Lecture #10: Equations of Plane in Space

10.1 General Equation of Plane in Space

Theorem (about general equation of plane) Suppose x, y, z are the coordinates of a point in the Cartesian coordinate system. Any linear equation $Ax + By + Cz + D = 0$, where $A^2 + B^2 + C^2 \neq 0$, is an equation of plane in space.

Proof. Suppose coordinates of point (x_0, y_0, z_0) satisfy the equation $Ax + By + Cz + D = 0$, and denote the coordinates of any other point satisfying this equation by (x, y, z). then

$$Ax + By + Cz + D = 0, \qquad (*)$$

$$Ax_0 + By_0 + Cz_0 + D = 0. \qquad (**)$$

After subtraction of equation (*) from equation (**) we have

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0. \qquad (***)$$

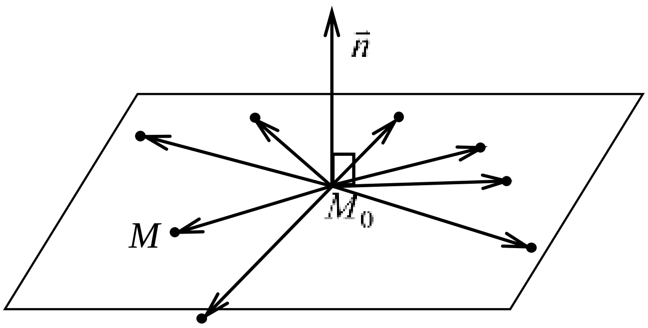

Figure 1

Equation (***) can be considered as zero scalar product of vector $\vec{n}(A, B, C)$ and vector $\overrightarrow{M_0M}(x - x_0, y - y_0, z - z_0)$. It means that for any point $M(x,y,z)$ with coordinates satisfying the equation (*) the vector $\overline{M_0M} \perp \vec{n}$, i.e. all point satisfying this linear equation belong to the plane perpendicular to the vector $\vec{n}(A, B, C)$ (Fig.1). Moreover, from (**) we have that $D = -Ax_0 - By_0 - Cz_0$, where (x_0, y_0, z_0) satisfies (*).

From the other side the opposite statement is valid as well, i.e. any point $M(x,y,z)$ of the plane satisfies the equation (*). Indeed, two points of this plane M and M_0 form vector in plane perpendicular to the vector $\vec{n}(A, B, C)$. So,

$$0 = A(x - x_0) + B(y - y_0) + C(z - z_0) = Ax + By + Cz - Ax_0 - By_0 - Cz_0 =$$
$$= Ax + By + Cz + D,$$

where $D = -Ax_0 - By_0 - Cz_0$. ***Theorem is proven.***

Definition. Vector $\vec{n}(A, B, C)$ is called the normal vector of plane.

Vector $\vec{n}$ gives *an orientation* of the plane.
To describe some certain plane we have to determine also *a location* which can be given by any point of this plane (Fig.2).

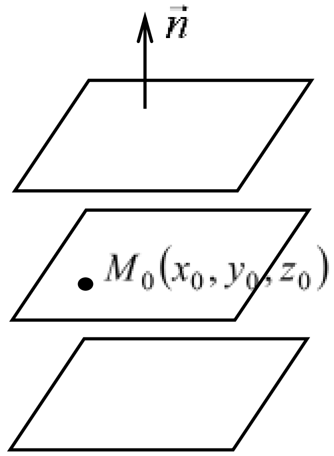

Figure 2

Definition. Equation

$$Ax + By + Cz + D = 0$$

is called *the general equation of the plane*.

Definition. Equation

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

is called the equation of the plane with given normal vector $\vec{n}(A, B, C)$ and a point of the plane $M_0(x_0, y_0, z_0)$.

Example. Let us find an equation of the plane perpendicular to the axis Oz and passing through the point $M_0(1;-2;3)$. Since this plane is perpendicular to the axis Oz It is perpendicular to the vector $\vec{k}(0,0,1)$ and this vector can be chosen as a normal vector of the plane. Therefore, $\vec{n}(A, B, C) = (0,0,1)$, $M_0(x_0, y_0, z_0) = (1;-2;3)$ and the equation of this plane looks like

$$0(x-1) + 0(y-(-2)) + 1(z-3) = 0 \Leftrightarrow z - 3 = 0 \Leftrightarrow z = 3.$$

10.2. Equation of Plane with Given Intercepts

Suppose $A \cdot B \cdot C \cdot D \neq 0$. Let us divide the general equation of the plane by $-D$. Then

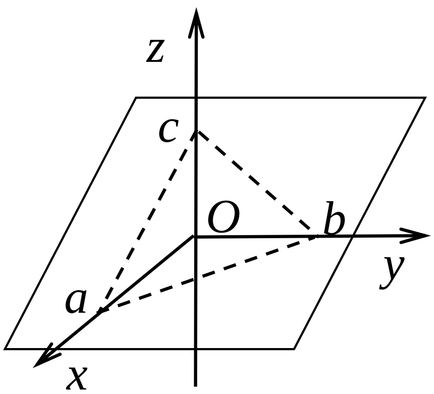

Figure 3

$$\frac{Ax}{-D} + \frac{By}{-D} + \frac{Cz}{-D} = 1$$

or

$$\frac{x}{-\frac{D}{A}} + \frac{y}{-\frac{D}{B}} + \frac{z}{-\frac{D}{C}} = 1$$

or

$$\boxed{\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1},$$

where $a=-\frac{D}{A}$, $b=-\frac{D}{B}$, $c=-\frac{D}{C}$ are the segments cut from the semi-axes of axes Ox,Oy,Oz or the intercepts (Fig.3).

The last equation is called the equation of plane with the given intercepts.

Example. Let us find an equation of the plane with equal intercepts and passing through the point $M_0(1;-2;3)$. Since the intercepts are equal the equation has a form:

$$\frac{x}{a}+\frac{y}{a}+\frac{z}{a}=1.$$

Since the point $M_0(1;-2;3)$ belongs to this plane the coordinates of this point satisfy the equation of the plane and therefore

$$\frac{1}{a}+\frac{-2}{a}+\frac{3}{a}=1 \Leftrightarrow \frac{2}{a}=1 \Leftrightarrow a=2.$$

Finally we obtain the equation

$$\frac{x}{2}+\frac{y}{2}+\frac{z}{2}=1 \quad \text{or} \quad x+y+z-2=0.$$

10.3. Angle Between Two Planes. Parallel and Perpendicular Planes

Definition. An angle between two planes is the angle between their normal vectors (Fig.4).

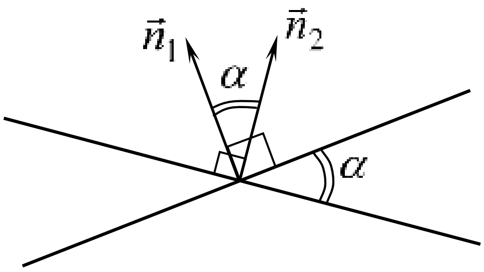

Figure 4

From definition we have:

$$\cos\alpha=\frac{(\bar{n}_1,\bar{n}_2)}{|\bar{n}_1||\bar{n}_2|} \Leftrightarrow \alpha=\arccos\frac{(\bar{n}_1,\bar{n}_2)}{|\bar{n}_1||\bar{n}_2|},$$

where $\bar{n}_1(A_1,B_1,C_1)$ and $\bar{n}_2(A_2,B_2,C_2)$ are the normal vectors of the planes

plane 1: $A_1x+B_1y+C_1z+D_1=0$,

plane 2: $A_2x+B_2y+C_2z+D_2=0$.

Therefore, conditions of parallel and perpendicular planes look like:

Plane 1 $\parallel$ Plane 2 $\Leftrightarrow \bar{n}_1 \| \bar{n}_2 \Leftrightarrow \bar{n}_1\times\bar{n}_2=0 \Leftrightarrow \frac{A_1}{A_2}=\frac{B_1}{B_2}=\frac{C_1}{C_2}$;

Plane 1 $\perp$ Plane 2 $\Leftrightarrow \bar{n}_1\perp\bar{n}_2 \Leftrightarrow (\bar{n}_1,\bar{n}_2)=0 \Leftrightarrow A_1A_2+B_1B_2+C_1C_2=0$.

Example 1. Find the value α such that the following planes are perpendicular: $\alpha x + y - 3z + 1 = 0$, $x + 5z - 19 = 0$. Since these planes are perpendicular then the scalar product of their normal vectors $\vec{n}_1(\alpha, 1, -3)$ and $\vec{n}_2(1,0,5)$ is equal to zero and we have

$$(\bar{n}_1, \bar{n}_2) = 0 = \alpha + 0 - 15 = \alpha - 15 \Leftrightarrow \alpha = 15.$$

Example 2. Find the values α and β such that two planes $\alpha x + y - 3z + 1 = 0$, $x - y + \beta z - 19 = 0$ are parallel. Since these planes are parallel then the coordinates of their normal vectors $\vec{n}_1(\alpha, 1, -3)$ and $\vec{n}_2(1, -1, \beta)$ are proportional and we have

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} \Leftrightarrow \frac{\alpha}{1} = \frac{1}{-1} = \frac{-3}{\beta} \Leftrightarrow \begin{cases} \frac{\alpha}{1} = -1 \\ \frac{-3}{\beta} = -1 \end{cases} \Leftrightarrow \begin{cases} \alpha = -1 \\ \beta = 3 \end{cases}$$

Example 3. Find the angle between the planes $x + y - 3z + 1 = 0$, $x - y + z - 19 = 0$. Here $\vec{n}_1(1,1,-3)$ and $\vec{n}_2(1,-1,1)$. Thus

$$\alpha = \arccos\frac{(\bar{n}_1, \bar{n}_2)}{|\bar{n}_1||\bar{n}_2|} = \arccos\frac{1 - 1 - 3}{\sqrt{1^2 + 1^2 + (-3)^2}\sqrt{1^2 + (-1)^2 + 1^2}} = \arccos\frac{-3}{\sqrt{11}\sqrt{3}} =$$

$$= \arccos\left(-\sqrt{\frac{3}{11}}\right) = \pi - \arccos\sqrt{\frac{3}{11}}.$$

10.4. Distance from Point to Plane

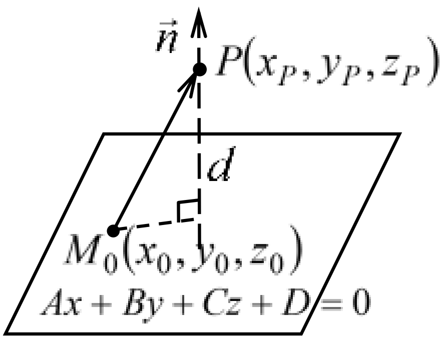

Figure 5

Let us find the distance from the point $P(x_P, y_P, z_P)$ to the plane $Ax + By + Cz + D = 0$.

Suppose $M_0(x_0, y_0, z_0)$ belongs to this plane. Then

$Ax_0 + By_0 + Cz_0 + D = 0$ or $D = -Ax_0 - By_0 - Cz_0$.

Distance from the point P to the plane can be found as (Fig.5)

$$d = \left|pr_{\vec{n}}\overline{M_0P}\right| = \left|\frac{(\bar{n}, \overline{M_0P})}{|\bar{n}|}\right| =$$

$$= \frac{|A(x_P - x_0) + B(y_P - y_0) + C(z_P - z_0)|}{\sqrt{A^2 + B^2 + C^2}} =$$

$$= \frac{|Ax_P + By_P + Cz_P - Ax_0 - By_0 - Cz_0|}{\sqrt{A^2 + B^2 + C^2}} = \frac{|Ax_P + By_P + Cz_P + D|}{\sqrt{A^2 + B^2 + C^2}}.$$

Thus

$$d = \frac{|Ax_P + By_P + Cz_P + D|}{\sqrt{A^2 + B^2 + C^2}}.$$

Example 1. Suppose $x_P = y_P = z_P = 0$. Then the distance from the origin to the plane is equal to

$$d = d_0 = \frac{|D|}{\sqrt{A^2 + B^2 + C^2}}.$$

Example 2. Find the distance from the point $P(1;-2;3)$ to the plane $x + 2y - 2z + 5 = 0$. By formula we have

$$d = \frac{|Ax_P + By_P + Cz_P + D|}{\sqrt{A^2 + B^2 + C^2}} = \frac{|1 + 2(-2) - 2 \cdot 3 + 5|}{\sqrt{1^2 + 2^2 + (-2)^2}} = \frac{|-4|}{\sqrt{9}} = \frac{4}{3}.$$

10.5. Normal Equation of Plane

Let us consider the general equation of the plane $Ax + By + Cz + D = 0$.

After division of the plane equation by $\sqrt{A^2 + B^2 + C^2}$ and renaming the coefficients we obtain the following:

$$\underbrace{\frac{A}{\sqrt{A^2 + B^2 + C^2}}}_{\cos\alpha} x + \underbrace{\frac{B}{\sqrt{A^2 + B^2 + C^2}}}_{\cos\beta} y + \underbrace{\frac{C}{\sqrt{A^2 + B^2 + C^2}}}_{\cos\gamma} z + \underbrace{\frac{D}{\sqrt{A^2 + B^2 + C^2}}}_{p} = 0$$

or

$$\cos\alpha \cdot x + \cos\beta \cdot y + \cos\gamma \cdot z + p = 0.$$

Here $\left(\frac{A}{\sqrt{A^2 + B^2 + C^2}}, \frac{B}{\sqrt{A^2 + B^2 + C^2}}, \frac{C}{\sqrt{A^2 + B^2 + C^2}} \right) = (\cos\alpha, \cos\beta, \cos\gamma)$ is the ort of the normal vector $\vec{n}^0$;

$|p| = \frac{|D|}{\sqrt{A^2+B^2+C^2}} = d_0$ is distance from the origin to the plane. So, p is a distance taken with the sign plus or minus depending on the sign of the coefficient D.

The obtained equation $\cos\alpha \cdot x + \cos\beta \cdot y + \cos\gamma \cdot z + p = 0$ is called *the normal equation of the plane*.

Example. Find the equations of the planes with distance from the plane $x + y - 3z + 1 = 0$ equal to $3\sqrt{11}$. These planes are parallel and therefore they have the same normal vector $\vec{n}(1,1,-3)$. Let us consider the normal equations of these three planes.

$|\vec{n}| = \sqrt{1+1+9} = \sqrt{11}$;

$$\frac{1}{\sqrt{11}}x + \frac{1}{\sqrt{11}}y - \frac{3}{\sqrt{11}}z + \frac{1}{\sqrt{11}} = 0;$$

$$\frac{1}{\sqrt{11}}x + \frac{1}{\sqrt{11}}y - \frac{3}{\sqrt{11}}z + p_1 = 0;$$

$$\frac{1}{\sqrt{11}}x + \frac{1}{\sqrt{11}}y - \frac{3}{\sqrt{11}}z + p_2 = 0.$$

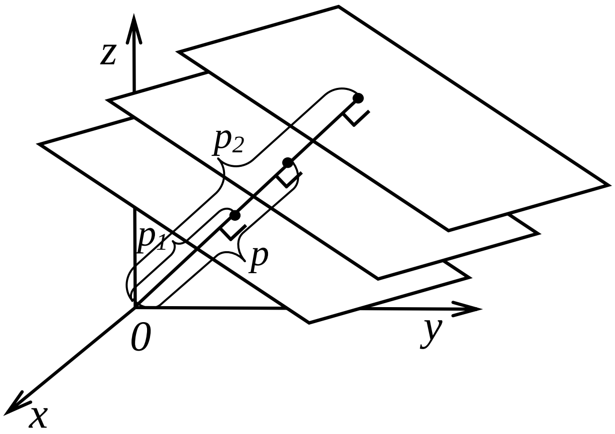

Figure 5

So, for the initial plane

$$p = \frac{1}{\sqrt{11}}.$$

From the condition and Fig.5 It follows that

$$p_1 = p - 3\sqrt{11} = \frac{1}{\sqrt{11}} - \frac{33}{\sqrt{11}} = -\frac{32}{\sqrt{11}},$$

$$p_2 = p + 3\sqrt{11} = \frac{1}{\sqrt{11}} + \frac{33}{\sqrt{11}} = \frac{34}{\sqrt{11}}$$

and therefore the asked equations are

$$\frac{1}{\sqrt{11}}x + \frac{1}{\sqrt{11}}y - \frac{3}{\sqrt{11}}z - \frac{32}{\sqrt{11}} = 0 \Leftrightarrow x + y - 3z - 32 = 0;$$

$$\frac{1}{\sqrt{11}}x + \frac{1}{\sqrt{11}}y - \frac{3}{\sqrt{11}}z + \frac{34}{\sqrt{11}} = 0 \Leftrightarrow x + y - 3z + 34.$$

10.6. Three Particular Cases for Plane Equations

Case 1 Suppose we know one point $M_0(x_0, y_0, z_0)$ of the plane and any two uncollinear vectors $\vec{a}, \vec{b}$ parallel to this plane (Fig.6*a*).

In this case we have a point and to get equation of the plane we should just find the normal vector. But

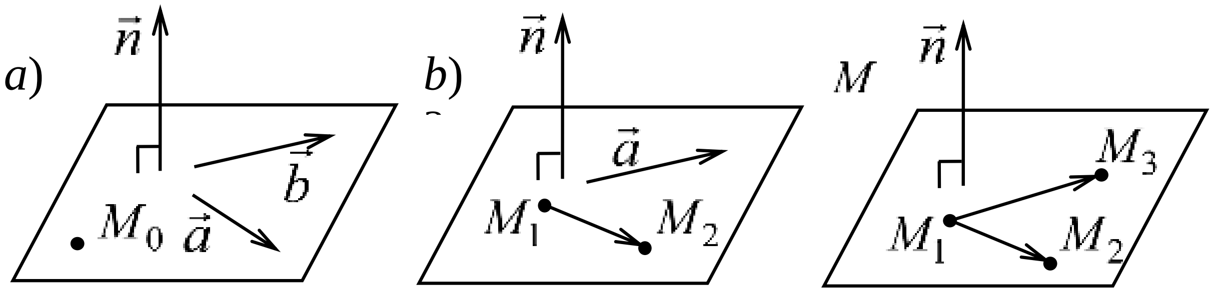

Figure 6

$$\begin{cases} \vec{n} \perp \vec{a} \\ \vec{n} \perp \vec{b} \end{cases} \Rightarrow \vec{n} \parallel \vec{a} \times \vec{b}.$$

Therefore as normal vector we can choose the vector

$$\vec{n} = \lambda \vec{a} \times \vec{b},$$

where $\lambda \in R$, $\lambda \neq 0$.

Case 2 Suppose we know two points $M_1(x_1, y_1, z_1)$ and $M_2(x_2, y_2, z_2)$ of the plane and a vector $\vec{a}$ parallel to this plane and uncollinear to the vector $\overrightarrow{M_1M_2}$ (Fig.6*b*).

In this case we have a point but do not have a normal vector. Since

$$\begin{cases} \vec{n} \perp \vec{a} \\ \vec{n} \perp \overrightarrow{M_1M_2} \end{cases} \Rightarrow \vec{n} \parallel \vec{a} \times \overrightarrow{M_1M_2}.$$

Therefore as normal vector we can choose the vector

$$\vec{n} = \lambda \vec{a} \times \overrightarrow{M_1M_2},$$

where $\lambda \in R$, $\lambda \neq 0$.

Case 3 Suppose we know three points $M_1(x_1, y_1, z_1)$, $M_2(x_2, y_2, z_2)$ and $M_2(x_2, y_2, z_2)$ of the plane such that vectors $\overrightarrow{M_1M_2}$ and $\overrightarrow{M_1M_3}$ are uncollinear (Fig.6*c*).

Since

$$\begin{cases} \vec{n} \perp \overrightarrow{M_1M_2} \\ \vec{n} \perp \overrightarrow{M_1M_3} \end{cases} \Rightarrow \vec{n} \parallel \overrightarrow{M_1M_2} \times \overrightarrow{M_1M_3}.$$

Therefore as normal vector we can choose the vector

$$\vec{n} = \lambda \overrightarrow{M_1M_2} \times \overrightarrow{M_1M_3},$$

where $\lambda \in R$, $\lambda \neq 0$. Then

$$\vec{n} = \lambda \overrightarrow{M_1M_2} \times \overrightarrow{M_1M_3} = \lambda \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = (A, B, C).$$

Thus, the equation of the plane passing through three given points has the form

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0 \text{ or}$$

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0.$$

Note. The last equation of the plane can be obtained directly from the condition of complanarity for vectors $\overrightarrow{M_1M_2}$, $\overrightarrow{M_1M_3}$, $\overrightarrow{M_1M}$, where point $M(x,y,z)$ is an arbitrary point of this plane.

Example. Find equation of the plane passing through the points $M_1(1,2,-1)$, $M_2(0,3,0)$ and $M_2(2,-1,1)$.

From the last formula we have

$$\begin{vmatrix} x-1 & y-2 & z+1 \\ 0-1 & 3-2 & 0+1 \\ 2-1 & -1-2 & 1+1 \end{vmatrix} = \begin{vmatrix} x-1 & y-2 & z+1 \\ -1 & 1 & 1 \\ 1 & -3 & 2 \end{vmatrix} =$$

$$= 5(x-1) + 3(y-2) + 2(z+1) = 5x + 3y + 2z - 9 = 0.$$