Lecture #9: Products of Vectors

9.1 A Dot (Scalar) Product

Definition. Scalar product (or *dot product*) of two vectors $\vec{a}$ and $\vec{b}$ is a number (scalar) equal to $|\vec{a}||\vec{b}|\cos\alpha$, where $\alpha = \vec{a}\wedge\vec{b}$ is an angle between vectors $\vec{a}$ and $\vec{b}$.

We denote the scalar product in two ways: $(\vec{a},\vec{b})$ or just $\vec{a}\vec{b}$. So,

$$(\vec{a},\vec{b}) = |\vec{a}||\vec{b}|\cos\alpha.$$

Since

$$|\vec{b}|\cos\alpha = pr_{\vec{a}}\vec{b}, \qquad |\vec{a}|\cos\alpha = pr_{\vec{b}}\vec{a},$$

we have

$$(\vec{a},\vec{b}) = |\vec{a}|\,pr_{\vec{a}}\vec{b} = |\vec{b}|\,pr_{\vec{b}}\vec{a},$$

$$pr_{\vec{b}}\vec{a} = \frac{(\vec{a},\vec{b})}{|\vec{b}|} \text{ and } pr_{\vec{a}}\vec{b} = \frac{(\vec{a},\vec{b})}{|\vec{a}|}$$

Algebraic properties of the scalar product:

1) $(\vec{a},\vec{b}) = (\vec{b},\vec{a})$**;**
2) $(\lambda\vec{a},\vec{b}) = (\vec{a},\lambda\vec{b}) = \lambda(\vec{a},\vec{b})$;
3) $(\vec{a}+\vec{b},\vec{c}) = (\vec{a},\vec{c}) + (\vec{b},\vec{c})$.

Proof.

1) $(\vec{a},\vec{b}) = |\vec{a}||\vec{b}|\cos\alpha = |\vec{b}||\vec{a}|\cos\alpha = (\vec{b},\vec{a})$;

2) $(\lambda\vec{a},\vec{b}) = |\vec{b}|\,pr_{\vec{b}}\lambda\vec{a} = \lambda|\vec{b}|\,pr_{\vec{b}}\vec{a} = \lambda(\vec{a},\vec{b})$;

3) $(\vec{a}+\vec{b},\vec{c}) = |\vec{c}|\ np_{\vec{c}}(\vec{a}+\vec{b}) = |\vec{c}|(pr_{\vec{c}}\vec{a} + pr_{\vec{c}}\vec{b}) = (\vec{a},\vec{c}) + (\vec{b},\vec{c})$.

Properties are proven.

From the definition It follows that

$$(\vec{a},\vec{a}) = |\vec{a}|^2 \quad \text{or} \quad |\vec{a}| = \sqrt{(\vec{a},\vec{a})}.$$

Thus, we obtain an additional fourth property of scalar product:

4) $(\vec{a},\vec{a}) \geq 0$ and $(\vec{a},\vec{a}) = 0 \Leftrightarrow \vec{a} = \vec{0}$.

Note. $(\vec{a},\vec{a})=\vec{a}^2$ is called a scalar square

5) ***Statement (Criterion of the perpendicularity)*** Two non-zero vectors are perpendicular if and only if their scalar product is equal to zero, i.e.

$$\bar{a}\perp\bar{b}\Leftrightarrow(\bar{a},\bar{b})=0.$$

Indeed,

$$\bar{a}\perp\bar{b}\Leftrightarrow\alpha=\frac{\pi}{2}\Leftrightarrow\cos\alpha=0\Leftrightarrow(\bar{a},\bar{b})=|\vec{a}||\vec{b}|\cos\alpha=0.$$

Example 1. It is known that $\vec{a}=5\vec{p}+2\vec{q}$, $\vec{b}=\vec{p}-3\vec{q}$, $|\vec{p}|=1$, $|\vec{q}|=2$, $\varphi=\left(\widehat{\vec{p},\vec{q}}\right)=\frac{\pi}{3}$. Find $|\bar{a}+\bar{b}|$.

By the last formula

$|\vec{a}+\vec{b}|^2=(\vec{a}+\vec{b},\vec{a}+\vec{b})=(5\vec{p}+2\vec{q}+\vec{p}-3\vec{q},5\vec{p}+2\vec{q}+\vec{p}-3\vec{q})=(6\vec{p}-\vec{q},6\vec{p}-\vec{q})=$

$=36(\vec{p},\vec{p})-6(\vec{p},\vec{q})-6(\vec{q},\vec{p})+(\vec{q},\vec{q})=$[By properties of scalar product]=

$=36|\vec{p}|^2-12(\vec{p},\vec{q})+|\vec{q}|^2=36\cdot1-12\cdot1\cdot2\cdot\cos\frac{\pi}{3}+2^2=36-12+4=28.$

Thus,

$$|\bar{a}+\bar{b}|=\sqrt{28}=2\sqrt{7}.$$

Example 2. Find the ort of the vector.

From the definition of ort it follows that $\vec{a}^\circ=\lambda\vec{a}$, where $\lambda>0$. Therefore

$$1=(\vec{a}^\circ,\vec{a}^\circ)=(\lambda\vec{a},\lambda\vec{a})=\lambda^2|\bar{a}|^2\Rightarrow\lambda=\frac{1}{|\bar{a}|}\Rightarrow$$

$$\boxed{\vec{a}^\circ=\frac{\vec{a}}{|\bar{a}|}}$$

Note, that we have obtained the same formula as obtained above through the direction cosines.

Let us find the formula to calculate the scalar product of vectors given by their coordinates in the orthonormal basis $\vec{i},\vec{j},\vec{k}$.

Since

$$|\vec{i}|=|\vec{j}|=|\vec{k}|=1$$

and

$$\vec{i}\perp\vec{j}, \vec{i}\perp\vec{k}, \vec{j}\perp\vec{k}\text{, i.e. } (\vec{i},\vec{j})=(\vec{i},\vec{k})=(\vec{j},\vec{k})=0,$$

we have

$$(\vec{a},\vec{b})=(a_x\vec{i}+a_y\vec{j}+a_z\vec{k}, b_x\vec{i}+b_y\vec{j}+b_z\vec{k})=a_xb_x+a_yb_y+a_zb_z.$$

It means that to find the scalar product we should multiply the corresponding coordinates of vectors and then summarize these products.

Note. This formula is valid for the vectors in plane (case, when $a_z=b_z=0$).

Example. It is known that $\vec{a}(1,2,3)$, $\vec{b}(-1,1,2)$, $\vec{c}(0,1,4)$. Find a value k such that $\vec{a}\perp\left(\vec{b}-k\vec{c}\right)$. By the criterion of the perpendicularity we have

$$0=\left(\vec{a},\vec{b}-k\vec{c}\right)=\left(\vec{a},\vec{b}\right)-k\left(\vec{a},\vec{c}\right)=$$
$$=1(-1)+2\cdot1+3\cdot2-k(1\cdot0+2\cdot1+3\cdot4)=$$
$$=7-14k=0.$$

Thus

$$k=\frac{7}{14}=\frac{1}{2}.$$

Note. The two and three-dimensional vector spaces with scalar product, satisfying four properties written above, are called *Euclidean vector spaces*.

Example Find the angle $\varphi=(\widehat{\overline{a},\overline{b}})$ between the vector $\overline{a}=(a_x; a_y; a_z)$ and $\overline{b}=(b_x, b_y, b_z)$

Since $$\left(\vec{a},\vec{b}\right)=|\vec{a}|\left|\vec{b}\right|\cos(\vec{a}\wedge\vec{b})$$

Then,

$$\cos\varphi=\frac{\overline{a}\cdot\overline{b}}{|\overline{a}||\overline{b}|}=\frac{a_xb_x+a_yb_y+a_zb_z}{\sqrt{a_x^2+a_y^2+a_z^2}\sqrt{b_x^2+b_y^2+b_z^2}}$$

u is defined by angles α, β and γ in

$$Pr_u\overline{a} = a_x \cos\alpha + a_y \cos\beta + a_z \cos\gamma$$

Example 4. Find a dot product of the vectors $\overline{a} = (2, -1, 4); \overline{b} = (0, 3, -5)$.

$\overline{a} \cdot \overline{b} = 2 \cdot 0 + (-1) \cdot 3 + 4 \cdot (-5) = -3 - 20 = -23.$

Example 5. A triangle with vertices is given A(1; -1; 3), B(7; -5; 4), C(0;3; -4). Find the inner angle at the vertex A.

The coordinates of the vectors are

$\overline{AB}$ = (7-1; -5+1; 4-3) = (6, -4, 1); $\overline{AC}$ = (0-1; 3+1; -4-3) = (-1, 4, -7).

Then,

$$|\overline{AB}| = \sqrt{6^2 + (-4)^2 + 1^2} = \sqrt{36 + 16 + 1} = \sqrt{53};$$

$$|\overline{AC}| = \sqrt{(-1)^2 + 4^2 + (-7)^2} = \sqrt{1 + 16 + 49} = \sqrt{66};$$

Finally

$$\cos\varphi = \frac{6 \cdot (-1) + (-4) \cdot 4 + 1 \cdot (-7)}{\sqrt{53} \cdot \sqrt{66}} = \frac{-6 - 16 - 7}{\sqrt{3498}} = \frac{-29}{\sqrt{3498}};$$

$$\varphi = arc\cos\frac{-29}{\sqrt{3498}} = \pi - arc\cos\frac{29}{\sqrt{3498}}.$$

9.2 A Cross (Vector) Product

Definition. The ordered triple of non-complanar vectors $\vec{a}, \vec{b}, \vec{c}$ form *a right–hand triple* if the shortest turn from the vector $\vec{a}$ to the vector $\vec{b}$ is made anticlockwise when their origins are connected and you observe this turn from the terminus of $\vec{c}$. In other case they form *a left – hand triple*.

Definition. Vector product (or *cross product*) of vectors $\vec{a}$ and $\vec{b}$ is a vector $\vec{c}$ satisfying the following three conditions:

1) $\vec{c} \perp \vec{a}$, $\vec{c} \perp \vec{b}$;

2) $|\vec{c}| = |\vec{a}||\vec{b}|\sin\alpha$, where α is an angle between $\vec{a}$ and $\vec{b}$;

3) $\vec{a}, \vec{b}, \vec{c}$ form the right–hand triple.

We denote vector product in two ways, namely $\vec{c} = \vec{a} \times \vec{b}$ or $\vec{c} = [\vec{a}, \vec{b}]$.

Algebraic properties of the vector product:

1) $[\vec{a}, \vec{b}] = -[\vec{b}, \vec{a}]$ (Property of anti-symmetry);
2) $[\lambda\vec{a}, \vec{b}] = \lambda[\vec{a}, \vec{b}] = [\vec{a}, \lambda\vec{b}]$;
3) $[\vec{a}, \vec{b} + \vec{c}] = [\vec{a}, \vec{b}] + [\vec{a}, \vec{c}]$.

Proof. Properties 1)-2) follow directly from conditions 2 and 3 of definition.

To prove the property 3) let us show first that there is another way to plot the result of vector product (Fig. 17). We connect the origins of two vectors, project the vector

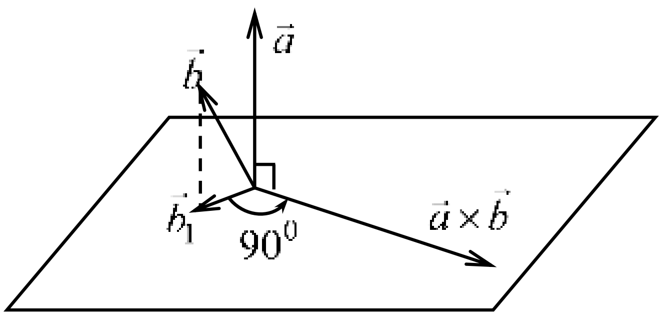

Figure 17

$\vec{b}$ on the plane perpendicular to the vector $\vec{a}$. Then we turn the obtained vector $\vec{b}_1$ anticlockwise on 90 degrees and multiply by $|\vec{a}|$. The result is $\vec{a} \times \vec{b}$ since it satisfies all conditions from the definition.

We are going to use this procedure to prove the third property. Consider the parallelogram I from the Fig.18 and project it on the plane perpendicular to $\vec{a}$.

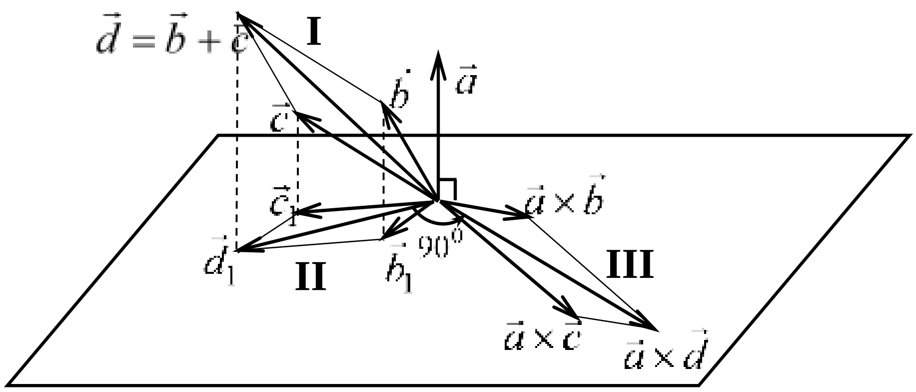

Figure 18

Obtained figure II is also parallelogram and, moreover, the diagonal $\vec{d} = \vec{b} + \vec{c}$ of the figure I is projected into the diagonal $\vec{d}_1 = \vec{b}_1 + \vec{c}_1$ of the figure II. To obtain the figure III we turn the figure II anticlockwise on 90 degrees and stretch it in $|\vec{a}|$ times. At that we again obtain the parallelogram where the diagonal of III is obtained by turn and stretching of the diagonal of II. It means that the obtained diagonal is the vector $\vec{a} \times \vec{d} = \vec{a} \times \left(\vec{b} + \vec{c}\right)$ equal to the sum of the parallelogram sides, i.e.

$$\vec{a} \times \vec{d} = \vec{a} \times \left(\vec{b} + \vec{c}\right) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}\,.$$

Properties are proven.

Geometrical properties of the vector product:

1) $\vec{a} \parallel \vec{b} \Leftrightarrow \vec{a} \times \vec{b} = 0$ (Criterion of collinearity of two non-zero vectors)

Indeed, $\vec{a} \parallel \vec{b} \Leftrightarrow \alpha = (\widehat{\vec{a}, \vec{b}}) = \left[\begin{array}{l} 0 \\ \pi \end{array}\right. \Leftrightarrow \sin\alpha = 0 \Rightarrow \left|\vec{a} \times \vec{b}\right| = 0 \Leftrightarrow \vec{a} \times \vec{b} = \vec{0}$.

Note. Another criterion of collinearity follows from definition, namely,

$$\vec{a} \parallel \vec{b} \Leftrightarrow \vec{a} = \lambda\vec{b} \Leftrightarrow a_x = \lambda b_x, a_y = \lambda b_y, a_z = \lambda b_z \Leftrightarrow \frac{a_x}{b_x} = \frac{a_y}{b_y} = \frac{a_z}{b_z},$$

i.e. *the coordinates of collinear vectors are proportional*.

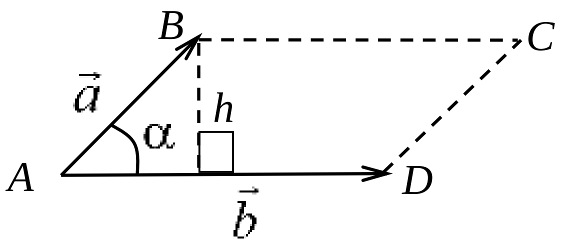

Figure 19

2) $S_{par} = \left|\vec{a} \times \vec{b}\right|$, i.e. the area of the parallelogram constructed on the vectors $\vec{a}$ and $\vec{b}$ is equal to the module of their vector product.

Indeed, from Fig. 19 we have

$$S_{par} = AB \cdot AD \cdot \sin\alpha = |\vec{a}|\left|\vec{b}\right| \sin\left(\widehat{\vec{a}, \vec{b}}\right) = \left|\vec{a} \times \vec{b}\right|.$$

3) The altitude of the parallelogram is equal to

$$h = \frac{\left|\vec{a} \times \vec{b}\right|}{\left|\vec{b}\right|}\,.$$

Indeed, from Fig.19 It follows that:

$$S_{par} = h \cdot AD \Rightarrow h = \frac{S_{par}}{AD} = \frac{\left|\vec{a} \times \vec{b}\right|}{\left|\vec{b}\right|}\,.$$

4) The area of the triangle, constructed on the vectors $\vec{a}$ and $\vec{b}$, is equal to a half of the module of their vector product. At the same time, the formula for the altitude dropped on the vector $\vec{b}$ is the same as for the parallelogram. So

$$S_{tr} = \frac{1}{2}\left|\bar{a}\times\bar{b}\right|, \qquad h = \frac{\left|\vec{a}\times\vec{b}\right|}{\left|\vec{b}\right|} .$$

Let us find the formula to calculate the vector product of vectors given by their coordinates in the orthonormal basis $\vec{i}, \vec{j}, \vec{k}$. Since

$$\begin{array}{lll} \vec{i}\times\vec{i} = 0 & \vec{i}\times\vec{j} = \vec{k} & \vec{i}\times\vec{k} = -\vec{j} \\ \vec{j}\times\vec{i} = -\vec{k} & \vec{j}\times\vec{j} = 0 & \vec{j}\times\vec{k} = \vec{i} \\ \vec{k}\times\vec{i} = \vec{j} & \vec{k}\times\vec{j} = -\vec{i} & \vec{k}\times\vec{k} = 0 \end{array}$$

the vector product of vectors $\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$ and $\vec{b} = b_x\vec{i} + b_y\vec{j} + b_z\vec{k}$ is equal to

$$\left[\vec{a},\vec{b}\right] = \left[a_x\vec{i} + a_y\vec{j} + a_z\vec{k}, b_x\vec{i} + b_y\vec{j} + b_z\vec{k}\right] = a_xb_x\vec{i}\times\vec{i} + a_xb_y\vec{i}\times\vec{j} + a_xb_z\vec{i}\times\vec{k} + a_yb_x\vec{j}\times\vec{i} +$$

$$+ a_yb_y\vec{j}\times\vec{j} + a_yb_z\vec{j}\times\vec{k} + +a_zb_x\vec{k}\times\vec{i} + a_zb_y\vec{k}\times\vec{j} + a_zb_z\vec{k}\times\vec{k} =$$

$$= \left(a_xb_y - a_yb_x\right)\vec{i}\times\vec{j} + \left(-a_xb_z + a_zb_x\right)\vec{k}\times\vec{i} + \left(a_yb_z - a_zb_y\right)\vec{j}\times\vec{k} =$$

$$= \vec{i}\left(a_yb_z - a_zb_y\right) + (-1)\vec{j}\left(a_xb_z - a_zb_x\right) + \vec{k}\left(a_xb_y - a_yb_x\right) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}.$$

So,

$$\bar{a}\times\bar{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}.$$

Example. Find area of the triangle with vertices in the points $A(1,1), B(2,-1), C(0,3)$ and vector $\vec{h}$ collinear to the altitude dropped on side *AB*.

Since the problem is formulated in plane we can not calculate vector product to find area. That is why before solving this problem we reformulate the task by expanding the coordinates of points to spatial case, i.e. we suppose that vertices have the following coordinates:

$$A(1,1,0), B(2,-1,0), C(0,2,0).$$

Then $\overrightarrow{AB}=(1,-2,0)$, $\overrightarrow{AC}=(-1,1,0)$,

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & -2 & 0\\ -1 & 1 & 0\end{vmatrix}=0\vec{i}-0\vec{j}+(-1)\vec{k}=(0,0,-1),$$

$$S_{tr}=\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\frac{1}{2}\sqrt{0^2+0^2+(-1)^2}=\frac{1}{2}\cdot 1=\frac{1}{2}.$$

Vector $\vec{h}$ is perpendicular to the vector $\overrightarrow{AB}$ and to the vector $\overrightarrow{AB}\times\overrightarrow{AC}$ (since this vector is perpendicular to any vector in the plane of triangle). It means that

$$\vec{h}=\left[\overrightarrow{AB},\overrightarrow{AB}\times\overrightarrow{AC}\right]=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & -2 & 0\\ 0 & 0 & -1\end{vmatrix}=2\vec{i}+\vec{j}+0\vec{k}=(2,1,0).$$

These coordinates are coordinates in space. To get final answer we should save only the first two coordinates, i.e. $\vec{h}(2,1)$.

9.3 A Mixed Product

Definition. Mixed product of vectors $\vec{a},\vec{b},\vec{c}$ is equal to the value obtained after scalar multiplication of the vector $\vec{c}$ by the vector product of vectors $\vec{a}$ and $\vec{b}$, i.e.

$$\left(\vec{a},\vec{b},\vec{c}\right)=\left(\vec{a}\times\vec{b},\vec{c}\right).$$

Theorem (Criterion of complanarity of three non-zero vectors) $\left(\vec{a},\vec{b},\vec{c}\right)=0 \Leftrightarrow \vec{a},\vec{b},\vec{c}$ are complanar.

Proof. $\left(\vec{a},\vec{b},\vec{c}\right)=\left(\vec{a}\times\vec{b},\vec{c}\right)=0 \Leftrightarrow \left[\begin{array}{l}\vec{a}\times\vec{b}\perp\vec{c}\\ \vec{a}\times\vec{b}=0\end{array}\right.$. It means that either $\vec{c}$ is parallel to the plane of $\vec{a}$ and $\vec{b}$ or $\vec{a}$ and $\vec{b}$ are collinear. In all these cases the vectors $\vec{a},\vec{b},\vec{c}$ are complanar. ***Theorem is proven.***

Note. If at least two factors coincide in the mixed product, this product is equal to zero. That is $(\vec{a},\vec{a},\vec{b})=0$.

Theorem (Mixed product of the right-hand triple) $\vec{a},\vec{b},\vec{c}$ form the right-hand triple if and only if $(\bar{a},\bar{b},\bar{c})>0$.

Proof. From Fig.20 it follows that if $\vec{a},\vec{b},\vec{c}$ form the right-hand triple then an angle α is acute. Thus

$$(\bar{a}\times\bar{b},\vec{c})=(\bar{a},\bar{b},\bar{c})>0.$$

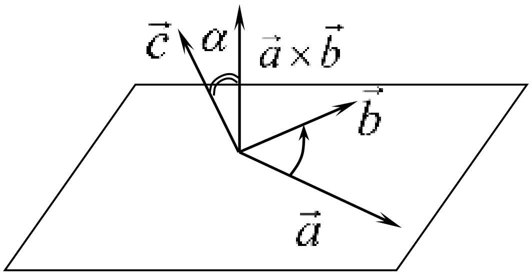

Figure 20

From the other hand, if $(\bar{a}\times\bar{b},\vec{c})>0\Rightarrow\cos\alpha>0\Rightarrow$

$\Rightarrow\alpha$ is acute $\Rightarrow\bar{a},\bar{b},\bar{c}$ form the right-hand triple. ***Theorem is proven.***

Corollary. $(\bar{a},\bar{b},\bar{c})<0\Leftrightarrow\vec{a},\vec{b},\vec{c}$ form the left-hand triple.

Theorem (Geometrical meaning of the mixed product) $V_{parallelepiped}=|(\bar{a},\bar{b},\bar{c})|$, i.e. the volume of the parallelepiped, constructed on the vectors $\vec{a},\vec{b},\vec{c}$, is equal to the module of their mixed product.

Proof. Suppose $\vec{a},\vec{b},\vec{c}$ is a right-hand triple (Fig.21). Then

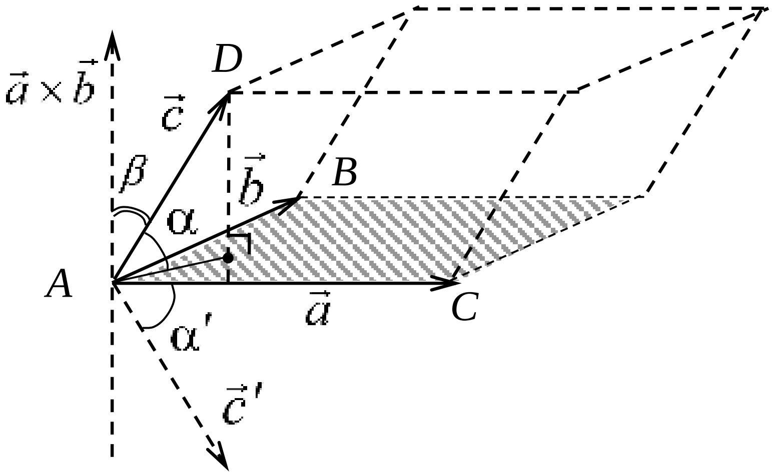

Figure 21

$V=S\cdot AD\sin\alpha=|\vec{a}\times\vec{b}||\vec{c}|\sin\alpha=$

$$=|\vec{a}\times\vec{b}||\vec{c}|\sin\left(\frac{\pi}{2}-\beta\right)=|\vec{a}\times\vec{b}||\vec{c}|\cos\beta=(\vec{a}\times\vec{b},\vec{c})=(\vec{a},\vec{b},\vec{c})=|(\bar{a},\bar{b},\bar{c})|.$$

If $\vec{a},\vec{b},\vec{c}$ form the left-hand triple (for this case $\vec{c}$ and α are shown as $\vec{c}\,',\alpha'$ on Fig.21) then

$$\sin\alpha = \sin\left(\beta - \frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2} - \beta\right) = -\cos\beta .$$

Therefore $V = -\left(\vec{a},\vec{b},\vec{c}\right) = \left|\left(\overline{a},\overline{b},\vec{c}\right)\right|$. ***Theorem is proven.***

Note. It is simple to check that if $\vec{a},\vec{b},\vec{c}$ is a right-hand triple then $\vec{c},\vec{a},\vec{b}$ and $\vec{b},\vec{c},\vec{a}$ form the right-hand triples, as well. Hence,

$$V = (\vec{a},\vec{b},\vec{c}) = (\vec{c},\vec{a},\vec{b}) = (\vec{b},\vec{c},\vec{a}) .$$

In the same way it can be shown that

$$V = -(\vec{b},\vec{a},\vec{c}) = -(\vec{c},\vec{b},\vec{a}) = -(\vec{a},\vec{c},\vec{b}) .$$

Moreover, from the obtained above it follows that

$$(\vec{a},\vec{b},\vec{c}) = (\vec{a}\times\vec{b},\vec{c}) = (\vec{b},\vec{c},\vec{a}) = (\vec{b}\times\vec{c},\vec{a}) = (\vec{a},\vec{b}\times\vec{c}) ,$$

i.e. to find mixed product we can multiply any two neighbour vectors in the vector way and then multiply the result vector by the third one in the scalar way.

Algebraic properties of the mixed product:

1.a) $\left(\vec{a},\vec{b},\vec{c}\right) = \left(\vec{c},\overline{a},\overline{b}\right) = \left(b,\vec{c},\overline{a}\right)$,

and

1.*b*) $\left(\vec{a},\vec{b},\vec{c}\right) = -\left(\overline{b},\vec{a},\vec{c}\right) = -\left(\vec{a},\vec{c},\vec{b}\right) = -\left(\vec{c},\vec{b},\vec{a}\right)$;

i.e. cyclic transposition of vectors does not change the value of the mixed product, but the transposition of any two neighbour vectors changes the sign of the mixed product. It follows from the last Note or from the properties of scalar and vector products.

2) $\left(\lambda\vec{a},\vec{b},\vec{c}\right) = \lambda\left(\vec{a},\vec{b},\vec{c}\right) = \left(\vec{a},\lambda\vec{b},\vec{c}\right) = \left(\vec{a},\vec{b},\lambda\vec{c}\right)$;

3) $\left(\vec{a}+\vec{b},\vec{c},\vec{d}\right) = \left(\vec{a},\vec{c},\vec{d}\right) + \left(\vec{b},\vec{c},\vec{d}\right)$.

Last two properties follow directly from the properties of scalar and vector products.

Geometrical properties of the mixed product:

1) $V_{parallelepiped} = \left|\left(\vec{a},\vec{b},\vec{c}\right)\right|$ (Fig.22)

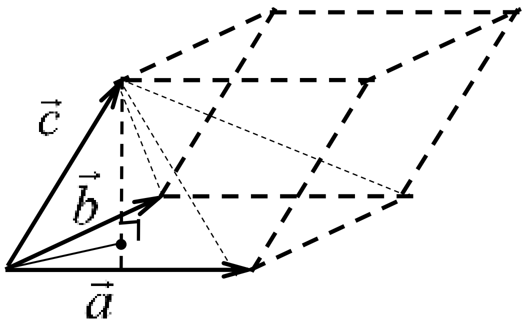

Figure 22

2) The altitude of the parallelepiped dropped on the base of vectors $\vec{a}$ and $\vec{b}$ is

$$h = \frac{V}{S} = \frac{\left|\left(\vec{a}, \vec{b}, \vec{c}\right)\right|}{\left|\vec{a} \times \vec{b}\right|}.$$

3) The volume of the tetrahedron constructed on vectors $\vec{a}, \vec{b}, \vec{c}$ (Fig.22) is equal to

$$V_{tetrahedron} = \frac{1}{6} V_{par.} = \frac{1}{6}\left|\left(\vec{a}, \vec{b}, \vec{c}\right)\right|.$$

The altitude of the tetrahedron coincides with the altitude of the parallelepiped, so it could be found by the same formula.

Let us find the formula to calculate the mixed product of vectors given by their coordinates in the orthonormal basis $\vec{i}, \vec{j}, \vec{k}$.

Suppose,

$$\vec{a} = a_x \vec{i} + a_y \vec{j} + a_z \vec{k} = \left(a_x, a_y, a_z\right)$$

$$\vec{b} = b_x \vec{i} + b_y \vec{j} + b_z \vec{k} = \left(b_x, b_y, b_z\right)$$

$$\vec{c} = c_x \vec{i} + c_y \vec{j} + c_z \vec{k} = \left(c_x, c_y, c_z\right)$$

Let us evaluate $\left(\bar{a}, \bar{b}, \vec{c}\right) = \left(\bar{a}, \bar{b} \times \bar{c}\right)$:

$$\bar{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = \vec{i}\begin{vmatrix} b_y & b_z \\ c_y & c_z \end{vmatrix} - \vec{j}\begin{vmatrix} b_x & b_z \\ c_x & c_z \end{vmatrix} + \vec{k}\begin{vmatrix} b_x & b_y \\ c_x & c_y \end{vmatrix} =$$

$$\left(\begin{vmatrix} b_y & b_z \\ c_y & c_z \end{vmatrix}, -\begin{vmatrix} b_x & b_z \\ c_x & c_z \end{vmatrix}, \begin{vmatrix} b_x & b_y \\ c_x & c_y \end{vmatrix} \right).$$

$$\left(\vec{a}, \vec{b} \times \vec{c}\right) = a_x \begin{vmatrix} b_y & b_z \\ c_y & c_z \end{vmatrix} - a_y \begin{vmatrix} b_x & b_z \\ c_x & c_z \end{vmatrix} + a_z \begin{vmatrix} b_x & b_y \\ c_x & c_y \end{vmatrix} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}.$$

Therefore,

$$\left(\vec{a}, \vec{b}, \vec{c}\right) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}.$$

Example 1. Find the coordinates of the vertex D of the tetrahedron $ABCD$ if the volume of this tetrahedron is equal to 10, D is situated on the positive semi-axis Oz and $A(1;2;3),\ B(-1;0;2),\ C(0,4,1)$.

From condition it follows that D has coordinates $D(0;0;z_D)$ and

$$V = 10 = \frac{1}{6}\left|\left(\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}\right)\right|, \text{ i.e. } \left|\left(\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}\right)\right| = 60.$$

But

$$\left(\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}\right) = \begin{vmatrix} -2 & -2 & 1 \\ -1 & 2 & -2 \\ -1 & -2 & z_D - 3 \end{vmatrix} = -4(z_D - 3) - 4 + 2 + 2 + 8 - 2(z_D - 3) =$$

$$= -6z_D + 26.$$

Therefore

$$-6z_D + 26 = \pm 60 \Leftrightarrow \left[\begin{array}{l} -6z_D = 34 \\ -6z_D = -86 \end{array}\right. \Leftrightarrow \left[\begin{array}{l} z_D = -34/6 = -17/3 \\ z_D = 43/3 \end{array}\right.$$

Since D is situated on the positive semi-axis Oz the answer is $D(0;0;43/3)$.

Example 2. Prove that four points are situated on the same plane if their coordinates are $A(1;1;1),\ B(1;2;3),\ C(2;3;4),\ D(0;2;4)$.

These points are from the same plane if and only if the vectors $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ are complanar. Let us check this statement.

$$\left(\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}\right) = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ -1 & 1 & 3 \end{vmatrix} = 0 - 3 + 2 + 4 - 3 - 0 = 0.$$

Therefore the vectors are complanar and points are situated on the same plane.

Example 3. Find $\left(\vec{a} + \vec{b}, \vec{b} + \vec{c}, \vec{c}\right)$ if $\left(\vec{a}, \vec{b}, \vec{c}\right) = 1$. By mixed product properties:

$$(\vec{a} + \vec{b}, \vec{b} + \vec{c}, \vec{c}) = (\vec{a}, \vec{b}, \vec{c}) + (\vec{a}, \vec{c}, \vec{c}) + (\vec{b}, \vec{b}, \vec{c}) + (\vec{b}, \vec{c}, \vec{c}) = 1 + 0 + 0 + 0 = 1.$$