Solving of Systems of Linear Algebraic Equations (SLAE) with the same number of equations and unknowns

4.2. Solving the Matrix Equations by Means of Inverse Matrix

Let us consider three types of matrix equations.

Type 1. $AX = B$, where A is square non-singular matrix.

By the theorem of the previous section, the matrix A has an inverse matrix A^{-1}. Let us multiply the equation by A^{-1} from the left hand side. Then

$$\begin{aligned} AX &= B \\ A^{-1}(AX) &= A^{-1}B \\ (A^{-1}A)X &= A^{-1}B \\ IX &= A^{-1}B \\ X &= A^{-1}B \end{aligned}$$

Type 2. $XA = B$, where A is square non-singular matrix.

By the theorem of the previous section, the matrix A has an inverse matrix A^{-1}. Let us multiply the equation by A^{-1} from the right hand side. Then

$$\begin{aligned} XA &= B \\ (XA)A^{-1} &= BA^{-1} \\ X(AA^{-1}) &= BA^{-1} \\ XI &= BA^{-1} \\ X &= BA^{-1} \end{aligned}$$

Type 3. $AXC = B$, where A and C are square non-singular matrix.

By the theorem of the previous section, the matrices A and C have the inverse matrices. Let us multiply the equation by A^{-1} from the left and by C^{-1} from the right hand sides. Then

$$\begin{aligned} AXC &= B \\ A^{-1}(AXC)C^{-1} &= A^{-1}BC^{-1} \\ (A^{-1}A)X(CC^{-1}) &= A^{-1}BC^{-1} \\ IXI &= A^{-1}BC^{-1} \\ X &= A^{-1}BC^{-1} \end{aligned}$$

Example. Solve the system $\begin{cases} x_1 + 2x_2 = 1 \\ 3x_1 + x_2 - 2x_3 = 5 \\ x_1 - x_2 - 4x_3 = 5 \end{cases}$.

Let us introduce some matrices:

$$A=\begin{pmatrix}1 & 2 & 0\\ 3 & 1 & -2\\ 1 & -1 & -4\end{pmatrix}$$, i.e. the matrix of the system;

$$B=\begin{pmatrix}1\\ 5\\ 5\end{pmatrix}$$, i.e. the column matrix of right sides;

$$X=\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}$$, i.e. the column matrix of the unknowns.

Then the initial system can be rewritten in the form

$$AX=B.$$

Since the determinant of the matrix *A*

$$\det A=\begin{vmatrix}1 & 2 & 0\\ 3 & 1 & -2\\ 1 & -1 & -4\end{vmatrix}=-4-4+0-0+24-2=14\neq 0$$

we can use the inverse matrix A^{-1} to obtain the solution of the system (Type 1). Let us calculate all cofactors of the matrix *A*:

$$A_{11}=(-1)^{1+1}\begin{vmatrix}1 & -2\\ -1 & -4\end{vmatrix}=-6,\ A_{21}=(-1)^{2+1}\begin{vmatrix}2 & 0\\ -1 & -4\end{vmatrix}=8,\ A_{31}=(-1)^{3+1}\begin{vmatrix}2 & 0\\ 1 & -2\end{vmatrix}=-4;$$

$$A_{12}=(-1)^{1+2}\begin{vmatrix}3 & -2\\ 1 & -4\end{vmatrix}=10,\quad A_{22}=(-1)^{2+2}\begin{vmatrix}1 & 0\\ 1 & -4\end{vmatrix}=-4,\ A_{32}=(-1)^{3+2}\begin{vmatrix}1 & 0\\ 3 & -2\end{vmatrix}=2;$$

$$A_{13}=(-1)^{1+3}\begin{vmatrix}3 & 1\\ 1 & -1\end{vmatrix}=-4,\quad A_{23}=(-1)^{2+3}\begin{vmatrix}1 & 2\\ 1 & -1\end{vmatrix}=3,\quad A_{33}=(-1)^{3+3}\begin{vmatrix}1 & 2\\ 3 & 1\end{vmatrix}=-5.$$

So

$$A^{-1}=\frac{1}{14}\begin{pmatrix}-6 & 8 & -4\\ 10 & -4 & 2\\ -4 & 3 & -5\end{pmatrix};$$

$$X=A^{-1}B=\frac{1}{14}\begin{pmatrix}-6 & 8 & -4\\ 10 & -4 & 2\\ -4 & 3 & -5\end{pmatrix}\begin{pmatrix}1\\ 5\\ 5\end{pmatrix}=\frac{1}{14}\begin{pmatrix}-6+40-20\\ 10-20+10\\ -4+15-25\end{pmatrix}=\frac{1}{14}\begin{pmatrix}14\\ 0\\ -14\end{pmatrix}=\begin{pmatrix}1\\ 0\\ -1\end{pmatrix},$$

i.e. $x_1=1,\ x_2=0,\ x_3=-1.$

4.3 Rule by Cramer

Another context in which the formula given in the Theorem on the invertible matrix is important is Cramer's Rule. Recall that we can represent a system of linear equations in the form $AX=B$, where the solutions to this system are given by X.

Cramer's Rule gives a formula for the solutions X in the special case that A is a square invertible matrix.

Note this rule does not apply if you have a system of equations in which there is a different number of equations than variables (in other words, when A is not square), or when A is not invertible.

Let us consider the system of linear algebraic equations (SLAE) with *n* equations and *n* unknowns:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \end{cases}$$

Hence, the solutions X to the system are given by $X = A^{-1}B$. Since we assume that A^{-1} exists, we can use the formula for A^{-1} given above. Substituting this formula into the equation for X , we have

$$X = A^{-1}B = \frac{1}{det(A)} adj(A)B$$

Let x_i be the $i-$th entry of X and b_j be the $j-$th entry of B. Then this equation becomes

$$x_i = \sum_{j=1}^{n} [a_{ij}]^{-1} b_j = \sum_{j=1}^{n} \frac{1}{det(A)} adj(A)_{ij} b_j$$

We can introduce the following notations:

$$\Delta = \begin{vmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \\ a_{n1} & \ldots & a_{nn} \end{vmatrix}$$

- it is called a determinant of the coefficients at the unknowns;

The following determinants Δ_i are determinants of the matrices A_i, where each matrix A_i is the matrix obtained by replacing the $i-$th column of A with the column matrix

$$\Delta_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} & \ldots & a_{1n} \\ b_2 & a_{22} & a_{23} & \ldots & a_{2n} \\ \vdots & & & & \\ b_n & a_{n2} & a_{n3} & \ldots & a_{nn} \end{vmatrix} = b_1A_{11} + b_2A_{21} + \ldots + b_nA_{n1} = \sum_{i=1}^{n} b_iA_{i1};$$

$$\Delta_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} & \dots & a_{1n} \\ a_{21} & b_2 & a_{23} & \dots & a_{2n} \\ \vdots & & & & \\ a_{n1} & b_n & a_{n3} & \dots & a_{nn} \end{vmatrix} = b_1 A_{12} + b_2 A_{22} + \dots + b_n A_{n2} = \sum_{i=1}^{n} b_i A_{i2};$$

…

$$\Delta_n = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & b_1 \\ a_{21} & a_{22} & a_{23} & \dots & b_2 \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \dots & b_n \end{vmatrix} = b_1 A_{1n} + b_2 A_{2n} + \dots + b_n A_{nn} = \sum_{i=1}^{n} b_i A_{in}.$$

Let us multiply the first equation of the given above system by A_{11}, the second equation by A_{21}, the third by A_{31}, … the n^{th} by A_{n1} and summarize these products collecting similar terms:

$$x_1(a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} + \dots + a_{n1}A_{n1}) + x_2(a_{12}A_{11} + a_{22}A_{21} + a_{32}A_{31} + \dots + a_{n2}A_{n1}) +$$

$$x_3(a_{13}A_{11} + a_{23}A_{21} + a_{33}A_{31} + \dots + a_{n3}A_{n1}) + \dots + x_n(a_{1n}A_{11} + a_{2n}A_{21} + a_{3n}A_{31} + \dots + a_{nn}A_{n1}) =$$

$$= b_1A_{11} + b_2A_{21} + \dots + b_nA_{n1}.$$

According to the Theorems of the previous section we can rewrite the equation above as follows:

$$x_1\Delta + x_2 0 + x_3 0 + \dots + x_n 0 = \Delta_1 \Leftrightarrow x_1\Delta = \Delta_1 \Rightarrow \left(\text{if } \Delta \neq 0 \text{ then } x_1 = \frac{\Delta_1}{\Delta}\right).$$

In the same way if one multiplies the first equation of the system by A_{1k}, the second equation by A_{2k}, the third by A_{3k}, … the n^{th} by A_{nk} one has:

$$x_1 0 + \dots + x_{k-1} 0 + x_k\Delta + x_{k+1} 0 + \dots + x_n 0 = \Delta_1 \Leftrightarrow x_k\Delta = \Delta_k \Rightarrow$$

$$\Rightarrow \text{if } \Delta \neq 0 \text{ then } x_k = \frac{\Delta_k}{\Delta}.$$

So we are able to prove the following theorem:

***Theorem* (Rule by Cramer)** Suppose A is an $n \times n$ invertible matrix and we wish to solve the system $AX = B$ for $X = [x_1, \cdots, x_n]^T$. Then Cramer's rule says

$$x_i = \frac{det(A_i)}{det(A)} \text{ or } x_i = \frac{\Delta_i}{\Delta},\ i = \overline{1, n}.$$

where A_i is the matrix obtained by replacing the i-th column of A with the column matrix

$$B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

Example. Solve the system $\begin{cases} x_1 + 2x_2 = 1 \\ 3x_1 + x_2 - 2x_3 = 5 \\ x_1 - x_2 - 4x_3 = 5 \end{cases}$.

Let us find the determinant of the system Δ:

$$\Delta = \begin{vmatrix} 1 & 2 & 0 \\ 3 & 1 & -2 \\ 1 & -1 & -4 \end{vmatrix} = -4 - 4 + 0 - 0 + 24 - 2 = 14.$$

Since the determinant is not equal to zero we can use the rule by Cramer:

$$\Delta_1 = \begin{vmatrix} 1 & 2 & 0 \\ 5 & 1 & -2 \\ 5 & -1 & -4 \end{vmatrix} = -4 - 20 + 0 - 0 + 40 - 2 = 14,$$

$$\Delta_2 = \begin{vmatrix} 1 & 1 & 0 \\ 3 & 5 & -2 \\ 1 & 5 & -4 \end{vmatrix} = -20 - 2 - 0 + 0 + 12 + 10 = 0,$$

$$\Delta_3 = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 1 & 5 \\ 1 & -1 & 5 \end{vmatrix} = 5 + 10 - 3 - 1 - 30 + 5 = -14.$$

So, $x_1 = \dfrac{\Delta_1}{\Delta} = \dfrac{14}{14} = 1,\ x_2 = \dfrac{\Delta_2}{\Delta} = \dfrac{0}{14} = 0,\ x_3 = \dfrac{\Delta_3}{\Delta} = \dfrac{-14}{14} = -1$