(D) = 1(-3)\begin{vmatrix} 11 & 22 \\ 14 & -17 \end{vmatrix} = 1485$$

and so $\det(A) = \left(\frac{1}{3}\right)(1485) = 495.$

Theorem (**Determinant of a Triangular Matrix**) Let A be an upper or lower triangular matrix. Then $det(A)$ is obtained by taking the product of the entries on the main diagonal.

The verification of this Theorem can be done by computing the determinant using Laplace Expansion along the first row or column.

Example: Calculate the determinant:

$$A = \begin{bmatrix} 1 & 2 & 3 & 77 \\ 0 & 2 & 6 & 7 \\ 0 & 0 & 3 & 33.7 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

We will expand along the first column. This gives

$$det(A) = 1\begin{vmatrix} 2 & 6 & 7 \\ 0 & 3 & 33.7 \\ 0 & 0 & -1 \end{vmatrix} + 0(-1)^{2+1}\begin{vmatrix} 2 & 3 & 77 \\ 0 & 3 & 33.7 \\ 0 & 0 & -1 \end{vmatrix} + 0(-1)^{3+1}\begin{vmatrix} 2 & 3 & 77 \\ 2 & 6 & 7 \\ 0 & 0 & -1 \end{vmatrix} + 0(-1)^{4+1}\begin{vmatrix} 2 & 3 & 77 \\ 2 & 6 & 7 \\ 0 & 3 & 33.7 \end{vmatrix}$$

and the only nonzero term in the expansion is

$$1\begin{vmatrix} 2 & 6 & 7 \\ 0 & 3 & 33.7 \\ 0 & 0 & -1 \end{vmatrix}$$

Now find the determinant of this 3×3 matrix, by expanding along the first column to obtain

$$det(A) = 1 \times \left(2 \times \begin{vmatrix} 3 & 33.7 \\ 0 & -1 \end{vmatrix} + 0(-1)^{2+1} \begin{vmatrix} 6 & 7 \\ 0 & -1 \end{vmatrix} + 0(-1)^{3+1} \begin{vmatrix} 6 & 7 \\ 3 & 33.7 \end{vmatrix}\right)$$

$$= 1 \times 2 \times \begin{vmatrix} 3 & 33.7 \\ 0 & -1 \end{vmatrix}$$

Putting all these steps together, we have

$$det(A) = 1 \times 2 \times 3 \times (-1) = -6$$

which is just the product of the entries down the main diagonal of the original matrix!

Note 4. The determinant of an upper (lower) triangle square matrix is equal to the product of their main diagonal elements.

$$= \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ 0 & 0 & \dots & a_{3n} \\ \vdots & & & a_{4n} \\ & & \dots & \\ \vdots & & & a_{(n-1)n} \\ 0 & 0 & \dots & a_{nn} \end{vmatrix} = [\text{by } theorem] = a_{11}A_{11} = a_{11}a_{22}A_{22} = \dots = a_{11}a_{22}a_{33} \cdot \dots \cdot a_{nn}$$

Note 5. Whenever computing the determinant, it is useful to consider all the possible methods and tools.