## Lecture #7: ANALYTIC GEOMETRY: Vector Algebra

### 7.1 Vectors. Basic Definitions and Concepts

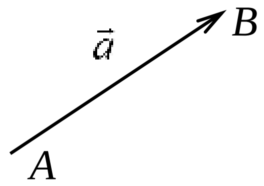


Figure 2

***Definition.*** The vector is a directed segment (Fig.2).

Notations: $\vec{a}$ or $\bar{a}$ or $\overrightarrow{AB}$ or $\overline{AB}$.

Point $A$ is called an origin of the vector.

Point $B$ is called a terminus.

***Definition.*** The distance between the origin and terminus is called a module or a length of this vector.

Notations: $|\vec{a}|$.

If the origin of the vector coincides with the terminus then $|\vec{a}| = 0$. Such a vector is called a zero-vector and denoted as $\vec{0}$ or just 0.

***Definition.*** Two vectors $\vec{a}$ and $\vec{b}$ are called equal if they have the same module and the same direction.

From the last definition It follows that the vectors obtained one from another by parallel shift are equal.

***Definition.*** Two vectors $\vec{a}$ and $\vec{b}$ are called collinear if they are parallel to the same straight line.

***Definition.*** Three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are called coplanar (or complanar) if they are parallel to the same plane.

***Definition.*** Vector of the unit length having the same direction with vector $\vec{a}$ is called the ort or the unit vector of the vector $\vec{a}$.

Notation: $\vec{a}^{\,o}$.

### 7.2 Linear Operations on Vectors

Linear operations on vectors are the multiplication of vector by scalar and the addition of the vectors.

***Definition.*** The vector $\vec{b} = \lambda\vec{a}$ is called the multiplication of the vector $\vec{a}$ by scalar $\lambda$ if:

1. $|\vec{b}| = |\lambda||\vec{a}|$;
2. $\vec{a}$ and $\vec{b}$ are collinear vectors;
3. $\vec{a}$ and $\vec{b}$ have the same direction for positive values of $\lambda$ and the opposite directions for negative values of $\lambda$.

See examples on Fig.3.

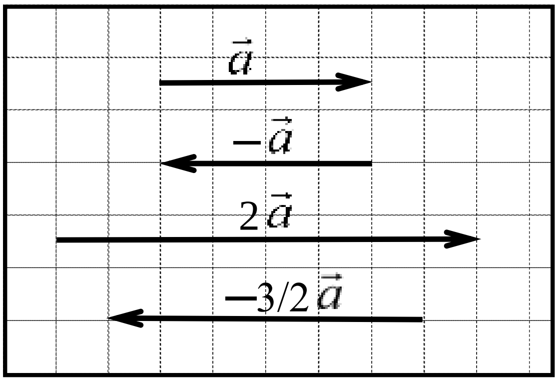


Figure 3

From definition It follows that two collinear vectors could be obtained one from another by multiplication by a scalar. So, we have the following *criterion of collinearity for two nonzero vectors*:

$$\vec{a} \parallel \vec{b} \Leftrightarrow \vec{a} = \lambda\vec{b} \Leftrightarrow \vec{b} = \mu\vec{a}, \quad \lambda, \mu \in R \setminus \{0\}$$

***Definition.*** The sum of vectors $\vec{a}$ and $\vec{b}$ is called a vector $\vec{c} = \vec{a} + \vec{b}$ which origin coincides with the origin of $\vec{a}$ and terminus coincides with the terminus of $\vec{b}$ if the terminus of $\vec{a}$ and the origin of $\vec{b}$ are connected (Fig.4).

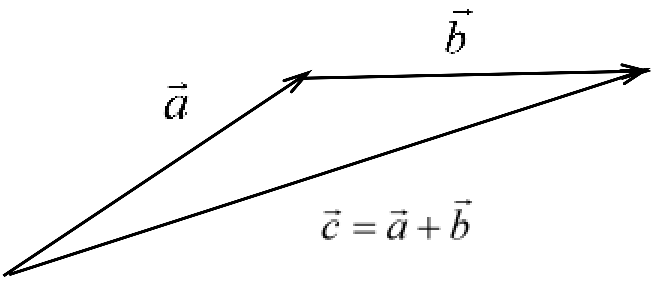


Figure 4. The rule of triangle

This rule to get sum is called the rule of triangle (Fig.4).

There is another rule to get sum called the rule of parallelogram (Fig.5). In this case you should construct a parallelogram on the vectors $\vec{a}$ and $\vec{b}$. The sum of the vectors coincides with the diagonal of this parallelogram directed from the origin of $\vec{a}$ to the terminus of $\vec{b}$.

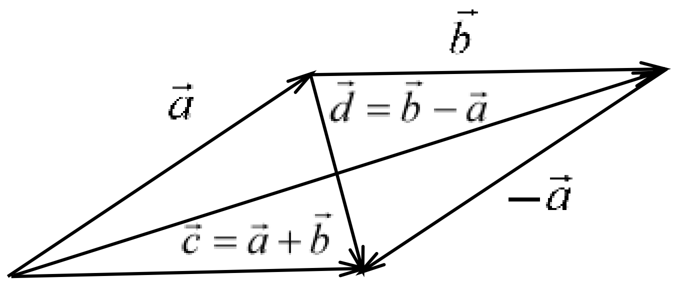


Figure 5. The rule of parallelogram

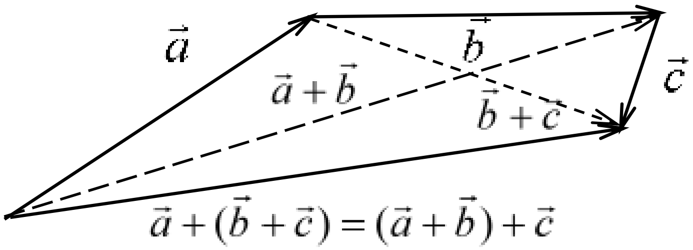


Figure 6

To get difference of vectors you should fulfill the following operations:

$$\vec{d} = \vec{b} - \vec{a} = \vec{b} + (-1)\vec{a} = \vec{b} + (-\vec{a}).$$

The difference of vectors coincides with the other diagonal of the parallelogram constructed on $\vec{a}$ and $\vec{b}$ (Fig.5). It is directed from minuend origin to subtrahend origin if their terminuses are connected.

**Basic properties of linear operations:**

1. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
2. $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$ (Fig.6)
3. $\lambda(\vec{a} + \vec{b}) = \lambda\vec{a} + \lambda\vec{b}$, $\lambda \in R$
4. $(\lambda + \mu)\vec{a} = \lambda\vec{a} + \mu\vec{a}$, $\lambda, \mu \in R$

***Example.*** Let us find a vector $\vec{c}$ with direction coinciding with a bisector of an angle between the vectors $\vec{a}$ and $\vec{b}$ (Fig.7).

A diagonal bisects the angle of a parallelogram only if this parallelogram is a rhomb. That is why the vector $\vec{c}$ bisects an angle between two vectors only if their lengths are equal to each other.

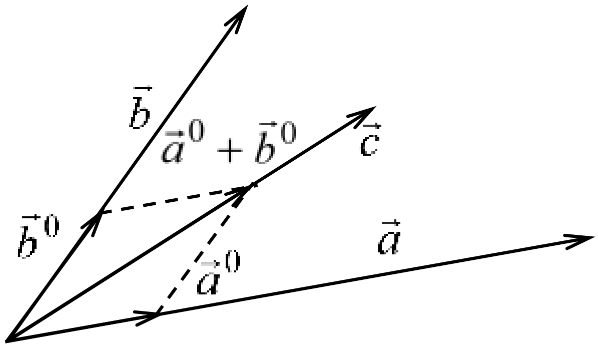

Figure 7

Let us consider the orts $\vec{a}^0$ and $\vec{b}^0$. Their lengths are equal to one and their directions coincide with directions of $\vec{a}$ and $\vec{b}$, relatively. Then the vector directed along the bisector of the angle between $\vec{a}$ and $\vec{b}$ has the same direction as $\vec{a}^0 + \vec{b}^0$. Therefore,

$$\vec{c} = \lambda(\vec{a}^0 + \vec{b}^0),\ \lambda > 0.$$

Note, that there are several other ways to construct the vectors of the equal length. For example, we can find the bisector as

$$\vec{c} = \lambda\left(\vec{a} \cdot \left|\vec{b}\right| + \vec{b} \cdot \left|\vec{a}\right|\right),\ \lambda > 0.$$

## 7.3. Concept of Basis. Decomposition of the Vector

***Definition.*** The set of vectors is a linear space, denoted by $L$, that is

I. There is an operation of multiplication of vectors by scalar such that

$$\forall \vec{a} \in L \Rightarrow \alpha\vec{a} \in L \ \ \forall \alpha \in R;$$

II. There is an operation of addition such that

$$\forall \vec{a}, \vec{b} \in L \Rightarrow \vec{a} + \vec{b} \in L;$$

III. These operations satisfy 8 conditions:

1. $\vec{a} + \vec{b} = \vec{b} + \vec{a} \quad \forall \vec{a}, \vec{b} \in L;$
2. $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}) \quad \forall \vec{a}, \vec{b}, \vec{c} \in L;$
3. $\exists 0 \in L: \ \vec{a} + 0 = 0 + \vec{a} = \vec{a} \quad \forall \vec{a} \in L;$
4. $\forall \vec{a} \in L \ \exists \vec{b} = -\vec{a} \in L: \ \vec{a} + \vec{b} = 0;$
5. $1 \cdot \vec{a} = \vec{a} \quad \forall \vec{a} \in L;$
6. $\lambda(\vec{a} + \vec{b}) = \lambda\vec{a} + \lambda\vec{b} \ \ \forall \vec{a}, \vec{b} \in L \ \forall \lambda \in R;$

7. $(\lambda+\mu)\vec{a}=\lambda\vec{a}+\mu\vec{a} \quad \forall\vec{a}\in L\ \forall\lambda,\mu\in R$;

8. $(\lambda\mu)\vec{a}=\lambda(\mu\vec{a})=\mu(\lambda\vec{a}) \quad \forall\vec{a}\in L\ \forall\lambda,\mu\in R$.

***Definition.*** The expression $\alpha_1\vec{a}_1+\alpha_2\vec{a}_2+\alpha_3\vec{a}_3+\ldots+\alpha_n\vec{a}_n,\ \alpha_1,\alpha_2,\ldots\alpha_n\in R$ is called linear combination (LC) of the vectors $\vec{a}_1,\vec{a}_2,\vec{a}_3,\ldots,\vec{a}_n$.

***Definition.*** The vectors $\vec{a}_1,\vec{a}_2,\vec{a}_3,\ldots,\vec{a}_n$ are called linearly independent (LI) if any their trivial (zero) linear combination has trivial coefficients, i.e.

$$\vec{a}_1,\vec{a}_2,\ldots,\vec{a}_n \text{ are LI} \Leftrightarrow \left(\alpha_1\vec{a}_1+\alpha_2\vec{a}_2+\ldots+\alpha_n\vec{a}_n=0 \quad\Rightarrow\quad \alpha_1=\alpha_2=\ldots=\alpha_n=0\right).$$

In other case they are called linearly dependent (LD), i.e.

$$\vec{a}_1,\vec{a}_2,\ldots,\vec{a}_n \text{ are LD} \Leftrightarrow \exists k:\alpha_k\neq 0 \text{ and } \alpha_1\vec{a}_1+\alpha_2\vec{a}_2+\alpha_3\vec{a}_3+\ldots+\alpha_n\vec{a}_n=0.$$

***Theorem (Linear dependence of vectors)*** The vectors $\vec{a}_1,\vec{a}_2,\vec{a}_3,\ldots,\vec{a}_n$ are linearly dependent if and only if one of these vectors is linear combination of other vectors.

By means of this Theorem we will prove the following 3 statements:

***Statement 1.*** Two vectors are linearly dependent if and only if they are collinear.

***Proof.*** $\vec{a},\vec{b}$ are LD $\Leftrightarrow$ [by Theorem 1] $\Leftrightarrow \vec{a}=\alpha\vec{b}\Leftrightarrow \vec{a},\vec{b}$ are collinear.

***Statement is proven.***

***Corollary.*** Two vectors are linearly independent if and only if they are not collinear.

***Statement 2.*** Three vectors are linearly dependent if and only if they are coplanar.

***Proof.*** Necessity. $\vec{a},\vec{b},\vec{c}$ are LD. By the Theorem 1 we get, for example, that $\vec{c}=\alpha\vec{a}+\beta\vec{b}$. Thus, $\vec{c}$ is a diagonal of the parallelogram constructed on $\alpha\vec{a}$ and $\beta\vec{b}$ and it belongs to the plane of this parallelogram as $\vec{a}$ and $\vec{b}$ do. So these vectors are coplanar.

To prove Sufficiency we need just to prove that for any three coplanar vectors one is linear combination of others.

Suppose $\vec{a},\vec{b}$ are collinear. Then

$$\vec{a}=\lambda\vec{b} \text{ or } \vec{a}=\lambda\vec{b}+0\cdot\vec{c},$$

so vectors  are LD by Theorem 1. Suppose now that $\vec{a},\vec{b}$ are not collinear. Then, accordingly to the Fig.8,

$$\vec{c} = \overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{AC} = \lambda\vec{a} + \mu\vec{b},$$

since *ABDC* is a parallelogram and $\overrightarrow{AB} \parallel \vec{a}$, $\overrightarrow{AC} \parallel \vec{b}$. ***Statement is proven.***

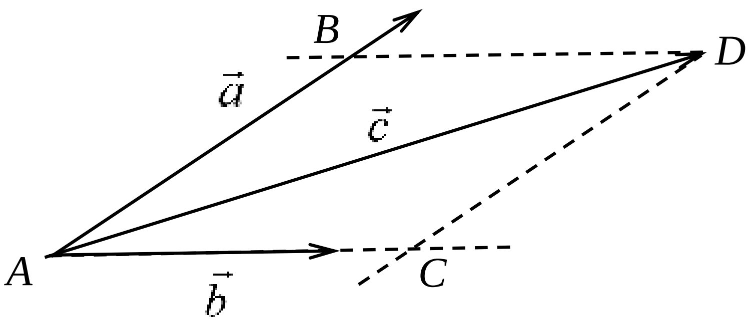


Figure 8

***Statement 3.*** Any four vectors in space are linearly dependent.

***Proof.*** Let us consider any four vectors in space. There are two cases.

Case 1. $\vec{a},\vec{b},\vec{c}$ are coplanar. Then they are LD, i.e. $\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} = \alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} + 0\cdot\vec{d} = 0$ with not all zero coefficients. Thus $\vec{a},\vec{b},\vec{c},\vec{d}$ are LD as well.

Case 2. $\vec{a},\vec{b},\vec{c}$ are not coplanar. Then let us draw a straight line through the terminus of the vector $\vec{d}$ (point *F*) parallel to the vector $\vec{c}$ to find point *D* which is an intersection of constructed straight line and plane of the vectors $\vec{a},\vec{b}$ (Fig.9). Obtained vector $\overrightarrow{AD}$ is coplanar with not collinear vectors $\vec{a},\vec{b}$, i.e. it could be presented as their linear combination. In other case, vector $\overrightarrow{DF} = \overrightarrow{AE}$ is collinear to $\vec{c}$, i.e.

$$\overrightarrow{DF} = \lambda\vec{c},\ \lambda \neq 0.$$

So,

$$\vec{d} = \overrightarrow{AF} = \overrightarrow{AE} + \overrightarrow{AD} = \overrightarrow{AE} + \overrightarrow{AB} + \overrightarrow{AC} = \lambda\vec{c} + \alpha\vec{a} + \beta\vec{b},$$

where $\lambda \neq 0$. It means that these vectors are linearly dependent. ***Statement is proven.***

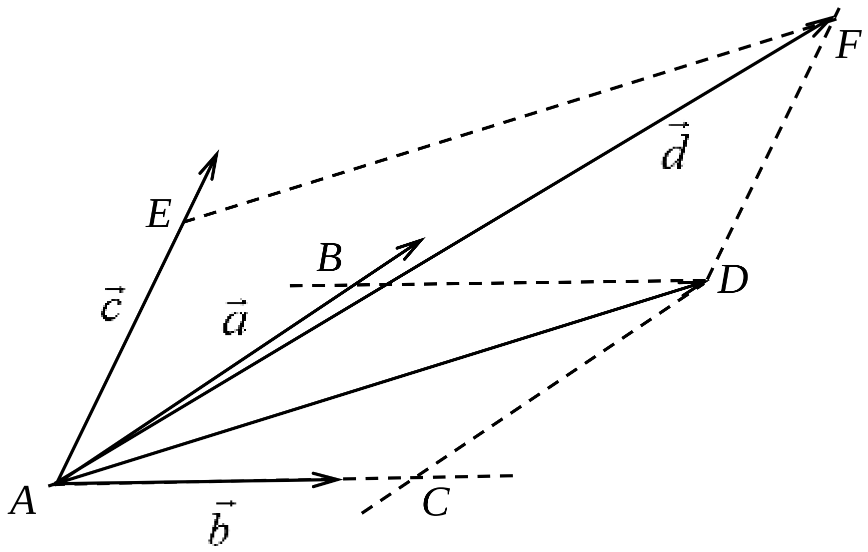


Figure 9

***Definition.*** Linear space *L* is called *n*-dimensional if there are *n* linearly independent elements and any (*n*+1) are linearly dependent.

***Definition.*** In *n*-dimensional linear space any *n* linearly independent elements are called basis of this space.

***Note 1.*** There could be a lot of different bases in the same *LS*.

***Note 2.*** From statements 1-3 follows that:

1. Plane is 2-dimensional *LS* and any two not collinear vectors form basis.
2. Space is 3-dimensional *LS* and any three not coplanar vectors form basis.

***Theorem (Decomposition of the vector in the basis)*** Any vector of *n*-dimensional linear space, $\vec{x}$ can be presented as linear combination of basic vectors, $\vec{e}_1, \vec{e}_2, \vec{e}_3, \ldots \vec{e}_n$ and this presentation is unique:

$$\vec{x} = \alpha_1 \vec{e}_1 + \alpha_2 \vec{e}_2 + \alpha_3 \vec{e}_3 + \ldots + \alpha_n \vec{e}_n$$

***Definition.*** The presentation of the vector as linear combination of basic vectors is called the decomposition of the vector $\vec{x}$ in the basis $\vec{e}_1, \vec{e}_2, \vec{e}_3, \ldots \vec{e}_n$. At the same time the coefficients of this decomposition are called the coordinates of the vector $\vec{x}$ in this basis.

***Note 1.*** If the basis is chosen, one can write only coordinates of vector instead of the whole decomposition, i.e. one can write that $\vec{x} = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n)$ instead of $\vec{x} = \alpha_1\vec{e}_1 + \alpha_2\vec{e}_2 + \alpha_3\vec{e}_3 + \ldots + \alpha_n\vec{e}_n$.

***Note 2.*** Suppose we have 2 or 3-dimentional space. From Statements 2 and 3 and the last Theorem It follows the way to find decomposition of the vector in any chosen basis:

$$\vec{c} = \lambda\vec{a} + \mu\vec{b}$$, where $\vec{a}, \vec{b}$ are basis vectors in 2-D space

and

$$\vec{d} = \alpha\vec{a} + \beta\vec{b} + \gamma\vec{c}$$, where $\vec{a}, \vec{b}, \vec{c}$ are basis vectors in 3-D space

***Definition.*** The basis is called orthogonal if every two vectors from basis are perpendicular to each other.

***Definition.*** The basis is called orthonormal if it is orthogonal and the module of every vector is equal to 1.