Lecture #6: Method by Gauss (or Method of Sequential Elimination of the Unknown Variables). Method by Jordan-Gauss

6.1 Method by Gauss

Method by Gauss is used to solve the system of the linear algebraic equations with arbitrary numbers of equations and unknowns. It includes sequential elimination of the variables from equations (i.e. vanishing of their corresponding coefficients in the equations) according to the following scheme:

Step 1. Write a system of linear equations as an augmented (extended) matrix:

- Each equation in the system becomes a row.
- Each variable in the system becomes a column.

Note. The variables are dropped and the coefficients are placed into a matrix. If the right hand side is included, it's called an augmented matrix. If the right hand side isn't included, it's called a coefficient matrix.

Step 2. Perform the elementary row operations to put the matrix into row-echelon form:

- Choose the leading equation and the leading variable, i.e. its coefficient in the leading equation has to be nonzero).
- Put the row corresponding to this equation on the first place in the extended matrix.
- Eliminate the leading variable from the other rows below the leading one (i.e. from other equations) by the elementary row operations.

Note. Repeat *Step 2* as much as need to obtain the matrix with columns rewritten in the order of the chosen leading variables and this matrix will be in row-echelon form.

Step 3. Determine the ranks of the system matrix A and the augmented matrix A^* and convert the matrix back into a system of linear equations:

- Each row in the matrix becomes an equation in the system.
- Each column in the matrix becomes a variable in the system.

Step 4. Use back substitution to obtain all the answers. There are three types of solutions which are possible when solving the obtained system of linear equations:

- If $rk(A) \neq rk(A^*)$, then the system has no solution – inconsistent (incompatible);
- If $rk(A) = rk(A^*) = n$, where n is a number of unknowns, then the system has only one solution – definite (independent).
- If $rk(A) = rk(A^*) < n$, then choose basic minor from the obtained augmented matrix, for example, consisting of the columns of the leading variables. Herewith, the unknowns, whose coefficients correspond to the basic minor are *basic (or main) variables*. All other are *free (or independent) variables*. Thereafter, solve the system expressing the main variables through the free ones. This is a *general solution* of the system – infinite number of solutions depending on the free unknowns. Assigning any values to free variables, a *particular solution* of the system will be got.

Note. In order to find the solution of the SLAE in *Step 4* we need to start from the last equation.

Example. Let us solve the system

$$\begin{cases} x_1 - 2x_2 + 3x_3 - x_4 = 4 \\ -2x_1 + 4x_2 - x_3 + 2x_4 = -3 \\ -x_1 + 2x_2 + 2x_3 + x_4 = 1 \end{cases}.$$

We will write down the extended matrix of the system and by means of elementary row operations reduce this matrix in column echelon form (a trapezoidal form). Since we work only with rows what is equivalent to elementary operations (summarizing, adding, subtracting, multiplying by nonzero numbers, changing of the order) on equations, the system stays the same.

$$A^* = \left(\begin{array}{cccc|c} 1 & -2 & 3 & -1 & 4 \\ -2 & 4 & -1 & 2 & -3 \\ -1 & 2 & 2 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & -2 & 3 & -1 & 4 \\ 0 & 0 & 5 & 0 & 5 \\ 0 & 0 & 5 & 0 & 5 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & -2 & 3 & -1 & 4 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Since $rk(A) = rk(A^*) = 2 < n = 4$ the system is consistent and indefinite.

It is appeared that the third equation in the initial system is linear combination of others equations. So to find solution it is enough to consider only the first two equations what was done in the previous Example 1.

To check the result, we write down the system corresponding to the last augmented matrix and compare solutions:

$$\begin{cases} x_1 - 2x_2 + 3x_3 - x_4 = 4 \\ 0x_1 + 0x_2 + 1x_3 + 0x_4 = 1 \end{cases} \Leftrightarrow \begin{cases} x_1 - 2x_2 + 3x_3 - x_4 = 4 \\ x_3 = 1 \end{cases}$$

[since the third unknown is defined from the second equation, it is a main unknown, then for the other main unknown one can choose any from the first equation such that the determinant of coefficients at the main unknowns to be nonzero, for instance, the first unknown, i.e. one can check $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$]. Then, find the main unknown

$\Leftrightarrow \begin{cases} x_1 = 4 + 2x_2 - 3x_3 + x_4 \\ x_3 = 1 \end{cases} \Leftrightarrow \begin{cases} x_1 = 1 + 2x_2 + x_4 \\ x_3 = 1 \end{cases}$, x_2, x_4 are arbitrary. So, answer is

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 = 2x_2 + x_4 + 1 \\ x_2 \ \ is \ \ free \\ x_3 = 1 \\ x_4 \ \ is \ \ free \end{pmatrix}, \text{ where } x_2 \ and \ x_4 \in R$$

6.2 Method by Jordan-Gauss

A modification of the method by Gauss where the leading variable is eliminated not only from the below rows (equations) but from all other rows (equations) is called *the method by Jordan-Gauss*.

Definition. A matrix is in *reduced row-echelon form* when all of the conditions of row-echelon form are met and leading elements equal to one and all elements above, as well as below, the leading ones are zero, i.e.

1. If there is a row of all zeros, then it is at the bottom of the matrix.
2. The first non-zero element of any row is a one. That element is called the leading one.
3. The leading one of any row is to the right of the leading one of the previous row.
4. All elements above and below a leading one are zero.

Example 3. Let us solve the homogeneous system of equation. Since the difference between the matrix of the system and extended matrix is in zero column we will work only with matrix of the system.

$$\begin{cases} x_1 + x_2 - x_3 - x_4 + x_5 = 0 \\ 2x_1 - 3x_2 - x_3 + 3x_4 - x_5 = 0 \\ x_1 - x_2 - x_3 + x_4 = 0 \end{cases}$$

$$rk(A)=\begin{pmatrix}1 & 1 & -1 & -1 & 1\\ 2 & -3 & -1 & 3 & -1\\ 1 & -1 & -1 & 1 & 0\end{pmatrix}\sim\begin{pmatrix}1 & 1 & -1 & -1 & 1\\ 0 & -5 & 1 & 5 & -3\\ 0 & -2 & 0 & 2 & -1\end{pmatrix}\sim$$

~[Add to the second row the third one multiplied by (–3)] ~

$$\sim\begin{pmatrix}1 & 1 & -1 & -1 & 1\\ 0 & 1 & 1 & -1 & 0\\ 0 & -2 & 0 & 2 & -1\end{pmatrix}\sim\begin{pmatrix}1 & 0 & -2 & 0 & 1\\ 0 & 1 & 1 & -1 & 0\\ 0 & 0 & 2 & 0 & -1\end{pmatrix}\sim$$

~[Add to the first row the third one multiplied by 1] ~

$$\sim\begin{pmatrix}1 & 0 & 0 & 0 & 0\\ 0 & 1 & 1 & -1 & 0\\ 0 & 0 & 2 & 0 & -1\end{pmatrix}\sim\begin{pmatrix}1 & 0 & 0 & 0 & 0\\ 0 & 1 & 1 & -1 & 0\\ 0 & 0 & -2 & 0 & 1\end{pmatrix}.$$

[One can see that the first, second and fifth columns composite a nonzero determinant], i.e. $\begin{vmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{vmatrix}=1\neq 0$.

So one can say that the leading variables x_1, x_2, x_5 are main and x_3, x_4 are free.

A new system is

$$\begin{cases}x_1=0\\ x_2+x_3-x_4=0\\ -2x_3+x_5=0\end{cases}\Leftrightarrow\begin{cases}x_1=0\\ x_2=-x_3+x_4\\ x_5=2x_3\end{cases}$$

and the general solution is

$$X=\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\end{pmatrix}=\begin{pmatrix}0\\ -x_3+x_4\\ x_3\\ x_4\\ 2x_3\end{pmatrix},\text{ where }\quad x_3, x_4\in R$$