

Lecture #11: Equations of Straight Line in Space

11.1 General Equation of Straight Line in Space

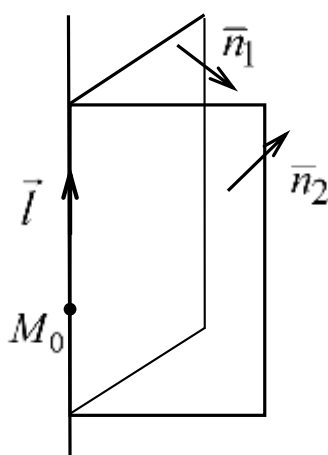


Figure 7

If two planes are not parallel, then they intersect and a line of their intersection is called *the straight line* (Fig.7).

From definition it follows that all points of straight line satisfy the following system of equations

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$$

This system is called *the general equation of the straight line*.

Definition. Vector \vec{l} parallel to the straight line is called the direction (or directing) vector of this straight line.

Vector $\vec{l}(m, n, p)$ determines *an orientation* of this straight line.

Since there are several straight lines with the same orientation, to describe straight line in the unique way we should determine any point $M_0(x_0, y_0, z_0)$ of this straight line (Fig.7). This point determines *a location* of this straight line.

There is a question: how to find the coordinates of point $M_0(x_0, y_0, z_0)$ and the direction vector $\vec{l}(m, n, p)$ from the general equation of the straight line?

Since two planes are not parallel their normal vectors are not collinear. Therefore

$$\begin{cases} \vec{l} \perp \vec{n}_1 \\ \vec{l} \perp \vec{n}_2 \end{cases} \Rightarrow \vec{l} = \lambda \vec{n}_1 \times \vec{n}_2.$$

Moreover, at least one of equalities $\frac{A_1}{A_2} = \frac{B_1}{B_2}$, $\frac{A_1}{A_2} = \frac{C_1}{C_2}$, $\frac{B_1}{B_2} = \frac{C_1}{C_2}$ is false, that is at

least one of numbers $\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}$, $\begin{vmatrix} A_1 & C_1 \\ A_2 & C_2 \end{vmatrix}$, $\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}$ is not 0 and therefore rank of the

matrix $\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix}$ is equal to 2 and the system $\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$ has infinite number of solutions.

So we can evaluate coordinates of point M_0 by assigning to one of coordinates some constant value and evaluating other coordinates.

Example. Find the direction vector and one point of the straight line

$$\begin{cases} 2x - y + 2z = 1 \\ -x + y + 3z = 4 \end{cases}$$

Here

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 2 \\ -1 & 1 & 3 \end{vmatrix} = -5\vec{i} - 8\vec{j} + \vec{k} = (-5, -8, 1).$$

Then as direction vector we can take, for example, the vector $\vec{l} = -\vec{n}_1 \times \vec{n}_2 = (5, 8, -1)$.

Let us write down the matrix of the system:

$$\begin{pmatrix} 2 & -1 & 2 \\ -1 & 1 & 3 \end{pmatrix}.$$

Since $\begin{vmatrix} 2 & 2 \\ -1 & 3 \end{vmatrix} \neq 0$ then variable y is free.

$$y := 4: \begin{cases} 2x + 2z = 5 \\ -x + 3z = 0 \end{cases} \Leftrightarrow \begin{cases} x = 3z \\ 6z + 2z = 5 \end{cases} \Leftrightarrow z = \frac{5}{8}, x = \frac{15}{8}.$$

Thus we obtained the point $M_0\left(\frac{15}{8}, 4, \frac{5}{8}\right)$.

11.2 Canonical Equations of Straight Line

Suppose we know the coordinates of the point $M_0(x_0, y_0, z_0)$ and the direction vector $\vec{l}(m, n, p)$ of the straight line. Then for any point $M(x, y, z)$ of this straight line we have (Fig.8):

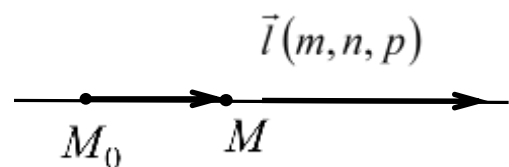


Figure 8

$$\overrightarrow{M_0M} \parallel \vec{l} \Leftrightarrow \boxed{\frac{x-x_0}{m} = \frac{y-y_0}{n} = \frac{z-z_0}{p}}$$

These equations are called *the canonical equations of the straight line*.

Note. One or two coordinates of the direction vector could be equal to zero. Canonical equations just show the proportionality of the coordinates of collinear vectors.

Example. Find the canonical equations of the straight line passing through the point $M_0(1;-3;0)$ and perpendicular to the vectors $\vec{a}(1,0,-3)$ and $\vec{b}(2,1,5)$.

The direction vector of this straight line is perpendicular to these vectors as well and therefore

$$\vec{l} = \lambda \vec{a} \times \vec{b}.$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -3 \\ 2 & 1 & 5 \end{vmatrix} = 3\vec{i} - 11\vec{j} + \vec{k} = (3, -11, 1).$$

Let $\vec{l} = 1 \cdot \vec{a} \times \vec{b} = (3, -11, 1)$. Then the canonical equations of this straight line are

$$\frac{x-1}{3} = \frac{y-(-3)}{-11} = \frac{z-0}{1} \quad \text{or} \quad \frac{x-1}{3} = \frac{y+3}{-11} = \frac{z}{1}.$$

11.3 Equation of Straight Line Passing through Two Points

Suppose we know the coordinates of two points $M_1(x_1, y_1, z_1)$ and $M_2(x_2, y_2, z_2)$ of the given straight line. Then

$$\vec{l} = (m, n, p) = \overrightarrow{M_1M_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

and the canonical equations of this straight line have the following form

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}.$$

Example. Find the canonical equations of the straight line passing through the points $M_1(1;-3;0)$ and $M_2(2;-1;4)$. By the given formula we have

$$\frac{x-1}{2-1} = \frac{y-(-3)}{-1-(-3)} = \frac{z-0}{4-0} \quad \text{or} \quad \frac{x-1}{1} = \frac{y+3}{2} = \frac{z}{4}.$$

11.4 Parametric Equations of Straight Line

From the canonical equations of straight line It follows that for any point of the straight line three values are equal to the same value (value of some parameter).

Let us denote this value as t . Then

$$\frac{x-x_0}{m} = \frac{y-y_0}{n} = \frac{z-z_0}{p} = t \in R \quad \Leftrightarrow \quad \begin{cases} x = mt + x_0 \\ y = nt + y_0 \\ z = pt + z_0 \end{cases} \quad t \in R$$

The last equations are called *the parametric equations of the straight line*.

Note. Parameter t plays a role of continuous index by means of which all points of straight line are numbered.

Example. Find the canonical equations of the straight line passing through the point $M_0(1;2;1)$ and parallel to the vector $\vec{a}(1,0,-3)$. Vector \vec{a} can be taken as the direction vector of this straight line and therefore we have

$$\begin{cases} x = 1t + 1 = t + 1 \\ y = 0t + 2 = 2 \\ z = -3t + 1 \end{cases} \quad t \in R.$$

11.5 Distance from Point to Straight Line

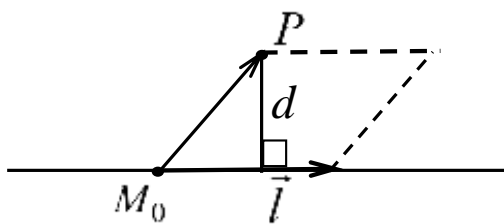


Figure 9

straight line (Fig.9). Therefore

To find distance from the point $P(x_P, y_P, z_P)$ to the straight line It is enough to find the altitude of the parallelogram constructed on vectors \vec{l} and $\overrightarrow{M_0P}$, where $\vec{l} = (m, n, p)$ is the direction vector of the straight line, $M_0(x_0, y_0, z_0)$ is any point of this

$$d = \frac{|\vec{l} \times \overrightarrow{M_0P}|}{|\vec{l}|}.$$

Example. Find the distance from the origin to the straight line $\frac{x-1}{1} = \frac{y+3}{2} = \frac{z}{4}$.

Here

$$P(0,0,0), \vec{l} = (1,2,4), M_0(1,-3,0), \overrightarrow{M_0P} = (-1,3,0),$$

$$|\vec{l}| = \sqrt{1^2 + 2^2 + 4^2} = \sqrt{1+4+16} = \sqrt{21},$$

$$\vec{l} \times \overrightarrow{M_0P} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 4 \\ -1 & 3 & 0 \end{vmatrix} = -12\vec{i} - 4\vec{j} + 5\vec{k} = (-12, -4, 5),$$

$$|\vec{l} \times \overrightarrow{M_0P}| = \sqrt{(-12)^2 + (-4)^2 + 5^2} = \sqrt{144+16+25} = \sqrt{185}.$$

Thus

$$d = \frac{|\vec{l} \times \overrightarrow{M_0P}|}{|\vec{l}|} = \sqrt{\frac{185}{21}}.$$

11.6 Arrangements of Straight Lines in Space

Definition. An angle between two straight lines is the angle between their direction vectors.

As It is shown on Fig.36

$$\cos \alpha = \frac{(\vec{l}_1, \vec{l}_2)}{|\vec{l}_1||\vec{l}_2|}$$

then

$$\alpha = \arccos \frac{(\vec{l}_1, \vec{l}_2)}{|\vec{l}_1||\vec{l}_2|}.$$

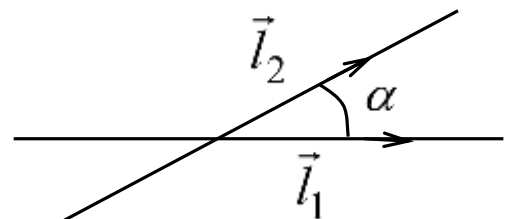


Fig.36

Moreover, for any two straight lines L_1, L_2 we have:

$$L_1 \parallel L_2 \Leftrightarrow \vec{l}_1 \parallel \vec{l}_2 \Leftrightarrow \frac{m_1}{m_2} = \frac{n_1}{n_2} = \frac{p_1}{p_2};$$

$$L_1 \perp L_2 \Leftrightarrow \vec{l}_1 \perp \vec{l}_2 \Leftrightarrow m_1m_2 + n_1n_2 + p_1p_2 = 0.$$

Note. The angle between two straight lines does not give to us full information about positional relationship between these straight lines.

We have three different situations, namely:

- a) crossing straight lines (Fig.37a);
- b) parallel or coinciding straight lines (Fig.37b);
- c) skew straight lines (Fig.37c).

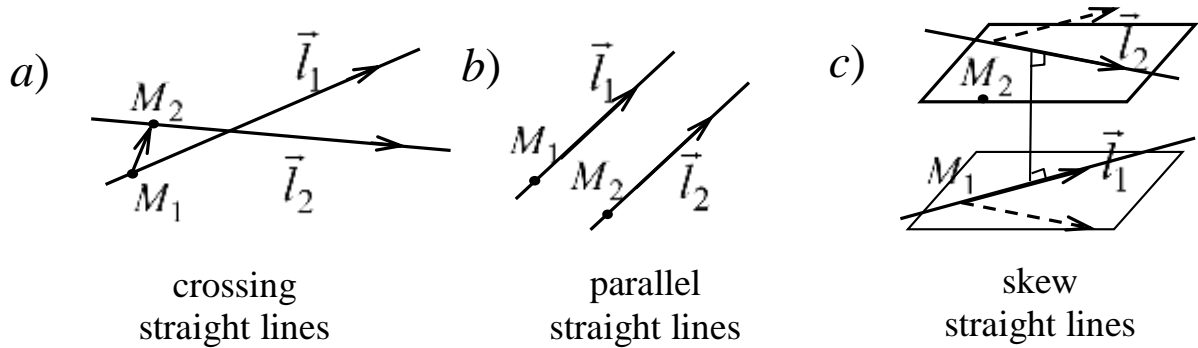


Figure 37

In cases a)-b) these straight lines lie in one plane, in case c) they lie in different planes.

It is simple to check that:

- 1) Straight lines lie in the same plane if and only if their direction vectors \vec{l}_1, \vec{l}_2 and vector $\overrightarrow{M_1M_2}$ connecting two different points of these straight lines are coplanar, i.e.

$$(\vec{l}_1, \vec{l}_2, \overrightarrow{M_1M_2}) = 0.$$

- 2) Two parallel lines coincide if and only if at least one point of one straight line belongs to other one.

It follows from discussed above that:

$$(\vec{l}_1, \vec{l}_2, \overrightarrow{M_1M_2}) \neq 0 \Leftrightarrow \text{straight lines are skew straight lines;}$$

$$(\vec{l}_1, \vec{l}_2, \overrightarrow{M_1M_2}) = 0 \Leftrightarrow \begin{cases} \text{straight lines are crossing, if } \vec{l}_1 \times \vec{l}_2 \neq 0 \\ \text{straight lines are parallel, if } \vec{l}_1 \times \vec{l}_2 = 0 \end{cases}$$

Example. Determine the positional relationship of two straight lines $\frac{x}{1} = \frac{y}{2} = \frac{z-1}{2}$

and $\frac{x-1}{-1} = \frac{y-3}{2} = \frac{z}{2}$. Since the direction vectors of these straight lines

$\vec{l}_1(1,2,2), \vec{l}_2(-1,2,2)$ are not collinear, these straight lines are not parallel. Let us find the angle between them and determine either they are intersecting straight lines or skew ones.

$$\alpha = \arccos \frac{(\vec{l}_1, \vec{l}_2)}{|\vec{l}_1||\vec{l}_2|} = \arccos \frac{-1+4+4}{\sqrt{1+4+4}\sqrt{1+4+4}} = \arccos \frac{7}{9}.$$

Here

$$M_1(0;0;1), M_2(1;3;0),$$

$$\overrightarrow{M_1M_2} = (1,3,-1),$$

$$(\vec{l}_1, \vec{l}_2, \overrightarrow{M_1M_2}) = \begin{vmatrix} 1 & 2 & 2 \\ -1 & 2 & 2 \\ 1 & 3 & -1 \end{vmatrix} = -2+4-6-4-2-6 = -16 \neq 0.$$

Therefore, these straight lines are skew straight lines.

11.7 Distance between Two Straight Lines

Case 1. Suppose we have two crossing straight lines (Fig.37a). Then the distance between them is equal to zero, i.e.

$$d = 0.$$

Case 2. Suppose we have two parallel or coinciding straight lines (Fig.37b). Then the distance between them is equal to the distance from any point of one straight line to other straight line. Thus it can be calculated by formula:

$$d = \frac{|\overrightarrow{M_1M_2} \times \vec{l}_1|}{|\vec{l}_1|}.$$

Case 3. Suppose we have two skew straight lines (Fig.37c). Let us plot from the points M_1 and M_2 direction vectors \vec{l}_1, \vec{l}_2 of both straight lines as it shown on Fig. 37c. We have formed two parallel planes. Common perpendicular of straight lines is perpendicular to these planes. Therefore to find distance between two skew straight lines It is enough to find distance between two parallel planes or just the altitude of the parallelepiped constructed on vectors $\vec{l}_1, \vec{l}_2, \overrightarrow{M_1M_2}$. Thus

$$d = \frac{\left| \left(\vec{l}_1, \vec{l}_2, \overrightarrow{M_1M_2} \right) \right|}{\left| \vec{l}_1 \times \vec{l}_2 \right|}.$$

Note. If two straight lines intersect then the numerator of the fraction in the last formula is equal to zero, while the denominator is not equal to zero. Therefore, we can use the last formula to find distance between any two unparallel straight lines (either intersecting or skew ones).

Example. Find the distance between two straight lines

$$\frac{x}{1} = \frac{y}{2} = \frac{z-1}{2} \quad \text{and} \quad \frac{x-1}{-1} = \frac{y-3}{2} = \frac{z}{2}.$$

Since the direction vectors of these straight lines $\vec{l}_1(1,2,2)$, $\vec{l}_2(-1,2,2)$ are not collinear, these straight lines are not parallel. So we should use the last formula to find distance. Here

$$M_1(0;0;1), M_2(1;3;0), \overrightarrow{M_1M_2} = (1,3,-1),$$

$$\vec{l}_1 \times \vec{l}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 2 \\ -1 & 2 & 2 \end{vmatrix} = 0\vec{i} - 4\vec{j} + 4\vec{k} = (0, -4, 4),$$

$$\left| \vec{l}_1 \times \vec{l}_2 \right| = \sqrt{0^2 + (-4)^2 + 4^2} = \sqrt{2 \cdot 16} = 4\sqrt{2},$$

$$\left(\vec{l}_1, \vec{l}_2, \overrightarrow{M_1M_2} \right) = \left(\vec{l}_1 \times \vec{l}_2, \overrightarrow{M_1M_2} \right) = 0 - 12 - 4 = -16.$$

Therefore we obtain

$$d = \frac{\left| \left(\vec{l}_1, \vec{l}_2, \overrightarrow{M_1M_2} \right) \right|}{\left| \vec{l}_1 \times \vec{l}_2 \right|} = \frac{16}{4\sqrt{2}} = 2\sqrt{2}.$$

11.8 Equation of Common Perpendicular to Skew Straight Lines

Since a direction vector of common perpendicular is perpendicular to both straight lines, we have

$$\vec{l} = \vec{l}_1 \times \vec{l}_2.$$

Common perpendicular is a straight line from intersection of two planes, namely (Fig.10):

1) plane passing through the first straight line and parallel to \vec{l} ;

2) plane passing through the second straight line and parallel to \vec{l} .

It follows from above that general equation of the common perpendicular is

$$\begin{cases} A_1(x - x_1) + B_1(y - y_1) + C_1(z - z_1) = 0 \\ A_2(x - x_2) + B_2(y - y_2) + C_2(z - z_2) = 0 \end{cases}$$

where

$$\vec{n}_1 = (A_1, B_1, C_1) = \vec{l}_1 \times \vec{l},$$

$$\vec{n}_2 = (A_2, B_2, C_2) = \vec{l}_2 \times \vec{l},$$

$M_1(x_1, y_1, z_1)$, $M_2(x_2, y_2, z_2)$ are points of the first and the second straight lines relatively.

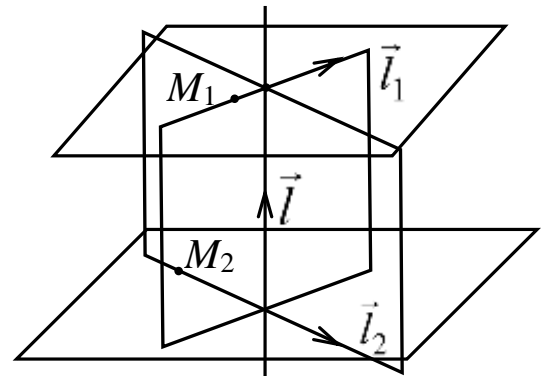


Figure 10

11.9 Arrangements of Plane and Straight Line

An angle α between straight line and a plane can be found in the following way (Fig. 11)

$$\alpha = \frac{\pi}{2} - \beta \text{ if } \beta = (\vec{l}, \vec{n}) \text{ is acute (Fig. 11a);}$$

$$\alpha = \beta - \frac{\pi}{2} \text{ if } \beta = (\vec{l}, \vec{n}) \text{ is obtuse (Fig. 11b).}$$

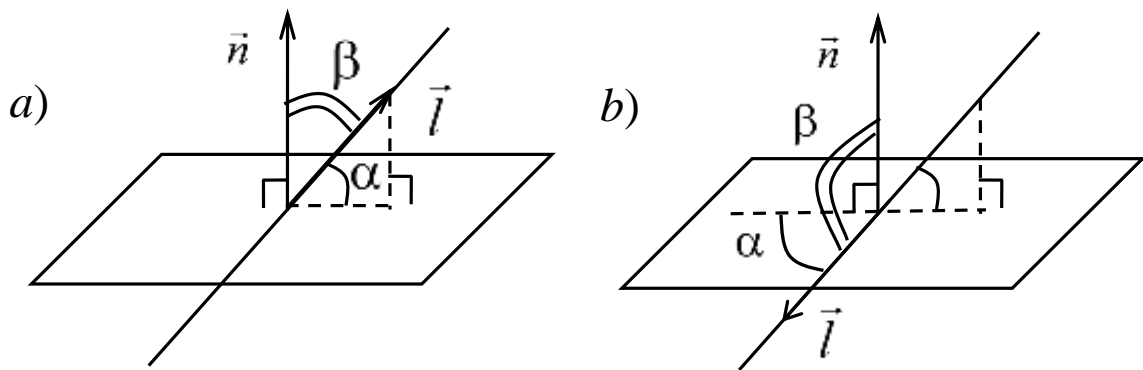


Figure 11

In other case,

$$\sin \alpha = |\cos \beta| = \frac{|(\vec{l}, \vec{n})|}{|\vec{l}| |\vec{n}|}, \quad \alpha = \arcsin \frac{|(\vec{l}, \vec{n})|}{|\vec{l}| |\vec{n}|}.$$

From the last formula we have

1. Straight line is parallel to the plane $\Leftrightarrow \vec{l} \perp \vec{n} \Leftrightarrow (\vec{l}, \vec{n}) = 0$;

2. Straight line is perpendicular to the plane $\Leftrightarrow \vec{l} \parallel \vec{n} \Leftrightarrow \vec{l} \times \vec{n} = 0 \Leftrightarrow \frac{m}{A} = \frac{n}{B} = \frac{p}{C}$;

3. Straight line crosses a plane $\Leftrightarrow (\vec{l}, \vec{n}) \neq 0$;

4. Straight line passing through the point M_0 belongs to the plane (Fig.12) \Leftrightarrow

$$1) \vec{n} \perp \vec{l} \Leftrightarrow (\vec{n}, \vec{l}) = 0 \Leftrightarrow Am + Bn + Cp = 0$$

$$2) M_0 : Ax_0 + By_0 + Cz_0 + D = 0$$

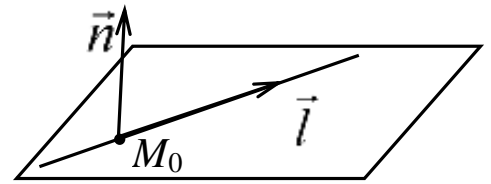


Figure 12

Example 1. Find an angle between the following straight line and plane:

$$\frac{x-1}{-2} = \frac{y}{3} = \frac{z-3}{4}, \quad -x + 6y - 4z + 1 = 0.$$

Here

$$\vec{l}(-2,3,4), \quad \vec{n}(-1,6,-4).$$

Therefore

$$\alpha = \arcsin \frac{(\vec{l}, \vec{n})}{|\vec{l}||\vec{n}|} = \arcsin \left| \frac{2+18-16}{\sqrt{4+9+16}\sqrt{1+36+16}} \right| = \arcsin \frac{4}{\sqrt{29}\sqrt{53}} = \arcsin \frac{4}{\sqrt{1537}}.$$

Example 2. Prove that straight line $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ lies in plane

$-x + 6y - 4z + 1 = 0$. Here

$$M_0(1,2,3), \quad \vec{l}(2,3,4), \quad \vec{n}(-1,6,-4).$$

$$(\vec{n}, \vec{l}) = 2(-1) + 3 \cdot 6 + 4(-4) = -2 + 18 - 16 = 0.$$

$$Ax_0 + By_0 + Cz_0 + D = -1 \cdot 1 + 6 \cdot 2 - 4 \cdot 3 + 1 = -1 + 12 - 12 + 1 = 0.$$

Therefore $\vec{n} \perp \vec{l}$ and M_0 belongs to this plane, i.e. this straight line lies in plane.

11.10 Point of Intersection of Straight Line and Plane

To find this point means to find solution of the following system:

$$\begin{cases} Ax + By + Cz + D = 0 \\ \frac{x - x_0}{mn} = \frac{y - y_0}{n} = \frac{z - z_0}{p} \end{cases} \Leftrightarrow \begin{cases} Ax + By + Cz + D = 0 \\ x = mt + x_0 \\ y = nt + y_0 \\ z = pt + z_0 \end{cases}$$

For this it is enough to substitute expressions for x , y , z into the equation of the plane and solve one equation for parameter t .

There are three possible situations:

- 1) The only solution \Leftrightarrow the only point of intersection \Leftrightarrow straight line crosses this plane;
- 2) No solutions \Leftrightarrow no points of intersection \Leftrightarrow straight line is parallel to this plane;
- 3) A lot of solutions \Leftrightarrow a lot of points of intersection \Leftrightarrow straight line belongs to this plane.

Example. Find the point of intersection of the straight line $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and plane $x + 2y + 3z - 4 = 0$.

First, we write down the parametric equations of this straight line: $x = 2t + 1$, $y = 3t + 2$, $z = 4t + 3$, $t \in \mathbb{R}$. Then we solve the system:

$$\begin{cases} x + 2y + 3z - 4 = 0 \\ x = 2t + 1 \\ y = 3t + 2 \\ z = 4t + 3 \end{cases} \Leftrightarrow \begin{cases} (2t + 1) + 2(3t + 2) + 3(4t + 3) - 4 = 20t + 10 = 0 \\ x = 2t + 1 \\ y = 3t + 2 \\ z = 4t + 3 \end{cases} \Leftrightarrow \begin{cases} t = -1/2 \\ x = 0 \\ y = 1/2 \\ z = 1 \end{cases}$$

Therefore the point of intersection is $P(0, 1/2, 1)$.