Lecture #12: Straight Line in Plane

12.1 Line in Plane

Consider the rectangular Cartesian coordinate system.

Definition. Locus of the points with coordinates satisfying an equation $\Phi(x, y) = 0$ is a *line in plane*.

Definition. Expression $\Phi(x, y) = 0$ is called an equation of the line in plane if coordinates of all points of this line, and only of those points, satisfy the equation.

Example. Suppose

$$\Phi(x, y) = x^2 + y^2 - 4 = 0,$$

i.e.

$$x^2 + y^2 = 4$$

That is an equation of the circle with radius $R = 2$ (Fig.15).

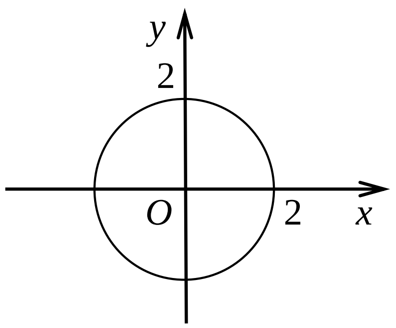

Figure 15

Equation of the line depends on the system of coordinates.

Let us consider, for example, *the polar system of coordinates*. Instead of to determine point in plane by Cartesian coordinates x, y we determine it by polar radius ρ and polar angle φ (Fig.16).

Herein, ρ is a distance from the point to the origin, i.e.

$$\rho = \left|\overrightarrow{OM}\right|;$$

φ is an angle between the positive semi-axis *Ox* and the radius-vector of the point. Herewith, φ is positive in the anticlockwise direction and negative in other direction.

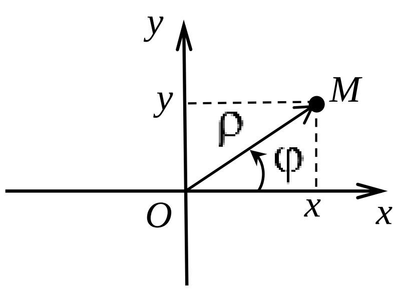

Figure 16

From Fig.16 It is simple to get that

$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \end{cases}$$

Example. Find the equation of the circle with center (0;0) and radius $R = 2$ in polar coordinates.

$$x^2 + y^2 = 4 \Leftrightarrow \rho^2 \cos^2 \varphi + \rho^2 \sin^2 \varphi = 4 \Leftrightarrow \rho^2(\cos^2 \varphi + \sin^2 \varphi) = 4 \Leftrightarrow$$

$$\Leftrightarrow \rho^2 = 4 \Leftrightarrow \rho = 2.$$

So, the equation of this circle in polar coordinates is

$$\rho = 2.$$

Another way to determine a line in plane is by means of the *parametric equations*:

$$\begin{cases} x = x(t) \\ y = y(y) \end{cases} \quad t \in R$$

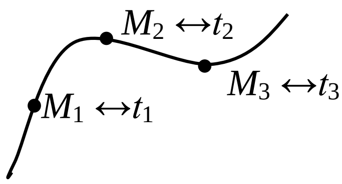

Figure 17

Here the coordinates of the points are functions of some parameter *t*. Parameter *t* is like a continuous number (index) of infinite number of points of this line (Fig.17).

Example. $\rho = 2 \Rightarrow \begin{cases} x = 2\cos \varphi \\ y = 2\sin \varphi \end{cases}$ where φ is a parameter.

12.2 Straight Line in Plane. General Equation of the Straight Line

Consider the rectangular Cartesian coordinate system.

Theorem (about general equation of the straight line) Equation $Ax + By + C = 0$, where $A^2 + B^2 \neq 0$, is an equation of the straight line in plane.

Proof. Let us consider an equation

$$Ax + By + C = 0 \qquad (*)$$

and suppose that $x = x_0, y = y_0$ satisfy the equation (*). i.e.

$$Ax_0 + By_0 + C = 0. \qquad (**)$$

The difference of equations (*) and (**) is

$$A(x - x_0) + B(y - y_0) = 0. \qquad (***)$$

Equation (***) means that for any point $M(x,y)$ with coordinates satisfying the equation (*) the vector $\overrightarrow{M_0M} \perp \vec{n}$, where $M_0(x_0, y_0)$, $\vec{n} = (A, B)$ (Fig.18). Thus M belongs to the straight line passing through the point M_0 and perpendicular to the vector $\vec{n}$. Note, that from (***) we have

$$Ax - Ax_0 + By - By_0 = 0 \Leftrightarrow Ax + By + (-Ax_0 - By_0) = 0 \Leftrightarrow Ax + By + C = 0,$$

where $C = -Ax_0 - By_0$.

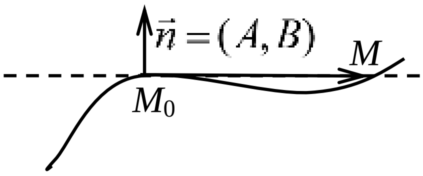

Figure 18

Let us consider now any point $M(x,y)$ that belongs to this straight line. Vector $\overrightarrow{M_0M}$ is the vector parallel to this straight line and therefore it is perpendicular to the vector $\vec{n} = (A, B)$, i.e.

$$0 = \left(\overrightarrow{M_0M}, \vec{n}\right) = A(x - x_0) + B(y - y_0) = Ax + By + (-Ax_0 - By_0) = Ax + By + D.$$

So, any point of this straight line satisfies the equation (*). ***Theorem is proven.***

Definition. Vector $\vec{n} = (A, B)$ is called a normal vector of the straight line $Ax + By + C = 0$.

Definition. Equation $Ax + By + C = 0$ is called *the general equation of the straight line*.

Definition. Equation $A(x - x_0) + B(y - y_0) = 0$ is called the equation of the straight line passing through the point $M_0(x_0, y_0)$ with normal vector $\vec{n}(A, B)$.

Example. Find the equation of the straight line parallel to the straight line $2x + y + 1 = 0$ and passing through the point $M_0(3;-2)$. Normal vector of the given straight line $\vec{n}(2,1)$ is perpendicular to the asked straight line as well, since these straight lines are parallel. Therefore we can take $\vec{n}(2,1)$ as normal vector of the straight line. So, we obtain

$$A(x - x_0) + B(y - y_0) = 2(x - 3) + 1(y - (-2)) = 2x - 6 + y + 2 = 2x + y - 4 = 0$$

or

$$2x + y - 4 = 0.$$

12.3 Equation of the Straight Line with Given Intercepts

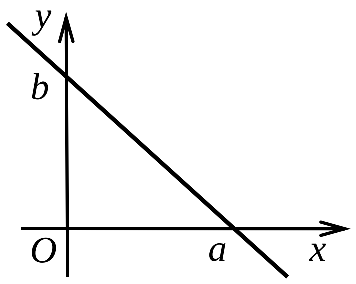

Figure 19

Suppose $A \cdot B \cdot C \neq 0$. Let us divide the general equation of the straight line by *C*. Then

$$Ax + By + C = 0 \Leftrightarrow \frac{A}{C}x + \frac{B}{C}y + 1 = 0 \Leftrightarrow$$

$$\Leftrightarrow \frac{x}{-\frac{C}{A}} + \frac{y}{-\frac{C}{B}} = 1 \Leftrightarrow \frac{x}{a} + \frac{y}{b} = 1,$$

where $a = -\frac{C}{A}$, $b = -\frac{C}{B}$ are the intercepts of the straight line on the exes (Fig.19).

Note. Intercepts a, b can be negative. It means that segments are cut from the negative semi-axes of *Ox*, *Oy*.

Example. Find the area of the triangle bounded by axes O*x*, O*y* and the straight line $\frac{x}{6} - \frac{y}{5} = 1$ (Fig.19). From the equation we have

$$a = 6,\ b = -5.$$

Therefore

$$S = \frac{1}{2}|a||b| = \frac{1}{2} \cdot 6 \cdot 5 = 15 \text{ square units.}$$

12.4 Canonical Equation of the Straight Line

As it was mentioned above, the straight line can be determined by its normal vector giving an orientation of the straight line and by its point giving a location of this straight line. But besides the normal vector the orientation of the straight line in plane can be given by the direction vector $\vec{l} = (m, n)$ (Fig.20). In this case

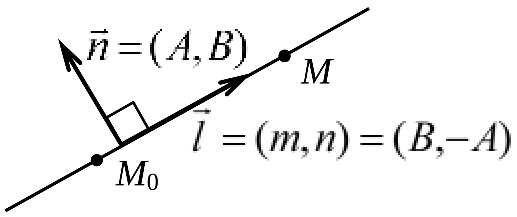

Figure 20

$$\vec{l} \parallel \overrightarrow{M_0M} \Leftrightarrow \frac{x - x_0}{m} = \frac{y - y_0}{n}.$$

The last equation is called *the canonical equation of the straight line in plane*.

Example. Find the equation of the straight line perpendicular to the straight line $2x+y+1=0$ and passing through the point $M_0(3;-2)$. Normal vector of the given straight line $\vec{n}(2,1)$ is parallel to the asked straight line, since these straight lines are perpendicular. Therefore we can take $\vec{n}(2,1)$ as the direction vector of the straight line. So, we obtain

$$\frac{x-x_0}{m}=\frac{y-y_0}{n}\Leftrightarrow\frac{x-3}{2}=\frac{y-(-2)}{1}\Leftrightarrow x-3=2(y+2)\Leftrightarrow$$
$$\Leftrightarrow x-3=2y+4\Leftrightarrow x-2y-3-4=0$$

or

$$x-2y-7=0.$$

12.5 Canonical Equation of the Straight Line Passing Through Two Points

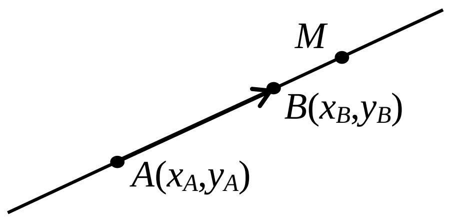

Figure 21

Suppose the points *A* and *B* belong to the straight line. Let us consider an arbitrary point of this line $M(x,y)$ (Fig.21). Then

$$\overrightarrow{AM}\parallel\overrightarrow{AB}\Leftrightarrow\frac{x-x_A}{x_B-x_A}=\frac{y-y_A}{y_B-y_A}.$$

The last equation is the equation of the *straight line passing through two points A* and *B*.

Example. Find the equation of the straight line passing through the origin and the point *B*(1;−1). In that case the second point is *A*(0;0). So we obtain

$$\frac{x-0}{1-0}=\frac{y-0}{-1-0}\Leftrightarrow x=-y.$$

Therefore the equation of the asked straight line is

$$x+y=0.$$

12.6 Parametric Equations of the Straight Line

Let us consider the canonical equation of the straight line and equate both sides of this equation to some value t:

$$\frac{x-x_0}{m}=\frac{y-y_0}{n}=t\,.$$

t is parameter of proportionality for point $M(x,y)$. From the last equation we get *the parametric equations of the straight line*:

$$\begin{cases} x=mt+x_0 \\ y=nt+y_0 \end{cases} \quad t\in R\,.$$

12.7 Distance between Point and Straight Line

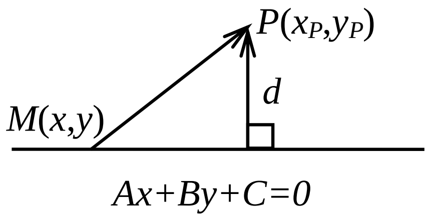

Figure 22

Distance from the point $P(x_P,y_P)$ to the straight line can be found as a module of the projection of the vector $\overrightarrow{MP}$ on normal direction (Fig.22):

$$d=\left|pr_{\vec{n}}\overrightarrow{MP}\right|=\left|\frac{\left(\vec{n},\overrightarrow{MP}\right)}{|\vec{n}|}\right|=\frac{|A(x_P-x)+B(y_P-y)|}{\sqrt{A^2+B^2}}=$$

$$=\frac{|Ax_P+By_P-(Ax+By)|}{\sqrt{A^2+B^2}}=\frac{|Ax_P+By_P+C|}{\sqrt{A^2+B^2}}\,.$$

So, the distance from the point $P(x_P,y_P)$ to the straight line $Ax+By+C=0$ is

$$d=\frac{|Ax_P+By_P+C|}{\sqrt{A^2+B^2}}\,.$$

Example 1. Find the distance from the origin to the straight line:

$$d_0=\frac{|Ax_p+By_p+C|}{\sqrt{A^2+B^2}}=\begin{bmatrix} x_p=0 \\ y_p=0 \end{bmatrix}=\frac{|A\cdot 0+B\cdot 0+C|}{\sqrt{A^2+B^2}}=\frac{|C|}{\sqrt{A^2+B^2}}\,.$$

Example 2. Find the altitude of the triangle with vertices $A(1;-2)$, $B(2;0)$, $C(-1;3)$ dropped on the side AC (Fig.23).

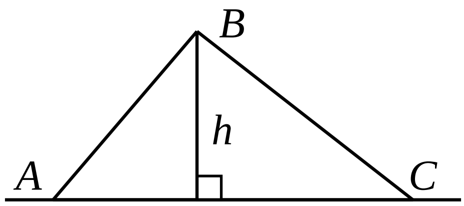

Figure 23

The value of this altitude can be found as a distance from the point *B* to the straight line, passing through the points *A* and *C*.

The equation of the straight line *AC* is

$$\frac{x-1}{-1-1}=\frac{y-(-2)}{3-(-2)}\Leftrightarrow 5(x-1)=-2(y+2)\Leftrightarrow$$

$$\Leftrightarrow 5x-5=-2y-4\Leftrightarrow 5x+2y-5+4=0\Leftrightarrow 5x+2y-1=0.$$

Therefore

$$h=d=\frac{|5\cdot 2+2\cdot 0-1|}{\sqrt{5^2+2^2}}=\frac{9}{\sqrt{29}}.$$

12.8 Normal Equation of the Straight Line

Let us consider the general equation of the straight line and divide it by the module of its normal vector:

$$\frac{A}{\sqrt{A^2+B^2}}x+\frac{B}{\sqrt{A^2+B^2}}y+\frac{C}{\sqrt{A^2+B^2}}=0\Leftrightarrow \cos\alpha\, x+\cos\beta\, y+p=0,$$

where $\cos\alpha=\frac{A}{\sqrt{A^2+B^2}},\cos\beta=\frac{B}{\sqrt{A^2+B^2}}$ are direction cosines of normal vector, i.e. coordinates of the ort $\vec{n}^0$,

$p=\frac{C}{\sqrt{A^2+B^2}}$ is a distance from the origin to this straight line taken with sign equal to the sign of the coefficient *C*.

The obtained equation

$$\cos\alpha\, x+\cos\beta\, y+p=0$$

is called the *normal equation* of this straight line.

Note. By means of the direction cosines we can rewrite the formula for point distance to straight line in the following way:

$$d=\frac{|Ax_p+By_p+C|}{\sqrt{A^2+B^2}}=$$

$$=\left|\frac{A}{\sqrt{A^2+B^2}}x_p+\frac{B}{\sqrt{A^2+B^2}}y_p+\frac{C}{\sqrt{A^2+B^2}}\right|=\left|\cos\alpha\ x_P+\cos\beta\ y_P+p\right|,$$

i.e.

$$d=\left|\cos\alpha\ x_P+\cos\beta\ y_P+p\right|.$$

Example. Find the equations of straight lines perpendicular to the vector $\vec{n}(3,4)$ and located on the distance 3 from the origin.

Since

$$|\vec{n}|=\sqrt{3^2+4^2}=\sqrt{25}=5,\ \vec{n}^0\left(\frac{3}{5},\frac{4}{5}\right)=(\cos\alpha,\cos\beta),$$

the normal equations of these straight lines look like

$$\cos\alpha\,x+\cos\beta\,y+p=\frac{3}{5}x+\frac{4}{5}y+p=0.$$

From condition of the example It follows that

$$p=\pm 3$$

and therefore the asked equations are

$$\frac{3}{5}x+\frac{4}{5}y\pm 3=0\Leftrightarrow 3x+4y\pm 15=0.$$

12.9 Equation of the Straight Line with Given Slope

Another way to determine the straight line is by the given slope and y-intercept (the segment cut from the axis Oy).

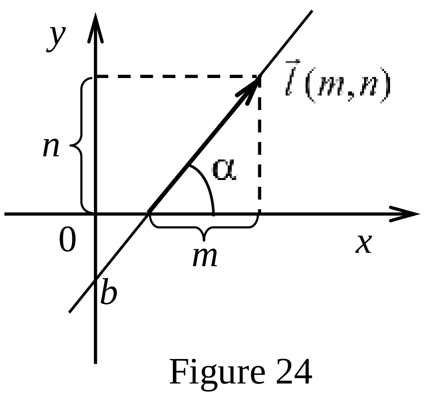

Figure 24

Definition. Slope is a tangent of the angle between the straight line and the positive semi-axis Ox. Notation: k.

So, if follows from Fig.24

$$k=\tan\alpha.$$

Let us obtain the equation of this straight line:

$$\frac{x-0}{m}=\frac{y-b}{n}\Leftrightarrow y-b=\frac{n}{m}x\Leftrightarrow$$

$$y=\frac{m}{n}x+b=\tan\alpha\,x+b\Leftrightarrow y=kx+b.$$

We obtained the equation of *the straight line with the given slope k and the intercept b:*

$$y = kx + b. \qquad (*)$$

Suppose (x_0, y_0) is a point of this straight line. Then

$$y_0 = kx_0 + b. \qquad (**)$$

The difference of the equations (*) and (**) gives *the equation of the straight line with slope k passing through the point* (x_0, y_0):

$$y - y_0 = k(x - x_0).$$

If you know any two points of the straight line $M_1(x_1, y_1)$ and $M_2(x_2, y_2)$ then:

$\vec{l} = \overrightarrow{M_1M_2} = (x_2 - x_1, y_2 - y_1) = (m, n)$,

$$k = \frac{m}{n} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Example. Find the equations of the triangle sides if the vertices of this triangle are $A(1;-2),\ B(2;0),\ C(-1;3)$.

Let us find the corresponding slops and equations:

$$k_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{0-(-2)}{2-1} = 2,$$

Side AB:

$$y - y_A = k_{AB}(x - x_A) \Leftrightarrow y - (-2) = 2(x-1) \Leftrightarrow 2x - y - 4 = 0;$$

$$k_{AC} = \frac{y_C - y_A}{x_C - x_A} = \frac{3-(-2)}{-1-1} = -\frac{5}{2},$$

Side AC:

$$y - y_A = k_{AC}(x - x_A) \Leftrightarrow y - (-2) = -\frac{5}{2}(x-1) \Leftrightarrow 5x + 2y - 1 = 0;$$

$$k_{CB} = \frac{y_B - y_C}{x_B - x_C} = \frac{0-3}{2-(-1)} = -1,$$

Side AB:

$$y - y_B = k_{CB}(x - x_B) \Leftrightarrow y - 0 = -1(x-2) \Leftrightarrow x + y - 2 = 0.$$

12.10 Angle between Two Straight Lines

An angle between two straight lines can be found in three ways: as angle between their normal vectors $\vec{n}_1, \vec{n}_2$, as angle between their direction vectors $\vec{l}_1, \vec{l}_2$ and by means of their slopes k_1, k_2.

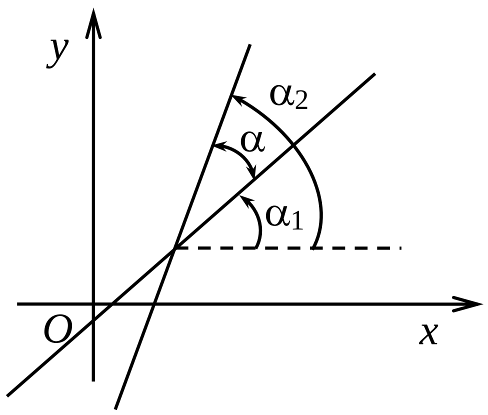

Figure 25

In the first two cases we calculate the angle by formulas known from vector algebra:

$$\cos\alpha = \frac{(\vec{l}_1, \vec{l}_2)}{|\vec{l}_1||\vec{l}_2|} \text{ or } \cos\alpha = \frac{(\vec{n}_1, \vec{n}_2)}{|\vec{n}_1||\vec{n}_2|}.$$

To get acute (or right) angle between two straight lines use formulas:

$$\cos\alpha = \left|\frac{(\vec{l}_1, \vec{l}_2)}{|\vec{l}_1||\vec{l}_2|}\right| \text{ or } \cos\alpha = \left|\frac{(\vec{n}_1, \vec{n}_2)}{|\vec{n}_1||\vec{n}_2|}\right|.$$

Let us consider the last variant (Fig.25).

$$k_1 = \tan\alpha_1, k_2 = \tan\alpha_2, \alpha = \alpha_2 - \alpha_1.$$

$$\tan\alpha = \tan(\alpha_2 - \alpha_1) = \frac{\tan\alpha_2 - \tan\alpha_1}{1 + \tan\alpha_1 \tan\alpha_2} = \frac{k_2 - k_1}{1 + k_1 k_2}.$$

To get acute (or right) angle between two straight lines use formula:

$$\tan\alpha = \left|\frac{k_2 - k_1}{1 + k_1 k_2}\right|.$$

Note 1. There are always two positive angles between two straight lines, namely α and $\pi - \alpha$ (Fig.26).

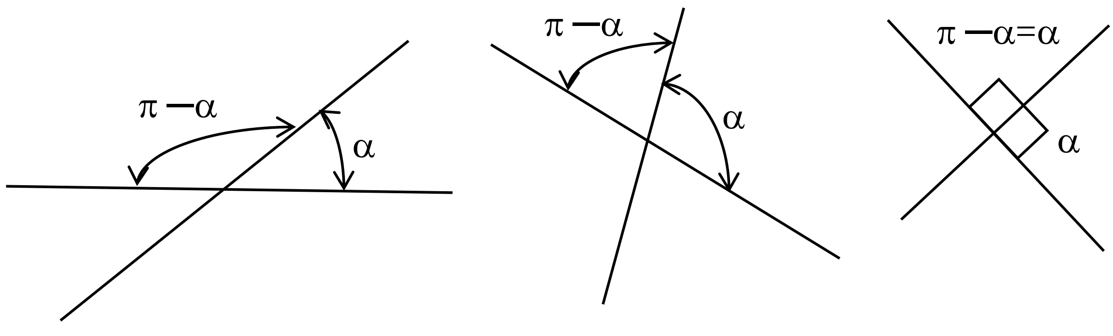

Figure 26

Note 2. From the obtained formulas we get some conditions for special positional relationships of straight lines:

1. *Two straight lines are perpendicular if and only if*

$$\vec{n}_1 \perp \vec{n}_2, \text{ i.e. } (\vec{n}_1, \vec{n}_2) = 0;$$

or

$$\vec{l}_1 \perp \vec{l}_2 \text{, i.e. } (\vec{l}_1, \vec{l}_2) = 0;$$

or

$$\cot\alpha = 0 \text{, i.e. } 1 + k_1 k_2 = 0 \Rightarrow k_2 = \frac{-1}{k_1}.$$

2. *Two straight lines are parallel if and only if*

$$\vec{n}_1 \parallel \vec{n}_2 \text{, i.e. } \frac{A_1}{A_2} = \frac{B_1}{B_2};$$

or

$$\vec{l}_1 \parallel \vec{l}_2 \text{, i.e. } \frac{m_1}{m_2} = \frac{n_1}{n_2};$$

or

$$\tan\alpha = 0 \text{, i.e. } k_2 - k_1 = 0 \Rightarrow k_1 = k_2.$$

Example. Find the equations of the straight lines parallel and perpendicular to the straight line $2x+y-3=0$ if they pass through the point $A(3;0)$ (Fig.27).

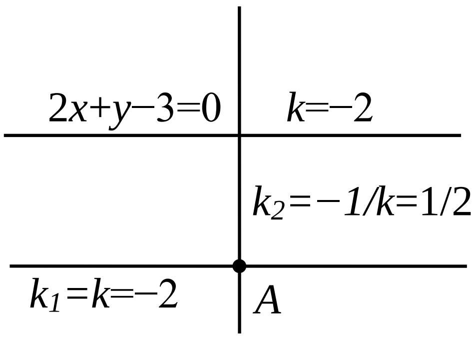

Figure 27

The slope of the given straight line is equal to the coefficient of x when y is expressed from the equation:

$$y = -2x + 3 \Rightarrow k = -2.$$

Since the first straight line is parallel to the initial straight line then its slope

$$k_1 = k = -2.$$

Therefore the equation of the first straight line is

$$y - y_A = k_1(x - x_A) \Leftrightarrow y - 0 = -2(x - 3) \Leftrightarrow 2x + y - 6 = 0.$$

Since the second straight line is perpendicular to the initial straight line then its slope

$$k_2 = -1/k = 1/2.$$

Therefore the equation of the first straight line is

$$y - y_A = k_2(x - x_A) \Leftrightarrow y - 0 = 1/2 \cdot (x - 3) \Leftrightarrow x - 2y - 3 = 0.$$