Lecture #5: Solving of Systems of Linear Algebraic Equations (SLAE) with *m* number of equations and *n* number unknowns

Let us consider the system of m linear algebraic equations (SLAE) with n unknown variables:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2 \\ \ldots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n = b_m \end{cases}$$

where we can denote that a matrix A is called *the matrix of the system*, B is a *column of right hand side*, X is *a column of unknowns,* that is

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \\ a_{m1} & \ldots & a_{mn} \end{pmatrix}, \; X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \; B = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

Let us denote each k-th row of the matrix A of the size m by n as $e_k \; (k = \overline{1,m})$,

$$e_1 = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \end{pmatrix},$$
$$e_2 = \begin{pmatrix} a_{21} & a_{22} & \ldots & a_{2n} \end{pmatrix},$$
$$\ldots$$
$$e_k = \begin{pmatrix} a_{k1} & a_{k2} & \ldots & a_{kn} \end{pmatrix},$$
$$\ldots$$
$$e_n = \begin{pmatrix} a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}$$

and each k-th column of this matrix as $t_k \; (k = \overline{1,n})$, i.e.

$$t_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}, \; t_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{pmatrix}, \; \ldots, \; t_k = \begin{pmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{nk} \end{pmatrix}, \; \ldots, \; t_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{pmatrix}.$$

Then the above system can be rewritten in the following equivalent forms:

$$AX = B$$

or

$$t_1x_1 + t_2x_2 + t_3x_3 + \ldots + t_nx_n = B.$$

Definition. If $B \neq 0$ then the system is called *inhomogeneous*. Otherwise, i.e. $B = 0$, it is called *homogeneous*.

Definition. Any set of numbers $x_1 = \alpha_1, x_2 = \alpha_2, \ldots, x_n = \alpha_n$ is called a *solution of the system*, if after substituting these numbers into the system, the identity will be obtained.

Definition. If the system has at least one solution, then it is called a *consistent system*.

Definition. If the system has no solutions, then it is called an *inconsistent system*.

Definition. If the system has only one solution, then it is called a *definite system*.

Definition. If the system has more then one solution, then it is called an *indefinite system*.

Definition. A matrix including both the matrix A and the column B, i.e.

$$A^* = \left(\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & & & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{array}\right)$$ is called an *augmented matrix* of the system.

5.1 The Conception of Linear Dependence/Independence

Example:

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$$

has the row vectors $e_1 = [3,0,2,2]$ (first row)

$e_2 = [-5,42,24,54]$ (second row)

$e_3 = [21,-21,0,-15]$ (third row)

We can combine:

$$6 \cdot e_1 - \frac{1}{2} \cdot e_2 = 6 \cdot [3,0,2,2] - \frac{1}{2} \cdot [-5,42,24,54] = [21,-21,0,-15] = e_3$$

So, e_3 is *a linear combination* of e_1 and e_2

Definition. The expression $\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \dots + \alpha_m e_m$, where α_k $(k = \overline{1,m})$ are real numbers (coefficients), is called a *linear combination of rows*, e_k $(k = \overline{1,m})$. Similar, the expression $\gamma_1 t_1 + \gamma_2 t_2 + \gamma_3 t_3 + \dots + \gamma_n t_n$, where γ_k $(k = \overline{1,m})$ are real numbers, is called *a linear combination of columns*, t_k $(k = \overline{1,n})$.

Definition. Rows of the matrix are *linearly dependent* (LD) if there is some zero linear combination of the rows with at least one nonzero coefficient, i.e.

$$e_1, e_2, \ldots, e_m \text{ are LD } \Leftrightarrow \left(\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \ldots + \alpha_m e_m = 0 \quad \text{and} \quad \exists i_0 : \alpha_{i_0} \neq 0\right).$$

Definition. Rows of the matrix are *linearly independent* (LI) if any linear combination, which is equal to zero has zero coefficients, i.e.

$$e_1, e_2, \ldots, e_m \text{ are LI } \Leftrightarrow \left(\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \ldots + \alpha_m e_m = 0 \quad \Rightarrow \alpha_i = 0 \; \forall i\right).$$

Note. The same definitions of linear dependence and linear independence can be introduced for columns.

Example. Let us consider matrix $A = \begin{pmatrix} -1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 9 & 12 \end{pmatrix}$. Here

$e_3 = \begin{pmatrix} 2 & 9 & 12 \end{pmatrix} = 2\begin{pmatrix} -1 & 2 & 3 \end{pmatrix} + \begin{pmatrix} 4 & 5 & 6 \end{pmatrix} = 2e_1 + e_2$. Thus, $2e_1 + e_2 - e_3 = 0$. We have linear combination of the rows equal to zero. But $\alpha_1 = 2 \neq 0$, $\alpha_2 = 1 \neq 0$, $\alpha_3 = -1 \neq 0$. It means that rows of this matrix are LD.

Theorem (Criterion of linear dependence for the rows) For rows of the matrix to be linearly dependent it is necessary and sufficient that one of them is linear combination of other rows.

Proof. Necessity. We know that rows of the matrix are linearly dependent. We should proof that one of them is linear combination of other rows. Suppose we have some zero linear combination of rows. From definition of linear dependence, we know that at least one coefficient is not equal to zero. Suppose it has number k, i.e. $\alpha_k \neq 0$. Let us divide this zero expression by $-\alpha_k$ and express the row e_k from the obtained equation:

$$\alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_k e_k + \ldots + \alpha_m e_m = 0 \Leftrightarrow$$

$$\Leftrightarrow -\frac{\alpha_1}{\alpha_k} e_1 - \frac{\alpha_2}{\alpha_k} e_2 - \ldots - e_k - \ldots - \frac{\alpha_m}{\alpha_k} e_m = 0$$

$$\Leftrightarrow e_k = -\frac{\alpha_1}{\alpha_k} e_1 - \frac{\alpha_2}{\alpha_k} e_2 - \ldots - \frac{\alpha_{k-1}}{\alpha_k} e_{k-1} - \frac{\alpha_{k+1}}{\alpha_k} e_{k+1} - \ldots - \frac{\alpha_m}{\alpha_k} e_m = 0 \Leftrightarrow$$

$$\Leftrightarrow e_k = -\sum_{\substack{i=1 \\ i \neq k}}^{m} \frac{\alpha_i}{\alpha_k} e_i .$$

Necessity is proven.

Sufficiency. Let $e_k = \sum_{\substack{i=1 \\ i \neq k}}^{m} \gamma_i e_i$. We should prove that rows are *linearly dependent*.

Let us put e_k to the right of the last equation. So we have $0 = \sum_{\substack{i=1 \\ i \neq k}}^{m} \gamma_i e_i - e_k$, i.e. zero linear combination of all rows with the coefficient $\gamma_k = -1 \neq 0$. From definition of linear dependence, it follows that the rows are LD. ***Theorem is proven.***

5.2 Rank of the Matrix

Definition. Minor of the *k*-th order M_k of the matrix *A* of the size *m* by *n* ($0 \le k \le \min(m,n)$) is the determinant consisting of the elements standing in the intersection of any *k* rows and any *k* columns of the matrix *A*.

Example. $A = \begin{pmatrix} 1 & 2 & 4 & 0 & -3 \\ 2 & 3 & 1 & -1 & 0 \\ -8 & 6 & 5 & 7 & 9 \end{pmatrix}$. The determinant $M_2 = \begin{vmatrix} 1 & 0 \\ -8 & 7 \end{vmatrix} = 37$ with elements from the first and the third rows and the first and the forth columns of *A* is one of the minors of the second order.

Definition. Rank of the matrix *A* is a maximum order of nonzero (nontrivial) minors of the matrix *A*. It is denoted as $rk(A)$ or $r(A)$.

Example. $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 9 \end{pmatrix}$. The biggest order of the existing minor is 3.

$M_3 = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -3 & -6 \end{vmatrix} = 0$, $M_2 = \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = 5 - 8 = -3 \neq 0$. Thus, $rk(A) = 2$.

Definition. Suppose $r = r(A)$. Nonzero minor of r^{th} order is called the basic minor and rows and columns of the matrix *A* composing this minor are called the basic rows and columns.

Note. Sometimes there are several basic minors in the matrix *A*.

Theorem (about basic minor) The following statements are valid:

(i) Basic rows (columns) are linearly independent;

(ii) Any row (column) of matrix A is a linear combination of basic rows (columns).

Proof. (i): Let us assume that basic rows are linearly dependent. It means that one of the rows in the basic minor is linear combination of other rows. From *property 9* of the determinants it follows that basic minor is equal to zero. We've got a contradiction with definition of the basic minor. Statement (i) is proven.

(ii): Without loss of generality we can assume that basic minor is situated in the upper left corner of the matrix A. So,

$$M_r = \begin{vmatrix} a_{11} & \dots & a_{1r} \\ \vdots & & \\ a_{r1} & \dots & a_{rr} \end{vmatrix} \neq 0, \text{ where } r = r(A).$$

Let us consider the following determinant obtained from M_r by adding the corresponding elements of k^{th} row and j^{th} column of A:

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{rr} & a_{1j} \\ a_{21} & a_{22} & \dots & a_{2r} & a_{2j} \\ \vdots & & & & \\ a_{r1} & a_{r2} & \dots & a_{rr} & a_{rj} \\ a_{k1} & a_{k2} & \dots & a_{kr} & a_{kj} \end{vmatrix}.$$

There are two situations:

1) $j \leq r$ or $k \leq r$. Then we have two identical rows or columns in the determinant, i.e. $\Delta = 0$.

2) $j > r$ and $k > r$. Then Δ is a minor of the $(r+1)^{th}$ order of A and equal to zero since r is maximum order of nonzero minors.

Thus, in any case $\Delta = 0$. Let us expand this determinant down the $(r+1)^{th}$ column:

$$0 = a_{1j}A_{1r+1} + a_{2j}A_{2r+1} + \dots + a_{rj}A_{1r+1} + a_{kj}A_{r+1r+1} = a_{1j}A_{1r+1} + \dots + a_{rj}A_{1r+1} + a_{kj}M_r.$$

Since $M_r \neq 0$, It follows that $0 = a_{1j}\dfrac{A_{1r+1}}{M_r} + \dots + a_{rj}\dfrac{A_{rr+1}}{M_r} + a_{kj} \Leftrightarrow$

$$\Leftrightarrow a_{kj} = -a_{1j}\frac{A_{1r+1}}{M_r} - \dots - a_{rj}\frac{A_{rr+1}}{M_r} = \gamma_1 a_{1j} + \gamma_2 a_{2j} + \dots \gamma_r a_{rj},$$

where coefficients γ_i depend on the elements of the k^{th} row and does not depend on the elements of the j^{th} column. Thus, each element of the k^{th} row is linear combination of the corresponding elements of basic rows, i.e. the k^{th} row is linear combination of basic rows.

In similar way we can prove these statements for the columns. ***Theorem is proven.***

Note. It follows from the theorem that the rank of the matrix is equal to the maximum number of linearly independent rows (columns) of this matrix.

5.3 Elementary Row/Column Operations

Definition. A matrix is in *row echelon form* if

- All nonzero rows, i.e. rows with at least one nonzero element are above any rows of all zeroes (all zero rows, if any, belong at the bottom of the matrix).
- The *leading coefficient* (entry), i.e. the first nonzero number from the left (also called the *pivot*) of a nonzero row is always strictly to the right of the leading coefficient of the row above it.
- All entries in a column below a leading entry are zeroes (implied by the first two criteria).

An example of a 3×5 matrix in row echelon form:

$$\begin{bmatrix} 1 & a_0 & a_1 & a_2 & a_3 \\ 0 & 0 & 2 & a_4 & a_5 \\ 0 & 0 & 0 & 3 & a_6 \end{bmatrix}$$

Note. A matrix is in column echelon form if its transpose is in row echelon form (see below).

$$\begin{bmatrix} 1 & 0 & 0 \\ a_0 & 0 & 0 \\ a_1 & 2 & 0 \\ a_2 & a_4 & 3 \\ a_3 & a_5 & a_6 \end{bmatrix}$$

Definition. *Elementary row operations and column operations* are a simple set of matrix operations that can be used to reduce a matrix to the upper (lower) triangle or a matrix in row (column) echelon form (trapezoidal form).

The next three actions with rows (columns) of a matrix are called elementary row (column) operations:

- interchange of any two rows (columns);
- multiplication of any row (column) by a nonzero number;

- addition of any row (column) multiplied by a nonzero number to another row (column).

Definition. The row operations, the column operations and the operation of matrix transposition are called together the *elementary matrix manipulations* (transformations).

Definition. If matrix B is obtained from matrix A by elementary matrix manipulations, then matrices A and B are called *equivalent* matrices and this relation of equivalence is denoted as A~B.

Note. It is obvious that matrix A can be obtained from B by the set of elementary manipulations which are inverse to initial manipulations applied to A to get B.

Theorem (about ranks of equivalent matrices) If matrices A and B are equivalent matrices then their ranks are *equal*.

Proof. Since all elementary matrix manipulations can not vanish the basic minor of matrix A according to determinant properties, then this determinant will be nonzero in the matrix B.

Let matrix B have the nonzero minor of the bigger order then order of basic minor in A. But it means that this determinant is nonzero in the matrix A, too. We got the contradiction with the definition of basic minor. ***Theorem is proven.***

Theorem (about the rank of matrix in row/column echelon form). The rank of matrix in the upper (lower) triangle or row (column) echelon forms is equal to the number of nonzero rows (columns) of this matrix.

Proof. Suppose the matrix has a row echelon form and the number of nonzero rows is equal to r. To proof this theorem, we should find the nonzero minor of the r-th order and to show that all minors of the bigger order are equal to zero.

Since there are only r nonzero rows then each minor of the bigger order if it exists has zero-row and thus it is equal to zero.

Let us consider the following determinant of the r^{th} order with elements from the first r nonzero rows where: the k^{th} column is the column consisting of the elements of the column of the first nonzero element of k^{th} row, k varies from 1 to r. At this choice of columns, we get the upper triangular determinant with nonzero elements on the main diagonal, i.e. nonzero determinant of the r^{th} order. It means that rank of the matrix is equal to r.

In the similar way this theorem can be proven for the matrices in the column echelon form. ***Theorem is proven.***

Corollary. Since the rank of the matrix does not change after elementary matrix manipulations we reduce a matrix to the *row echelon form* or *column echelon form*, because once this form is computed, it is easier to determine the rank.

Example. $A = \begin{pmatrix} 1 & -1 & 0 & 3 & 2 \\ 3 & -1 & 1 & 7 & 5 \\ -1 & 3 & 1 & -5 & -3 \end{pmatrix}$ ~ [we add the first row multiplied by (−3) to the second row and then add the same row multiplied by 1 to the third row]~

$$\sim \begin{pmatrix} 1 & -1 & 0 & 3 & 2 \\ 0 & 2 & 1 & -2 & -1 \\ 0 & 2 & 1 & -2 & -1 \end{pmatrix} \sim$$

~[we add to the third row the second one multiplied by (−1)]~

$$\sim \begin{pmatrix} 1 & -1 & 0 & 3 & 2 \\ 0 & 2 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We used only elementary row operations to get matrix in the row echelon form. Since there are two nonzero rows the rank of this matrix is equal to 2, $rk(A) = 2$.

In this case the minor $\begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix}$, for example, can be chosen as basic minor.

5.4 The Theorem by Kronecker-Kapelly

Theorem (Theorem by Kronecker-Kapelly) In order to SLAE be consistent it is necessary and sufficient for the ranks of coefficient *A* and augmented A^* matrices to be equal, i.e. $rk(A) = rk(A^*)$.

Proof.

<u>Necessity</u>: Let's the SLAE have a solution $x_1 = \alpha_1, x_2 = \alpha_2, \ldots, x_n = \alpha_n$, then we can write

$$t_1\alpha_1 + t_2\alpha_2 + t_3\alpha_3 + \ldots + t_n\alpha_n = B,$$

i.e. *B* which is the last column of A^* is linear combination of the other columns of A^*. It means that *B* does not increase the number of linear independent columns of A^* with respect to *A*, so $rk(A) = rk(A^*)$.

Sufficiency: Let's $rk(A) = rk(A^*) = r$. It means that basic minor of A can be chosen as basic minor of A^*. But from the theorem about basic rows and columns it means that B is a linear combination of the basic columns, i.e. of some columns of A:

$$t_{i_1}\alpha_{i_1} + t_{i_2}\alpha_{i_2} + \ldots + t_{i_r}\alpha_{i_r} = B.$$

Let us complete the sum from the left side of expression to full sum of columns by missing columns multiplied by zeros. Then according to the definition of the solution the coefficients of the obtained sum are solution of the system and the system is consistent. ***Theorem is proven.***

Note. It is simple to prove by means of the rule by Cramer that:

- If $rk(A) = rk(A^*) = n$ then system is definite;
- If $rk(A) = rk(A^*) < n$ then system is indefinite.

In general, to investigate a SLAE on consistence and to find solutions for the consistent system we have to follow the next plan described as the diagram

<table>
<tr><td colspan="3" align="center">System Linear Algebraic Equations (SLAE)
with
m equations
n unknowns</td></tr>
<tr><td colspan="2" align="center">If $rk(A) = rk(A^*)$ then
SLAE is consistent</td><td align="center">If $rk(A) \neq rk(A^*)$ then
SLAE is inconsistent</td></tr>
<tr><td align="center">If $rk(A) = n$ then
SLAE is definite</td><td align="center">If $rk(A) < n$ then
SLAE is indefinite</td><td></td></tr>
</table>

Let us demonstrate this on the next example.

Example 1. Let us solve the system

$$\begin{cases} x_1 - 2x_2 + 3x_3 - x_4 = 4 \\ -2x_1 + 4x_2 - x_3 + 2x_4 = -3 \end{cases}.$$

Since the number of unknowns is greater than the number of equations then $rg(A) < n$ and if there are any solutions then the system is indefinite. Let us write down the augmented matrix of the system and, then, find out its rank

$$A^* = \left(\begin{array}{cccc|c} 1 & -2 & 3 & -1 & 4 \\ -2 & 4 & -1 & 2 & -3 \end{array}\right), \begin{vmatrix} 1 & -2 \\ -2 & 4 \end{vmatrix} = 4 - 4 = 0, \text{ but } \begin{vmatrix} 1 & 3 \\ -2 & -1 \end{vmatrix} = -1 + 6 = 5 \neq 0.$$

Thus $rk(A) = rk(A^*) = 2 < n = 4$ and system is consistent and indefinite.

Let us rewrite the system by leaving to the left only the unknowns corresponding to *basic columns*:

$$\begin{cases} x_1 + 3x_3 = 4 + 2x_2 + x_4 \\ -2x_1 - x_3 = -3 - 4x_2 - 2x_4 \end{cases}.$$

Since the determinant of the obtained system for variables x_1, x_3 is not equal to zero it can be solved by rule by Cramer.

$$x_1 = \frac{\begin{vmatrix} 4 + 2x_2 + x_4 & 3 \\ -3 - 4x_2 - 2x_4 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ -2 & -1 \end{vmatrix}} = \frac{-4 - 2x_2 - x_4 - 3(-3 - 4x_2 - 2x_4)}{5} = 1 + 2x_2 + x_4,$$

$$x_3 = \frac{\begin{vmatrix} 1 & 4 + 2x_2 + x_4 \\ -2 & -3 - 4x_2 - 2x_4 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ -2 & -1 \end{vmatrix}} = \frac{-3 - 4x_2 - 2x_4 + 2(4 + 2x_2 + x_4)}{5} = 1,$$ where x_2, x_4 are arbitrary.

Finally, we can write the solution of the SLAE as follows:

$$X^T = (1 + 2x_2 + x_4, x_2, 1, x_4)$$

Note that the basic unknowns were expressed through the arbitrary ones. Thereby, by assigning any values to x_2, x_4 (arbitrary unknowns) we get a lot of particular solutions of this system.

For instance, let's the arbitrary unknowns be $x_2 = -2, x_4 = 5$, then we have one of infinite number of solutions as follows:

$$\tilde{X}^T = (2, -2, 1, 5)$$

It follows from the mentioned above solution of the SLAE that next definitions can be introduced:

Definition 1. *Basic (or main) unknowns* are called unknowns corresponding to basic columns of the augmented matrix of the SLAE, i.e. the determinate for coefficients of these unknowns is not equal to zero. Otherwise, any other unknowns of the SLAE are called *arbitrary (or free) unknowns*.

Definition 2. The solution of the indefinite system written as a function of some arbitrary values is called *a general solution* of the system.

Definition 3. Any solution calculated from the general solution by substituting some certain values instead of arbitrary unknowns is called *a particular solution*.

5.5 Homogeneous Systems. Construction of the Fundamental System of Solutions

Let us consider the homogeneous system of m linear algebraic equations with n unknown variables

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = 0 \\ \ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = 0 \end{cases}$$

or

$$AX = 0$$

or

$$t_1x_1 + t_2x_2 + \ldots + t_nx_n = 0$$

Since $B = 0$ in the homogeneous system (HS) and zero column does not increase the number of linear independent columns in the augmented matrix with respect to matrix of the system, *the homogeneous system is always consistent.*

Actually, it is obvious, since the homogeneous system always has a zero (trivial) solution. The question is when does it have nontrivial solution?

Theorem. For the homogeneous system to have nontrivial solution it is necessary and sufficient that $rk(A) < n$.

Proof. Necessity: If we have nontrivial solution then

$$t_1\alpha_1 + t_2\alpha_2 + t_3\alpha_3 + \ldots + t_n\alpha_n = 0,\ |\alpha_1| + |\alpha_2| + \ldots + |\alpha_n| \neq 0.$$

But it means that columns of the matrix A are linear dependent so $rk(A) \neq n$ and thus $rk(A) < n$.

Sufficiency: If $rk(A) < n$ then n columns of the matrix A are linear dependent and there is a set of numbers such that

$$|\alpha_1| + |\alpha_2| + \ldots + |\alpha_n| \neq 0 \text{ and } t_1\alpha_1 + t_2\alpha_2 + t_3\alpha_3 + \ldots + t_n\alpha_n = 0.$$

It means that this set of numbers is a nontrivial solution of the system. ***Theorem is proven.***

Note. It follows from the theorem, that for the homogeneous system of n equations with n variables to have nontrivial solution it is necessary and sufficient that the determinant of the system matrix is equal to zero, i.e. the homogeneous system with square matrix is indefinite if and only if $\det(A)=0$.

So, if $rk(A)<n$ then the system $AX=0$ is indefinite and has infinite number of solutions. But how many of them are linearly independent?

Note 1. When we say about the linear dependence of solutions we consider solutions as columns $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ and investigate linear dependence of columns.

Note 2. Linear combination of the solutions of homogeneous system is also a solution of this system. Indeed, suppose Y_1, Y_2 are the solutions of the system $AX=0$, i.e. $AY_1=0$, $AY_2=0$. Then

$$A(\alpha Y_1+\beta Y_2)=A(\alpha Y_1)+A(\beta Y_2)=\alpha AY_1+\beta AY_2=\alpha 0+\beta 0=0,$$

i.e. $\alpha Y_1+\beta Y_2$ is also a solution.

Definition. *Fundamental system of solutions* (FSS) of the homogeneous system is any maximum set of linearly independent solutions.

Note. It follows from the definition that:

1) Only indefinite homogeneous systems have FSS.
2) Choice of the FSS is not unique.

Theorem (About Fundamental System of Solutions)

(i) If $r=rk(A)<n$ then the homogeneous system has a fundamental system of (n-r) solutions;

(ii) Any solution of the system is a linear combination of solutions following from FSS.

Proof. Suppose the basic minor stands in the upper left corner of the matrix A. Then the first r rows are linearly independent and all other rows (equations) are linear combination of the basic rows and, thus, do not contain helpful information to find a solution. So let us consider only the first r rows written in the following form:

$$\begin{pmatrix} a_{11} \\ \vdots \\ a_{r1} \end{pmatrix} x_1 + \begin{pmatrix} a_{12} \\ \vdots \\ a_{r2} \end{pmatrix} x_2 + \ldots + \begin{pmatrix} a_{1r} \\ \vdots \\ a_{rr} \end{pmatrix} x_r = -\begin{pmatrix} a_{1r+1} \\ \vdots \\ a_{rr+1} \end{pmatrix} x_{r+1} - \ldots - \begin{pmatrix} a_{1n} \\ \vdots \\ a_{rn} \end{pmatrix} x_n .$$

The determinant of the obtained system for the unknowns $x_1, x_2, \dots, x_r$ is not equal to zero, i.e. it is basic minor, and we can find values of the unknowns $x_1, x_2, \dots, x_r$ as functions of other unknowns by means of rule by Cramer. In this case substituting instead of unknowns $x_{r+1}, x_{r+2}, \dots, x_n$ some values, we get particular solutions of the initial system. Let us consider the following set of (*n*-*r*) particular solutions:

$$\begin{matrix} x_{r+1}=1 \\ x_{r+2}=0 \\ x_{r+3}=0 \\ \vdots \\ x_n=0 \end{matrix} \Rightarrow X_1 = \begin{pmatrix} \alpha_{11} \\ \alpha_{12} \\ \vdots \\ \alpha_{1r} \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \begin{matrix} x_{r+1}=0 \\ x_{r+2}=1 \\ x_{r+3}=0 \\ \vdots \\ x_n=0 \end{matrix} \Rightarrow X_2 = \begin{pmatrix} \alpha_{21} \\ \alpha_{22} \\ \vdots \\ \alpha_{2r} \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \begin{matrix} x_{r+1}=0 \\ x_{r+2}=0 \\ x_{r+3}=1 \\ \vdots \\ x_n=0 \end{matrix} \Rightarrow X_3 = \begin{pmatrix} \alpha_{31} \\ \alpha_{32} \\ \vdots \\ \alpha_{3r} \\ 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{matrix} x_{r+1}=0 \\ x_{r+2}=0 \\ x_{r+3}=0 \\ \vdots \\ x_n=1 \end{matrix} \Rightarrow X_{n-r} = \begin{pmatrix} \alpha_{n-r1} \\ \alpha_{n-r2} \\ \vdots \\ \alpha_{n-rr} \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

The matrix of the order *n* by (*n*-*r*) constructed on these columns has rank equal to (*n*-*r*) since there is unit matrix of the (*n*-*r*)th order in the bottom of it. It means that all these columns (solutions) are linearly independent.

Let us consider now an arbitrary solution of the system $X_0 = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}$. Then

$$Y = X_0 - q_{r+1}X_1 - q_{r+2}X_2 - \dots - q_nX_{n-r} = = \begin{pmatrix} q_1 - q_{r+1}\alpha_{11} - q_{r+2}\alpha_{21} - q_{r+3}\alpha_{31} - \dots - q_n\alpha_{n-r1} \\ q_2 - q_{r+1}\alpha_{12} - q_{r+2}\alpha_{22} - q_{r+3}\alpha_{32} - \dots - q_n\alpha_{n-r2} \\ \vdots \\ q_r - q_{r+1}\alpha_{1r} - q_{r+2}\alpha_{2r} - q_{r+3}\alpha_{3r} - \dots - q_n\alpha_{n-rr} \\ q_{r+1} - q_{r+1}\cdot 1 - q_{r+2}\cdot 0 - q_{r+3}\cdot 0 - \dots - q_n\cdot 0 \\ q_{r+2} - q_{r+1}\cdot 0 - q_{r+2}\cdot 1 - q_{r+3}\cdot 0 - \dots - q_n\cdot 0 \\ q_{r+3} - q_{r+1}\cdot 0 - q_{r+2}\cdot 0 - q_{r+3}\cdot 1 - \dots - q_n\cdot 0 \\ \vdots \\ q_n - q_{r+1}\cdot 0 - q_{r+2}\cdot 0 - q_{r+3}\cdot 0 - \dots - q_n\cdot 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \gamma_1 & \gamma_2 & \dots & \gamma_r & 0 & \dots & 0 \end{pmatrix}^T$$

is a solution as well, and thus

$$\gamma_1 t_1 + \gamma_2 t_2 + \dots + \gamma_r t_r + 0t_{r+1} + \dots + 0t_n = \gamma_1 t_1 + \gamma_2 t_2 + \dots + \gamma_r t_r = 0.$$

Since we obtained zero linear combination of the basic linearly independent columns then $\gamma_1 = 0, \gamma_2 = 0, \dots, \gamma_r = 0$, i.e.

$$Y = X_0 - q_{r+1}X_1 - q_{r+2}X_2 - \dots - q_nX_{n-r} = 0 \quad \text{and} \quad X_0 = q_{r+1}X_1 + q_{r+2}X_2 + \dots + q_nX_{n-r}.$$

It means that any other solution is linear combination of X_1, X_2, …X_{n-r} and can not increase number of linearly independent columns. Thus X_1, X_2, …X_{n-r} form FSS and any solution is a a linear combination of solutions from FSS.

Theorem is proven.

Theorem (about general solution of inhomogeneous system). General solution of the inhomogeneous system $AX = B$ is a sum of the particular solution of the inhomogeneous system and linear combination of solutions from the FSS of homogeneous system $AX = 0$.

Proof. Suppose X is an arbitrary solution and X_0 is some particular solution of the system $AX = B$.

Then $AY = A(X - X_0) = AX - AX_0 = B - B = 0$ and $Y = X - X_0$ is the solution of the homogeneous system and thus equal to the linear combination of solutions of the FSS.

Thus, $X = Y + X_0$ is a sum of the particular solution of the inhomogeneous system and linear combination of the FSS. ***Theorem is proven.***