Lecture #4:

4.1 Inverse Matrix

Definition. A square matrix A is called a *non-singular matrix* if $\det A \neq 0$. In other case it is called a singular.

Definition. A square matrix A^{-1} is called an *inverse matrix* to the square non-singular matrix A if $A^{-1}A = AA^{-1} = E$.

In this case, the matrix A is called ***invertible***

Example.

Suppose $A = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$, $B = \begin{pmatrix} \cos(-t) & -\sin(-t) \\ \sin(-t) & \cos(-t) \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$.

Then $AB = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} =$

$$= \begin{pmatrix} \cos^2 t + \sin^2 t & \cos t \sin t - \sin t \cos t \\ \sin t \cos t - \cos t \sin t & \sin^2 t + \cos^2 t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$BA = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} =$$

$$= \begin{pmatrix} \cos^2 t + \sin^2 t & -\cos t \sin t + \sin t \cos t \\ -\sin t \cos t + \cos t \sin t & \sin^2 t + \cos^2 t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, $B = A^{-1}$, i.e. the matrix B is an inverse matrix to the matrix A.

Analyzing this results one can see that an inverse matrix A^{-1} will have the same size as the matrix A. It is very important to observe that the inverse of a matrix, if it exists, is unique. Another way to think of this is that if it acts like the inverse, then it is the inverse.

Theorem: Uniqueness of Inverse

Suppose A is an $n \times n$ matrix such that an inverse A^{-1} exists. Then there is only one such inverse matrix. That is, given any matrix B such that $AB = BA = I$, $B = A^{-1}$.

Proof

In this proof, it is assumed that I is an $n \times n$ identity matrix. Let A, B be $n \times n$ matrices such that A^{-1} exists and $AB = BA = I$. We want to show that $A^{-1} = B$.

Now using properties, we have seen, we get:

$$A^{-1} = A^{-1}I = A^{-1}(AB) = (A^{-1}A)B = IB = B$$

Hence, $A^{-1} = B$ which tells us that the inverse is unique.

The next example demonstrates how to check the inverse of a matrix.

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. Show $\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ is the inverse of A

To check this, multiply

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

and

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Properties of the Inverse

Let A be an $n \times n$ matrix and I the usual identity matrix

1. I is invertible and $I^{-1} = I$
2. If A is invertible then so is A^{-1}, and $(A^{-1})^{-1} = A$
3. If A is invertible then so is A^k, and $(A^k)^{-1} = (A^{-1})^k$
4. If A is invertible and p is a nonzero real number, then pA is invertible and $(pA)^{-1} = \frac{1}{p}A^{-1}$

Further, we explore how to find A^{-1}.

First, we need to introduce a new matrix called the ***cofactor matrix*** of A.

Definition. The cofactor matrix of A is the matrix whose ij −th entry is the ij −th cofactor of A. The formal definition is as follows:

Let $A = [a_{ij}]$ be an $n \times n$ matrix. Then the cofactor matrix of A, denoted $cof(A)$, is defined by $cof(A) = [cof(A)_{ij}]$ where $cof(A)_{ij}$ is the ij −th cofactor of A:

$$cof(A) = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & & & \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}$$

Note that $cof(A)_{ij}$ denotes the ij −th entry of the cofactor matrix.

We will use the cofactor matrix to define the ***adjugate*** of A. We can also call this matrix the ***classical adjoint*** of A.

Definition. The adjugate or classical adjoint of a square matrix A is the transpose of its cofactor matrix and we denote it by

$$adj(A):\ adj(A) = (cof\,(A))^T = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & & A_{nn} \end{pmatrix}.$$

Theorem **(*Necessary condition for the matrix to have an inverse matrix*)** If matrix A has an inverse matrix A^{-1} then $\det A \neq 0$.

Proof. Let us prove the theorem from the contrary. Let $\det A = 0$. By property 10 of the determinants:

$1 = \det E = \det(AA^{-1}) = \det A \det A^{-1} = 0 \det A^{-1} = 0$. We have got a contradiction.

Theorem is proven.

Definition. The matrix A is called non-singular if its determinant does not equal to zero, otherwise the matrix is singular.

Theorem **(*Sufficient condition for the matrix to have an inverse matrix*)** Any non-singular matrix A has an inverse matrix and only one.

Proof. Let us prove that matrix $B = \dfrac{1}{\det A} A^{ad}$ is inverse matrix to the matrix A.

$$AB = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \frac{1}{\det A} \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & & A_{nn} \end{pmatrix} =$$

$$\frac{1}{\det A} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & & A_{nn} \end{pmatrix} =$$

=[by Theorem on determinant decomposition and Theorem about sum of products of row/column elements by algebraic cofactors of elements from another row/columns]=

$$= \frac{1}{\det A}\begin{pmatrix} \det A & 0 & \dots & 0 \\ 0 & \det A & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & \det A \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & 1 \end{pmatrix} = E\,.$$

In the same way it can be proven that $BA = E$. It means that B is inverse matrix of A. Let us prove that A does not have other inverse matrices.

Suppose C is another inverse matrix of A, i.e. $AB = BA = E$ and $CA = AC = E$. Then $C = CE = CAB = EB = B$. It means that B is the only inverse matrix.

The theorem is proven.

So, in this case we have that the inverse of the matrix can be found by the formula:

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & & & \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{pmatrix}^T = \frac{1}{\det A}\begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & & A_{nn} \end{pmatrix} \text{ if } \det A \neq 0$$

That is

$$A^{-1} = \frac{1}{det(A)} adj(A)$$

Example. Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

According to Theorem

$$A^{-1} = \frac{1}{det(A)} adj(A)$$

First, we will find the determinant of this matrix. Using Theorems, we can first simplify the matrix through row operations. First, add -3 times the first row to the second row. Then add -1 times the first row to the third row to obtain

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -6 & -8 \\ 0 & 0 & -2 \end{bmatrix}$$

By Theorem $det(A) = det(B)$. By Theorem, $det(B) = 1 \times -6 \times -2 = 12$. Hence, $det(A) = 12$.

Now, we need to find $adj(A)$. To do so, first we will find the cofactor matrix of A. This is given by

$$\operatorname{cof}(A) = \begin{bmatrix} -2 & -2 & 6 \\ 4 & -2 & 0 \\ 2 & 8 & -6 \end{bmatrix}$$

Here, the ij −th entry is the ij-th cofactor of the original matrix A which you can verify. Therefore, from Theorem, the inverse of A is given by

$$A^{-1} = \frac{1}{12}\begin{bmatrix} -2 & -2 & 6 \\ 4 & -2 & 0 \\ 2 & 8 & -6 \end{bmatrix}^T = \begin{bmatrix} -\frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{2}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}$$

Remember that we can always verify our answer for A^{-1}. Compute the product AA^{-1} and $A^{-1}A$ and make sure each product is equal to I.

Compute $A^{-1}A$ as follows

$$A^{-1}A = \begin{bmatrix} -\frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{2}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence our answer is correct.

Example. Find the inverse of the matrix

$$A = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{6} & \frac{1}{3} & -\frac{1}{2} \\ -\frac{5}{6} & \frac{2}{3} & -\frac{1}{2} \end{bmatrix}$$

First we need to find $det(A)$. This step is left as an exercise and you should verify that $\det(A) = \frac{1}{6}$. The inverse is therefore equal to

$$A^{-1} = \frac{1}{(1/6)}adj(A) = 6adj(A)$$

We continue to calculate as follows. Here we show the 2×2 determinants needed to find the cofactors.

$$A^{-1} = 6\begin{bmatrix} \begin{vmatrix} \frac{1}{3} & -\frac{1}{2} \\ \frac{2}{3} & -\frac{1}{2} \end{vmatrix} & -\begin{vmatrix} -\frac{1}{6} & -\frac{1}{2} \\ -\frac{5}{6} & -\frac{1}{2} \end{vmatrix} & \begin{vmatrix} -\frac{1}{6} & \frac{1}{3} \\ -\frac{5}{6} & \frac{2}{3} \end{vmatrix} \\ -\begin{vmatrix} 0 & \frac{1}{2} \\ \frac{2}{3} & -\frac{1}{2} \end{vmatrix} & \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{5}{6} & -\frac{1}{2} \end{vmatrix} & -\begin{vmatrix} \frac{1}{2} & 0 \\ -\frac{5}{6} & \frac{2}{3} \end{vmatrix} \\ \begin{vmatrix} 0 & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{2} \end{vmatrix} & -\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{6} & -\frac{1}{2} \end{vmatrix} & \begin{vmatrix} \frac{1}{2} & 0 \\ -\frac{1}{6} & \frac{1}{3} \end{vmatrix} \end{bmatrix}^T$$

Expanding all the 2×2 determinants, this yields

$$A^{-1} = 6\begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & -2 & 1 \end{bmatrix}$$

Again, you can always check your work by multiplying $A^{-1}A$ and AA^{-1} and ensuring these products equal I.

$$A^{-1}A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{6} & \frac{1}{3} & -\frac{1}{2} \\ -\frac{5}{6} & \frac{2}{3} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Inverses of Transposes and Products

By means of the obtained above formula for inverse matrix, it is simple to prove the following properties of the inverse matrices and products:

Let A, B, and A_i for $i = 1, \ldots, k$ be $n \times n$ matrices

1. If A is an invertible matrix, then $(A^{-1})^{-1} = A$;
2. If A is an invertible matrix, then $(A^T)^{-1} = (A^{-1})^T$;
3. If A and B are invertible matrices, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$. Moreover, If $A_1, A_2, \ldots, A_k$ are invertible, then the product $A_1A_2 \cdots A_k$ is invertible, and $(A_1A_2 \cdots A_k)^{-1} = A_k^{-1}A_{k-1}^{-1} \cdots A_2^{-1}A_1^{-1}$
4. $\det A^{-1} = (\det A)^{-1}$