Lecture #8: Linear Operations on Vectors Given by Their Coordinates

8.1 Projection of the Vector on Axis

Let us consider an arbitrary vector $\overrightarrow{AB}$ and an axis with direction given by the vector $\vec{u}$ (Fig.10). To get points A^*, B^* we drop perpendiculars from the origin and terminus of the vector on the axis.

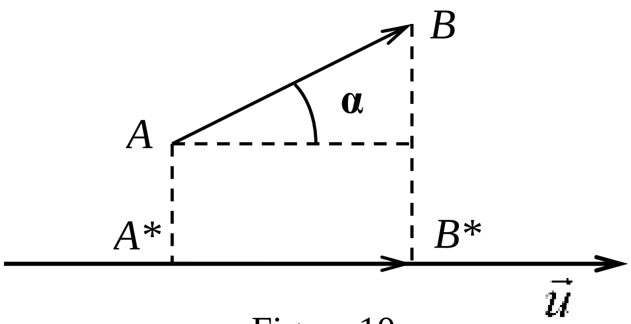


Figure 10

Definition. The length of the segment A^*B^* taken with sigh "+" if $\overrightarrow{A^*B^*}$ has the same direction with $\vec{u}$ or with the sigh "–" if $\overrightarrow{A^*B^*}$ has the opposite direction with $\vec{u}$ is called projection of the vector $\overrightarrow{AB}$ on $\vec{u}$ (or on the axis with direction $\vec{u}$). Notation: $pr_{\vec{u}}\overrightarrow{AB}$.

Note. From the definition and Fig.10 it follows that

$$pr_{\vec{u}}\overrightarrow{AB} = \left|\overrightarrow{A^*B^*}\right| = \left|\overrightarrow{AB}\right|\cos(\widehat{\overrightarrow{AB},\vec{u}}) = \left|\overrightarrow{AB}\right|\cos\alpha .$$

Properties of the projections:

1. $pr_{\vec{u}}\lambda\vec{a} = \lambda pr_{\vec{u}}\vec{a}$;
2. $pr_{\vec{u}}(\vec{a}+\vec{b}) = pr_{\vec{u}}\vec{a} + pr_{\vec{u}}\vec{b}$.
3. $pr_{\lambda\vec{u}}\vec{a} = pr_{\vec{u}}\vec{a}$ for positive λ;

and $pr_{\lambda\vec{u}}\vec{a} = -pr_{\vec{u}}\vec{a}$ for negative λ.

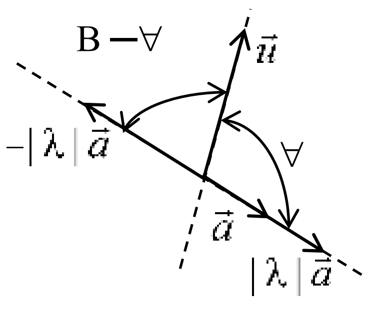


Figure 11

Proof. 1. Let $\alpha = (\widehat{\vec{a},\vec{u}})$ (Fig.11). Then

$$pr_{\vec{u}}\lambda\vec{a} = |\lambda\vec{a}|\cos(\widehat{\lambda\vec{a},\vec{u}}) = |\lambda||\vec{a}|\cos(\widehat{\lambda\vec{a},\vec{u}}) =$$

$$=\begin{cases} |\lambda\|\vec{a}|\cos\alpha \ if\ \lambda\ge 0 \\ |\lambda\|\vec{a}|\cos(\pi-\alpha)\ if\ \lambda<0 \end{cases} = \begin{cases} |\lambda\|\vec{a}|\cos\alpha \ if\ \lambda\ge 0 \\ -|\lambda\|\vec{a}|\cos\alpha\ if\ \lambda<0 \end{cases} = \lambda|\vec{a}|\cos\alpha = \lambda pr_{\vec{u}}\vec{a}.$$

2. Let us prove this property geometrically. There are six different cases (Fig.12).

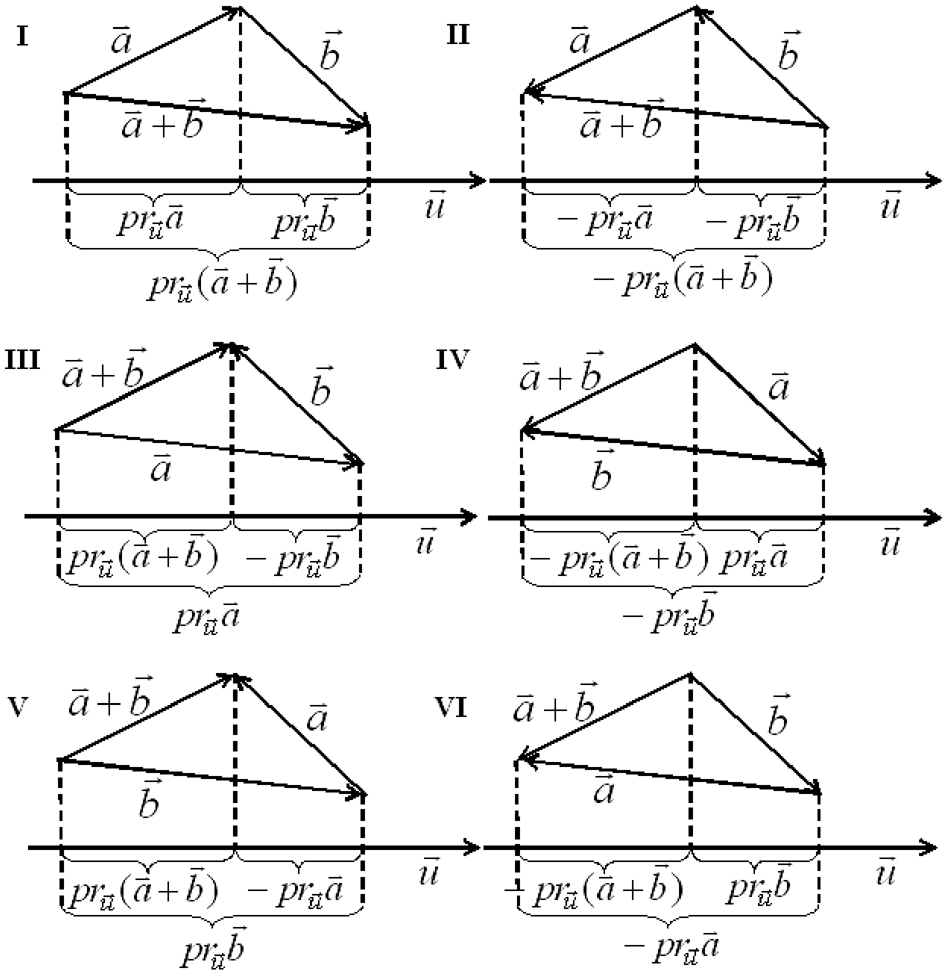


Figure 12

It follows from case I that

$$pr_{\vec{u}}(\vec{a}+\vec{b}) = pr_{\vec{u}}\vec{a} + pr_{\vec{u}}\vec{b}.$$

It follows from case II that

$$-pr_{\vec{u}}(\vec{a}+\vec{b}) = -pr_{\vec{u}}\vec{a} + -pr_{\vec{u}}\vec{b}, \text{ i.e. } pr_{\vec{u}}(\vec{a}+\vec{b}) = pr_{\vec{u}}\vec{a} + pr_{\vec{u}}\vec{b}.$$

It follows from case III that

$$pr_{\vec{u}}\vec{a} = pr_{\vec{u}}(\vec{a}+\vec{b}) - pr_{\vec{u}}\vec{b}, \text{ i.e. } pr_{\vec{u}}(\vec{a}+\vec{b}) = pr_{\vec{u}}\vec{a} + pr_{\vec{u}}\vec{b}.$$

It follows from case IV that

$$-pr_{\vec{u}}\vec{b} = -pr_{\vec{u}}(\vec{a}+\vec{b}) + pr_{\vec{u}}\vec{a}, \text{ i.e. } pr_{\vec{u}}(\vec{a}+\vec{b}) = pr_{\vec{u}}\vec{a} + pr_{\vec{u}}\vec{b}.$$

It follows from case V that

$$pr_{\vec{u}}\vec{b} = pr_{\vec{u}}(\vec{a}+\vec{b}) - pr_{\vec{u}}\vec{a}, \text{ i.e. } pr_{\vec{u}}(\vec{a}+\vec{b}) = pr_{\vec{u}}\vec{a} + pr_{\vec{u}}\vec{b}.$$

It follows from case VI that

$$-\,pr_{\vec{u}}\vec{a} = -pr_{\vec{u}}(\vec{a}+\vec{b}) + pr_{\vec{u}}\vec{b}\text{, i.e. } pr_{\vec{u}}(\vec{a}+\vec{b}) = pr_{\vec{u}}\vec{a} + pr_{\vec{u}}\vec{b}\,.$$

3. Since for the positive λ the direction of the axis stays the same, the projection of the vector saves its value. For the negative λ we obtain the opposite direction of the axis and therefore the opposite sign of the projection.

Properties are proven.

Note. One additional property of vector projection follows directly from the definition:

Example. It is known that $pr_{\vec{c}}\vec{a} = 10$, $pr_{\vec{c}}\vec{b} = 5$. Find $pr_{-\vec{c}}(3\vec{a}-2\vec{b})$.

By the projection properties we have

$$pr_{-\vec{c}}(3\vec{a}-2\vec{b}) = -pr_{\vec{c}}(3\vec{a}-2\vec{b}) = -3pr_{\vec{c}}\vec{a} + 2pr_{\vec{c}}\vec{b} = -3\cdot 10 + 2\cdot 5 = -30+10 = -20\,.$$

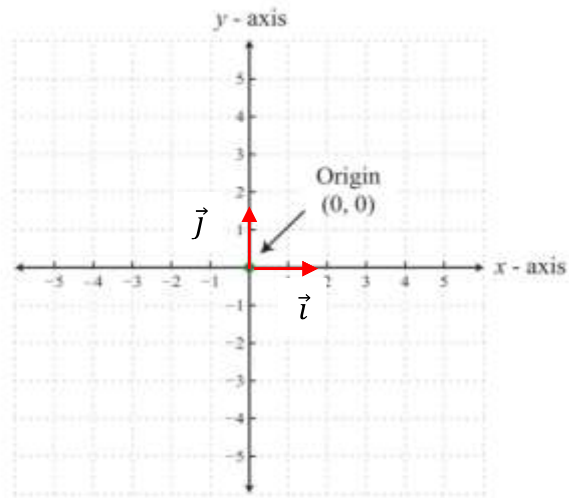
Thus, the vector $-\vec{c}$ and the vector-projection of the vector $3\vec{a}-2\vec{b}$ have the opposite directions.

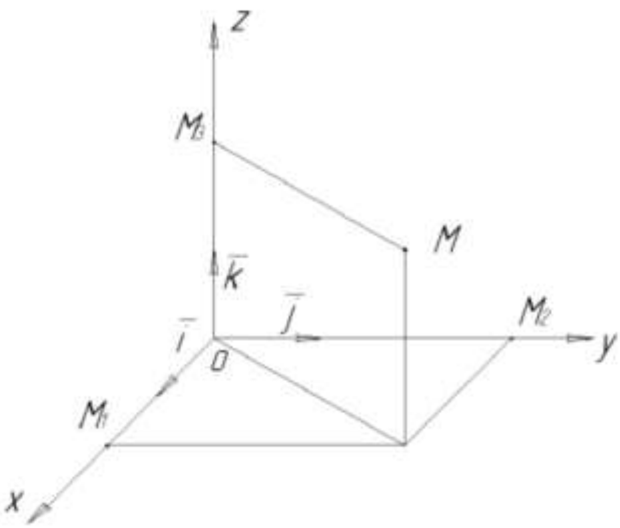
8.2 Cartesian Coordinate System

Definition. Cartesian coordinate system consists on a point *O* called an origin and perpendicular directed coordinate axes passing through the origin.

Cartesian coordinate system with two (three) axes is called coordinate system in plane (in space).

Traditionally, the axes in plane are called axis of abscissas (axis O*x*) and axis of ordinates (axis O*y*) and directed in the way that the shortest turn from positive semi-axis O*x* to positive semi-axis O*y* is made anticlockwise.





Definition. The axes in space are called axis of abscissas (axis Ox), axis of ordinates (axis Oy) and applicate axis (axis Oz) and directed in the way that the shortest turn from positive semi-axis Ox to positive semi-axis Oy is made anticlockwise if you look from the positive semi-axis Oz.

Definition. Natural bases in plane and in space are bases formed from the unit vectors directed along the positive semi-axes.
Namely, natural basis in plane is set of vectors

$$\vec{i}\,(1,0),\ \vec{j}\,(0,1);$$

and natural basis in space is set of vectors

$$\vec{i}\,(1,0,0),\ \vec{j}\,(0,1,0), \vec{k}\,(0,0,1).$$

Herewith,

$$|\vec{i}\,|=|\,\vec{j}\,|=|\,\vec{k}\,|=1$$

and

$$\vec{i} \perp \vec{j}, \vec{i} \perp \vec{k}, \vec{j} \perp \vec{k}\,,\ \text{i.e. } (\vec{i}\,,\vec{j}\,)=(\vec{i}\,,\vec{k}\,)=(\vec{j}\,,\vec{k}\,)=0\,,$$

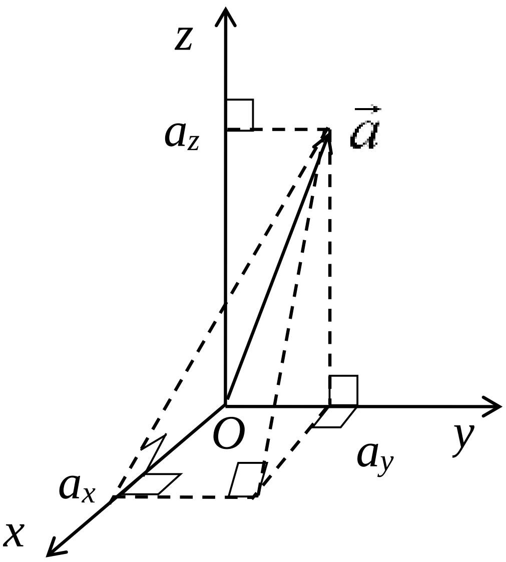


Figure13

From the Theorem about vector decomposition it follows that to find coordinates of the vector in the mentioned above bases we should connect the origin of the vector with point O and drop perpendiculars on the exes to find the vector-projections of this vector on basis vectors.

In this case the vector is equal to sum of obtained vector-projections (Fig.13).

Thus, we have:

in plane Oxy

$$\vec{a} = a_x\vec{i} + a_y\vec{j} = (a_x, a_y);$$

in space $Oxyz$

$$\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k} = (a_x, a_y, a_z)\,,$$

where

$$a_x = pr_{\vec{i}}\vec{a} = |\vec{a}|\cos\alpha, \quad a_y = pr_{\vec{j}}\vec{a} = |\vec{a}|\cos\beta, \quad a_z = pr_{\vec{k}}\vec{a} = |\vec{a}|\cos\gamma,$$

and $\alpha = (\widehat{\vec{a},\vec{i}})$, $\beta = (\widehat{\vec{a},\vec{j}})$, $\gamma = (\widehat{\vec{a},\vec{k}})$ are angles between the vector and positive semi-axes *Ox*,*Oy*,Oz.

Definition. $\cos\alpha, \cos\beta, \cos\gamma$ are called *the direction cosines* of the vector.

From Fig.13 it follows that

1) By Pythagorean Theorem

$$\boxed{|\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}}$$

2) $1 = \dfrac{|\vec{a}|^2}{|\vec{a}|^2} = \dfrac{|\vec{a}|^2\cos^2\alpha + |\vec{a}|^2\cos^2\beta + |\vec{a}|^2\cos^2\gamma}{|\vec{a}|^2} = \cos^2\alpha + \cos^2\beta + \cos^2\gamma$, i.e.

$$\boxed{\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1}$$

3) Vector $(\cos\alpha, \cos\beta, \cos\gamma)$ is a vector of unit length with the same with vector $\vec{a}$ direction. Thus, this vector is ort of the vector $\vec{a}$, i.e.

$$\boxed{\vec{a}^0 = (\cos\alpha, \cos\beta, \cos\gamma) = \frac{\vec{a}}{|\vec{a}|}}$$

Example. It is known that $|\vec{a}| = 2$, $\cos\alpha = 1/2$, $\cos\gamma = -1/2$ and an angle between the axis O*y* and $\vec{a}$ is acute. Find the coordinates of the vector $\vec{a}$.

Since the angle β is acute then $\cos\beta > 0$ and

$$\cos\beta = \sqrt{1 - \cos^2\alpha - \cos^2\gamma} = \sqrt{1 - \frac{1}{4} - \frac{1}{4}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}.$$

Therefore

$$a_x = |\vec{a}|\cos\alpha = 1, \quad a_y = |\vec{a}|\cos\beta = \sqrt{2}, \quad a_z = \vec{a}|\cos\gamma = -1;$$

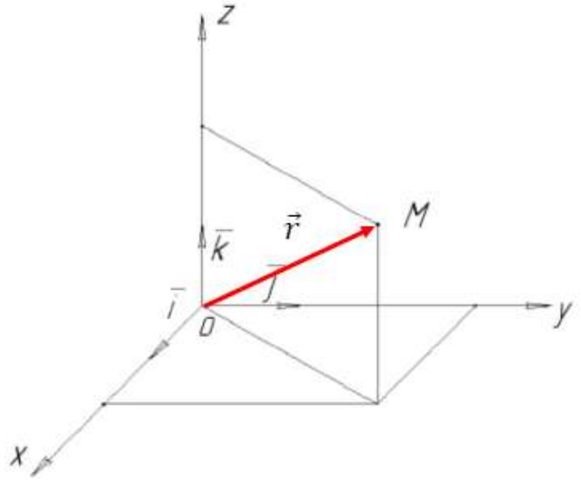
$$\vec{a} = \left(1; \sqrt{2}; -1\right).$$

8.3 Radius-vector of the Point

Definition. Suppose we have Cartesian coordinate system. Vector $\overrightarrow{OM}$ with origin in the point O and a terminus M is called a radius-vector of the point M, denoted $\vec{r}_M$ or simply $\vec{r}$

Coordinates of the point in the Cartesian coordinate system by definition are coordinates of its radius-vector, i.e.

$$\text{If} \quad \overrightarrow{OM} = \vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = (x, y, z) \quad \text{then} \quad M(x, y, z).$$



8.4 Coordinates of Vectors in Orthonormal Basis

Let us find coordinates of the vector $\overrightarrow{AB}$ through the coordinates of A and B (Fig.14).

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x_B, y_B, z_B) - (x_A, y_A, z_A) = (x_B - x_A, y_B - y_A, z_B - z_A)$$

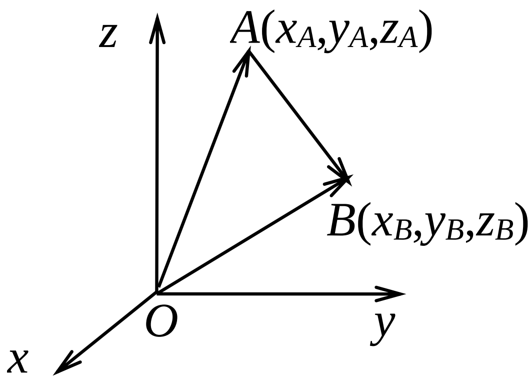


Figure 14

Definition. It means that to find coordinates of the vector we should subtract from the coordinates of the terminus the coordinates of the origin.

At the same time, since module of the vector $\overrightarrow{AB}$ is equal to the distance between two points, we state the following:

Definition. The distance between two points *A* and *B* is equal to

$$d = |\overrightarrow{AB}| = \sqrt{(x_{\mathrm{B}} - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2} \ .$$

Example. It is known that $\vec{a} = \overrightarrow{AB} = (1;2;-1)$, $A(1;1;0)$. Find the coordinates of the point *B* and distance between the points *A* and *B*.

$$a_x = x_{\mathrm{B}} - x_A \Rightarrow x_B = a_x + x_A = 1 + 1 = 2\,;$$

$$a_y = y_{\mathrm{B}} - y_A \Rightarrow y_B = a_y + y_A = 2 + 1 = 3;$$

$$a_z = z_{\mathrm{B}} - z_A \Rightarrow z_B = a_z + z_A = -1 + 0 = -1\,.$$

Therefore, *B*(2;3;-1).

The distance between the points *A* and *B* is equal to the length of the vector $\overrightarrow{AB}$:

$$d = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6}\ .$$

8.5 Linear Operations on Vectors Given by Their Coordinates

Suppose we consider some 3-dimensional linear space with basis $\vec{e}_1, \vec{e}_2, \vec{e}_3$, and vectors

$$\vec{x} = \alpha_1\vec{e}_1 + \alpha_2\vec{e}_2 + \alpha_3\vec{e}_3 = (\alpha_1, \alpha_2, \alpha_3)\,,$$

$$\vec{y} = \beta_1\vec{e}_1 + \beta_2\vec{e}_2 + \beta_3\vec{e}_3 = (\beta_1, \beta_2, \beta_3)\,.$$

Then one can calculate

- multiplication of a vector by a scalar

$$\lambda\vec{x} = \lambda(\alpha_1\vec{e}_1 + \alpha_2\vec{e}_2 + \alpha_3\vec{e}_3) = \lambda\alpha_1\vec{e}_1 + \lambda\alpha_2\vec{e}_2 + \lambda\alpha_3\vec{e}_3 = (\lambda\alpha_1, \lambda\alpha_2, \lambda\alpha_3)\,.$$

- sum of vectors

$$\vec{x} + \vec{y} = \alpha_1\vec{e}_1 + \alpha_2\vec{e}_2 + \alpha_3\vec{e}_3 + \beta_1\vec{e}_1 + \beta_2\vec{e}_2 + \beta_3\vec{e}_3 =$$

$$= (\alpha_1 + \beta_1)\vec{e}_1 + (\alpha_2 + \beta_2)\vec{e}_2 + (\alpha_3 + \beta_3)\vec{e}_3 = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \alpha_3 + \beta_3)\,.$$

- subtraction of vectors

$$\vec{x} - \vec{y} = \alpha_1\vec{e}_1 + \alpha_2\vec{e}_2 + \alpha_3\vec{e}_3 - (\beta_1\vec{e}_1 + \beta_2\vec{e}_2 + \beta_3\vec{e}_3) =$$

$$= (\alpha_1 - \beta_1)\vec{e}_1 + (\alpha_2 - \beta_2)\vec{e}_2 + (\alpha_3 - \beta_3)\vec{e}_3 = (\alpha_1 - \beta_1, \alpha_2 - \beta_2, \alpha_3 - \beta_3)$$

Conclusions:

1. To multiply vector by scalar means to multiply all its coordinates by this scalar;
2. To add (subtract) two vectors means to add (subtract) their corresponding coordinates.

Example. Find $3\overline{a}$, and $\overline{a}+\overline{b}$. if the vectors $\overline{a}$ = (1, -2, 4); $\overline{b}$ = (0, -3, 1)

$$3\overline{a} = (3\cdot 1,\ 3\cdot(-2),\ 3\cdot 4) = (3, -6, 12)$$

$$\overline{a}+\overline{b} = (1+0, -2-3, 4+1) = (1, -5, 5)$$

Definition. Two vectors $\overline{a} = (a_x; a_y; a_z)$ and $\overline{b} = (b_x; b_y; b_z)$ are equal if their coordinates are equal

$$a_x = b_x,\quad a_y = b_y,\quad a_z = b_z$$

Definition. Two vectors $\overline{a} = (a_x; a_y; a_z)$ and $\overline{b} = (b_x; b_y; b_z)$ are collinear if their coordinates are proportional:

$$\frac{a_x}{b_x} = \frac{a_y}{b_y} = \frac{a_z}{b_z}.$$

Proof: If $\overline{a}$ and $\overline{b}$ are collinear then a scalar λ occur such that $\overline{a}=\lambda\overline{b}$. Therefore,

$$a_x = \lambda b_x;\quad a_y = \lambda b_y;\quad a_z=\lambda b_z,$$

from which the proportionality of the coordinates arises.

Example. Check collinearity of vectors $\overline{a} = (2; -7; 4)$ i $\overline{b} = (4; -14; 5)$

$$\frac{2}{4} = \frac{-7}{-14} \neq \frac{4}{5}$$

The vectors are not collinear.