Lecture #13: Second Order Lines in Plane

13.1 General Definitions

Definition. Locus of the points with coordinates satisfying the general equation of the second degree is the curve of the second order.
At the same time the equation

$$\Phi(x, y) = Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0,$$

where $A^2 + B^2 + C^2 \neq 0$, is called *the general equation of the second order curve*.

The second order curves are, for example, circle, ellipse, hyperbola, parabola, pair of straight lines, etc.

13.2 Circle

Definition. Circle is a locus of the point which moves so that its distance from a fixed point, called the center, is equal to a given distance (Fig.28). The given distance is called the radius of the circle

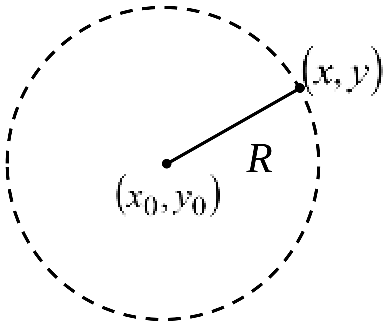

Figure 28

$$d = \sqrt{(x - x_0)^2 + (y - y_0)^2} = R \text{ or}$$

$$(x - x_0)^2 + (y - y_0)^2 = R^2.$$

This equation is a *canonical equation of the circle*.

Distinguishing features of circle equation:

1. The coefficients of x^2 and y^2 are equal to each other.
2. The coefficients of xy is equal to zero.

Suppose we have equation

$$\Phi(x, y) = x^2 + y^2 + 2Dx + 2Ey + F = 0.$$

When will it be a circle equation?

First we should rearrange the terms and complete the square in both variables *x* and *y*.

Completing the square is the procedure that consists basically of adding and subtracting certain quantities to the second-degree equation to form the sum of two perfect squares. When both the first- and the second-degree members of the same variable are known, the square of one-half the coefficient of the first-degree term should be added and subtracted. This will allow the quadratic equation to be factored into the sum of two perfect squares.

Therefore the given above equation may be rewritten in the following way:

$$x^2 + 2Dx + D^2 - D^2 + y^2 + 2Ey + E^2 - E^2 + F = 0$$

$$(x + D)^2 + (y + E)^2 = D^2 + E^2 - F$$

There are three different cases:

1) $D^2 + E^2 - F > 0$.

Then this equation is the equation of the circle with the origin in the point $(-D, -E)$ and of the radius equal to $\sqrt{D^2 + E^2 - F}$.

2) $D^2 + E^2 - F = 0 \Leftrightarrow$ It is a point $(-D, -E)$

3) $D^2 + E^2 - F < 0 \Rightarrow$ Radius of the circle is imaginary. In this case the given equation does not represent any real geometrical locus.

Example. Suppose we have the equation $x^2 + y^2 - 2x + 4y - 11 = 0$.

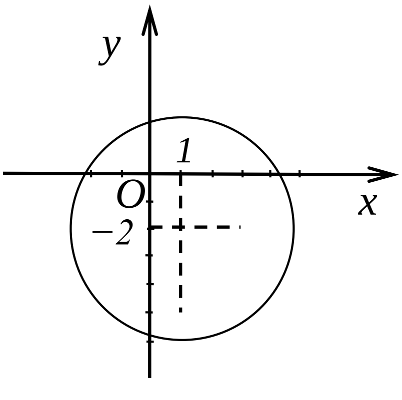

Figure 29

If that is equation of the circle then find the origin and the radius of this circle.

After completing the squares we obtain

$$(x - 1)^2 + (y + 2)^2 = 16.$$

Thus, the origin is $(x_0, y_0) = (1, -2)$, the radius is $R = 4$ (Fig.29).

13.3 Conic sections

Definition. The locus of a point *P*, which moves so that its distance from a fixed point is always in a constant ratio to its perpendicular distance from a fixed straight line is called a *conic section*.

The fixed point is called the *Focus* (pl. focuses or foci) and is usually denoted by F.

The constant ratio is called the *Eccentricity* and is denoted by ε.

The fixed straight line is called the *Directrix*.

That is why this property of points of conic sections to save ratio of distances is called *the focal-directorial property* of the conic sections.

Definition 1. When $\varepsilon = 1$ the conic section is called *parabola*.

Definition 2. When $0 \le \varepsilon < 1$ the conic section is called *ellipse*.

Definition 3. When $\varepsilon > 1$ the conic section is called *hyperbola*.

Note 1. At $\varepsilon = 0$ the ellipse becomes the circle.

Note 2. The name Conic Section is derived from the fact that these curves were first obtained as plane sections of a right circular cone (Fig.30). A circle is formed when a cone is cut perpendicular to its axis. An ellipse is produced when the cone is cut

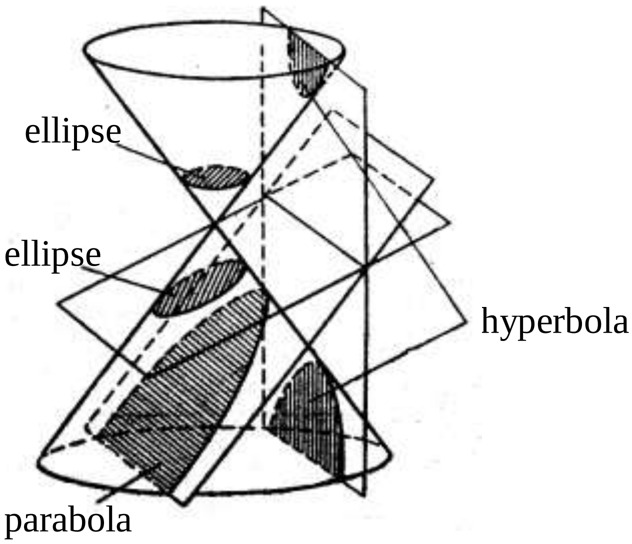

Figure 30

obliquely to the axis and the surface. A hyperbola results when the cone is intersected by a plane parallel to the axis, and a parabola results when the intersecting plane is parallel to an element of the surface.

Note 3. It will be shown later that conic sections are curves of the second order.

13.4 Canonical Equation of Parabola

Definition. Parabola is a locus of point with its distance from some fixed point F equal to its distance from some straight line.

It is clear that new definition is equivalent to the old one.

Point F is called a focus.

Straight line is called a directrix.

To derive equation of the parabola we consider point $F(p/2,0)$ as focus and straight line $x=-p/2$ (Fig.31).

$$\varepsilon=\frac{r_1}{r_2}=1\Leftrightarrow r_1^2=r_2^2$$

$$r_1^2=\left(x-\frac{p}{2}\right)^2+(y-0)^2$$

$$r_2^2=\left(x+\frac{p}{2}\right)^2$$

$$x^2-px+\frac{p^2}{2^2}+y^2=x^2+px+\frac{p^2}{4}$$

$$\boxed{y^2=2px}$$

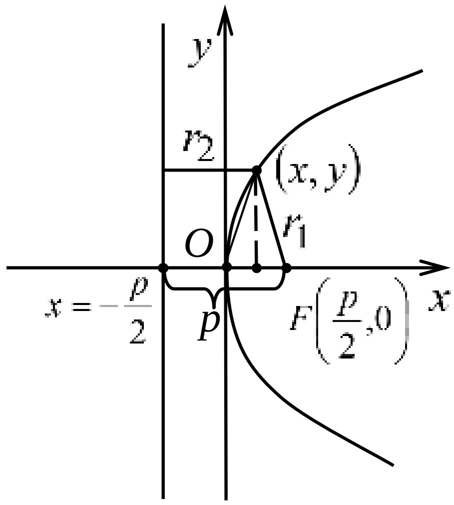

Figure 31

That is *a canonical equation of parabola*.

Properties of parabola graph:

1) Parabola is situated in a half plane with positive abscissa. Indeed,

$$y^2\geq 0\Leftrightarrow x\geq 0.$$

2) $|y|$ is increasing if x is increasing.

3) Ox is axis of symmetry. Indeed, for one value of x we have two values for y which differ only in sign.

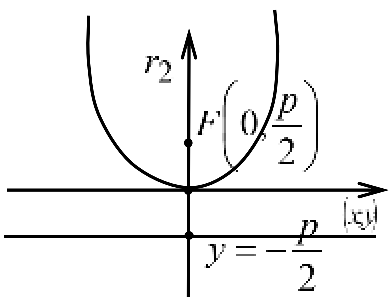

Figure 32

If we change x by y and y by x in the canonical equation of parabola, then we get

$$\boxed{x^2=2py}$$

That is the canonical equation of parabola with axis of symmetry Oy (Fig.32).

Point O is called vertex of parabola.

Value p is distance between vertex and directrix.

Note. If point (x_0, y_0) is a vertex of parabola and directrix is parallel to one of axes then canonical equation of parabola has form

$$(y-y_0)^2 = 2p(x-x_0) \text{ or } (x-x_0)^2 = 2p(y-y_0).$$

And new parabola is obtained by shift of

$$y^2 = 2px \quad (x^2 = 2py)$$

on x_0 along the axis Ox and on y_0 along the axis Oy.

Example. Reduce the equation of the parabola $x^2 + 2x + 4y - 7 = 0$ to the canonical form and plot the graph of this parabola.

After completing the squares we obtain

$$(x+1)^2 + 4y - 8 = 0 \text{ or } (x+1)^2 + 4(y-2) = 0 \text{ or}$$

$$(x+1)^2 = -4(y-2).$$

Thus, the vertex is $(x_0, y_0) = (-1,2)$ and $p = -2$. Graph of this parabola is presented on Fig.33.

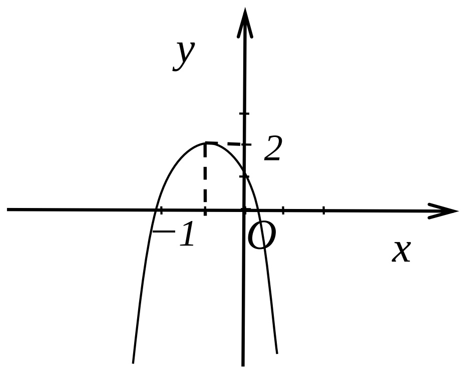

Figure 33

13.5 Canonical Equation of Ellipse

Definition. Ellipse is a locus of points with constant sum of distances from two fixed points called foci.

Let us find canonical equation of Ellipse. Suppose, foci are in points $F_1(-c,0), F_2(c,0)$ and the sum of distances is equal to 2a, where a>c (Fig.34). Then

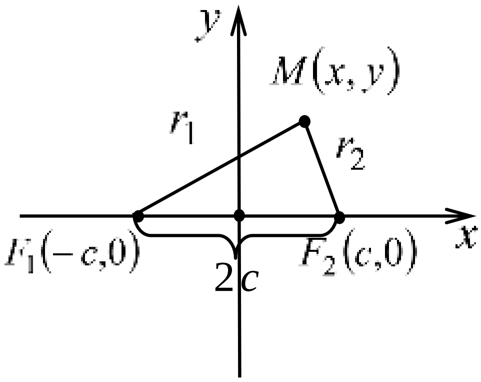

Figure 34

$$r_1 + r_2 = 2a,$$
$$r_1^2 = (x+c)^2 + y^2,$$
$$r_2^2 = (x-c)^2 + y^2,$$
$$\sqrt{(x+c)^2+y^2} = 2a - \sqrt{(x-c)^2+y^2},$$
$$\left(\sqrt{(x+c)^2+y^2}\right)^2 = \left(2a - \sqrt{(x-c)^2+y^2}\right)^2,$$

$$(x+c)^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2+y^2} + (x-c)^2 + y^2,$$
$$x^2 + 2xc + c^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2+y^2} + x^2 - 2xc + c^2 + y^2,$$
$$2xc = 4a^2 - 4a\sqrt{(x-c)^2+y^2} - 2xc,$$
$$4xc - 4a^2 = -4a\sqrt{(x-c)^2+y^2},$$
$$\left(xc - a^2\right)^2 = \left(-a\sqrt{(x-c)^2+y^2}\right)^2,$$
$$x^2c^2 - 2a^2xc - a^4 = a^2(x-c)^2 + a^2y^2,$$
$$x^2c^2 - 2a^2xc + a^4 = a^2x^2 - 2a^2xc + a^2c^2 + a^2y^2,$$
$$a^4 - a^2c^2 = a^2x^2 - x^2c^2 + a^2y^2,$$
$$a^2\left(a^2 - c^2\right) = \left(a^2 - c^2\right)x^2 + a^2y^2,$$
$$1 = \frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2},$$

Let us denote by $b^2 = a^2 - c^2$, since $a>c$. Then the last equation has the following form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

called *the canonical equation of the ellipse*.

Note. that $a>b$. At that

the value a is called *a major semi-axis*;

the value b is called *a minor semi-axis*.

Let us check that we do not have extraneous roots obtained at calculating the second power of expressions.

From canonical equation: $r_1 = \sqrt{(x+c)^2 + y^2} = \sqrt{(x+c)^2 + b^2\left(1 - \frac{x^2}{a^2}\right)} =$

$= \sqrt{x^2\left(1 - \frac{b^2}{a^2}\right) + 2cx + a^2} = \sqrt{x^2 \frac{c^2}{a^2} + 2cx + a^2} = \sqrt{\left(a + \frac{c}{a}x\right)^2} = \left|a + \frac{c}{a}x\right|.$

But $\frac{c}{a} < 1$ and $\left(\frac{x^2}{a^2} \le 1 \Rightarrow |x| \le a\right)$.

Therefore, $r_1 = a + \frac{c}{a}x$.

In a similar way we get $r_2 = a - \frac{c}{a}x$.

Thus, $r_1 + r_2 = 2a$.

Note. If $a = b \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2} = 1 \Leftrightarrow x^2 + y^2 = a^2$, i.e. the ellipse with equal semi-axes is a circle.

$$c^2 = a^2 - b^2 = 0 \Leftrightarrow \text{For circle } F_1 = F_2 = (0,0)$$

Properties of ellipse graph:

1) Since variables in ellipse equation are squared then if $M(x, y)$ belongs to ellipse then points $(-x, y)$, $(x, -y)$, $(-x, -y)$ belong to ellipse, too. It means that ellipse has two axes of symmetry, namely *Ox* and *Oy*.

Center of symmetry is *a center* of the ellipse.

Points of intersections with axes O*x* and O*y* are *the vertices* of the ellipse.

$$(-a,0), (a,0), (0,b), (0,-b).$$

2) From canonical equation: $\frac{x^2}{a^2} \le 1$ and $\frac{y^2}{b^2} \le 1 \Rightarrow |x| \le a, \;\; |y| \le b$.

It means that a graph of ellipse is situated inside the rectangle.

3) Let $\bar{x} = x$, $\bar{y} = \frac{a}{b}y$ be new variables.

If $\bar{x}^2 + \bar{y}^2 = a^2$, then $x^2 + \frac{a^2}{b^2}y^2 = a^2 \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

It means that graph of ellipse can be obtained by pressing the circle in the direction of axis *Oy*. Result of the analysis made above is presented on Fig.35.

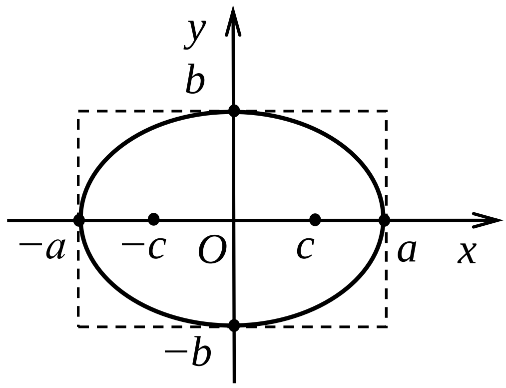

Figure 35

Eccentricity of ellipse ε can be found as the ratio of value of major axis and distance between foci, i.e.

$$\varepsilon = \frac{2c}{2a} = \frac{c}{a}.$$

From here it follows that

$$0 < \varepsilon < 1;$$
$$r_1 = a + \varepsilon x;$$
$$r_2 = a - \varepsilon x.$$

Two straight lines perpendicular to the major axis and situated symmetrically with distance a/ε from center are directrices of ellipse.

Let us show that *the focal-directorial property* is valid for such a definition of ellipse, i.e.

$$\frac{r}{d} = \varepsilon .$$

Since $\varepsilon < 1$, then $\frac{a}{\varepsilon} > a$. It means that directrices $x = \pm\frac{a}{\varepsilon}$ are situated outside the rectangle of ellipse. So (Fig.36),

$$\frac{r_1}{d_1} = \frac{a+\varepsilon x}{\frac{a}{\varepsilon} + x} = \frac{a+\varepsilon x}{\frac{a+\varepsilon x}{\varepsilon}} = \varepsilon;$$

$$\frac{r_2}{d_2} = \frac{a-\varepsilon x}{\frac{a}{\varepsilon} - x} = \frac{a-\varepsilon x}{\frac{a-\varepsilon x}{\varepsilon}} = \varepsilon.$$

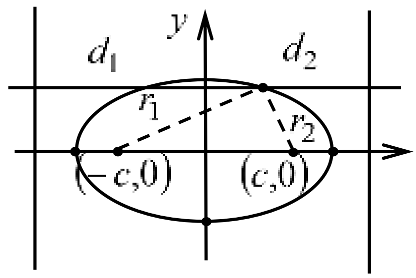

Figure 36

Thus, the property is valid.

Suppose that foci are situated on the axes *Oy* and the sum of distances from an ellipse point to fici is equal to 2*b* (Fig. 37). Then in the similar way we can get:

$$r_1 + r_2 = 2b \Leftrightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ where } \quad b>a \text{ (Fig.37)},$$

$$c^2 = b^2 - a^2,\ \varepsilon = \frac{c}{b},$$

We can denote

$$r_1 = b + \varepsilon y,$$

$$r_2 = b - \varepsilon y;$$

where

$y = \pm\dfrac{b}{\varepsilon}$ are directrices.

Here

a is called a minor semi-axis,

b is called a major semi-axis.

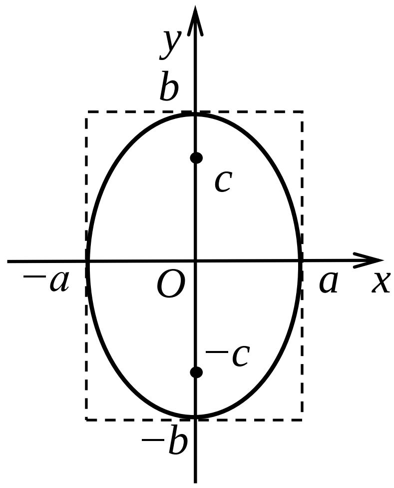

Figure 37

Note 1. If center of the ellipse is in the point (x_0, y_0) but ellipse axes are parallel to coordinate axes, then the canonical equation of this ellipse has the following form (Fig.38):

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} = 1.$$

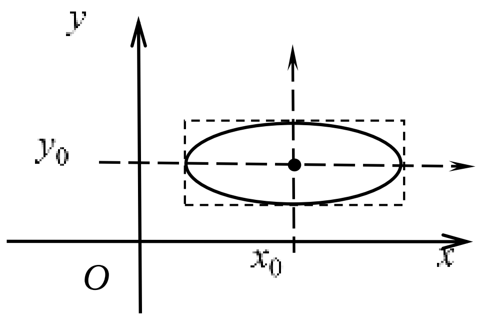

Figure 38

In this case,

$F_1(-c + x_0,\ 0 + y_0),\ F_2(c + x_0,\ 0 + y_0)$ are foci,

$x = x_0 \pm \dfrac{a}{\varepsilon}$ are directrices.

Example. Plot the graphs of the following ellipses:

1) $\dfrac{x^2}{20} + \dfrac{y^2}{5} = 1$; 2) $4x^2 + y^2 = 36$; 3) $x^2 - 4x + 4y^2 + 8y - 28 = 0$.

1) The first ellipse is an ellipse with semi-axes $a = \sqrt{20} = 2\sqrt{5}$, $b = \sqrt{5}$. Its graph is presented on Fig.39*a*.

2) To plot the second ellipse we should first reduce its equation to the canonical form dividing the equation by **36**:

$$\frac{4x^2}{36} + \frac{y^2}{36} = \frac{36}{36} \Leftrightarrow \frac{x^2}{9} + \frac{y^2}{36} = 1.$$

That is an ellipse with semi-axes $a = \sqrt{9} = 3$, $b = \sqrt{36} = 6$. Its graph is presented on

Fig.39*b*.

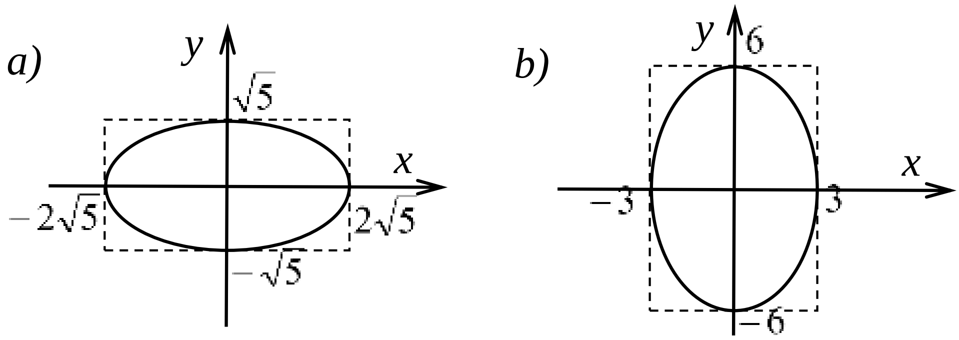

Figure 39

3) To plot the last ellipse we should complete the squares in x and y in the equation:

$$x^2-4x+4y^2+8y-28=x^2-4x+4\left(y^2+2y\right)-28=$$

$$=x^2-4x+4-4+4\left(y^2+2y+1-1\right)-28=$$

$$=(x-2)^2-4+4(y+1)^2-4-28==(x-2)^2+4(y+1)^2-36=0.$$

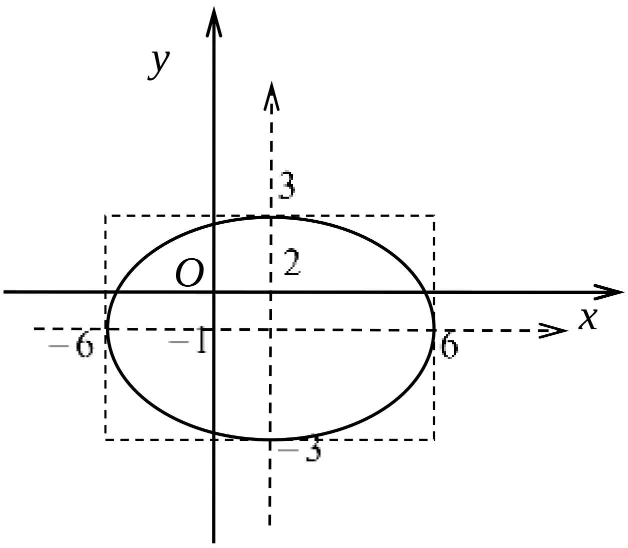

Figure 40

After transposing free term **36** to the right-hand side and dividing the equation by it we get:

$$\frac{(x-2)^2}{36}+\frac{(y+1)^2}{9}=1.$$

That is an ellipse with semi-axes $a=\sqrt{36}=6$, $b=\sqrt{9}=3$ and the vertex (2;–1). Its graph is presented on Fig.40.

Note 2. Canonical equation of the ellipse could be found directly from the initial definition, namely as a curve with $0\le\varepsilon<1$. Let us complete this derivation.
Suppose

$$PO=P'O=a,\ FO=c,\ MO=d_0$$

where F is the focus, O is the center, P and P' are points of the ellipse situated on the straight line passing through focus and center, M is a point of directrix situated on the same straight line (Fig. 41). Then from the definition of eccentricity we have

$$\frac{a-c}{d_0-a}=\varepsilon \text{ or } a-c=\varepsilon d_0-\varepsilon a,$$

$$\frac{a+c}{d_0+a}=\varepsilon \text{ or } a+c=\varepsilon d_0+\varepsilon a.$$

Subtraction and addition of these two equations give

$$2c=2a\varepsilon \text{ or } c=a\varepsilon,$$

$$2a=2d_0\varepsilon \text{ or } d_0=\frac{a}{\varepsilon}.$$

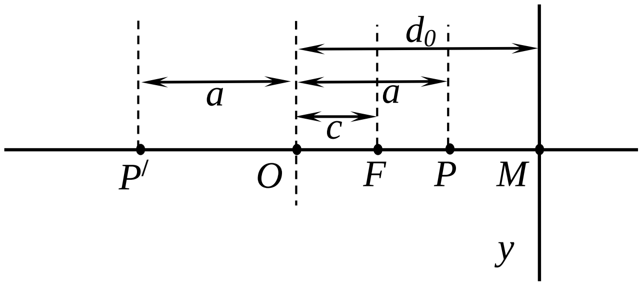

Figure 41

Note All these formulas are true for any value of eccentricity.

Let us place the center of the ellipse at the origin so that the focus lies on the positive semi-axis O*x* (Fig.42). Then for any arbitrary point (x, y) of ellipse we have

$$\varepsilon=\frac{\sqrt{(x-a\varepsilon)^2+y^2}}{\frac{a}{\varepsilon}-x} \text{ or}$$

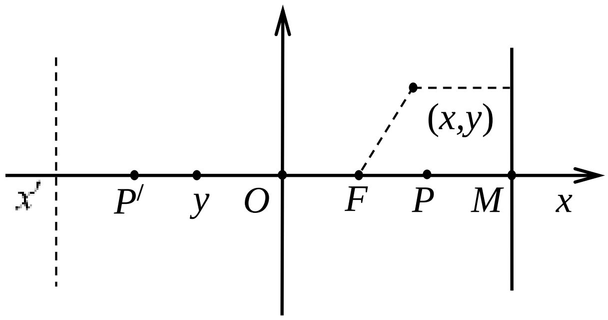

Figure 42

$$\sqrt{(x-a\varepsilon)^2+y^2}=a-x\varepsilon.$$

Squaring and expanding both sides give

$$x^2-2xa\varepsilon+a^2\varepsilon^2+y^2=a^2-2ax\varepsilon+x^2\varepsilon^2.$$

Canceling like terms and transposing terms in *x* to the left-hand side of the equation give

$$x^2-x^2\varepsilon^2+y^2=a^2-a^2\varepsilon^2.$$

Removing a common factor gives

$$x^2(1-\varepsilon^2)+y^2=a^2(1-\varepsilon^2).$$

Dividing both sides of this equation by the right-hand member gives

$$\frac{x^2}{a^2}+\frac{y^2}{a^2(1-\varepsilon^2)}=1.$$

From equation we obtain *y*-intercept of ellipse

$$b^2=a^2(1-\varepsilon^2) \text{ or } b=a\sqrt{1-\varepsilon^2}.$$

so that the equation becomes

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \text{ where } a\le b.$$

This is the equation of an ellipse in the canonical form.

Note 3. In Note 2 we, actually, have found only the right branch of ellipse. The left branch could be found in the similar way but with the focus $F'(-a\varepsilon, 0)$ and the directrix $x = -d_0 = -\dfrac{a}{\varepsilon}$. Equation of the left branch gives the same canonical equation.

13.6 Canonical Equation of Hyperbola

Definition. Hyperbola is locus of points with constant absolute value of difference of distances from two fixed points called foci.

Suppose, the foci are situated on the axis O*x* symmetrically with respect to the origin (Fig.43), $F_1(c,0)$, $F_2(-c,0)$ and

$$|r_1 - r_2| = 2a,$$

where

$$r_1 = \sqrt{(x+c)^2 + y^2}, \quad r_2 = \sqrt{(x-c)^2 + y^2}.$$

Squaring and expanding both sides give

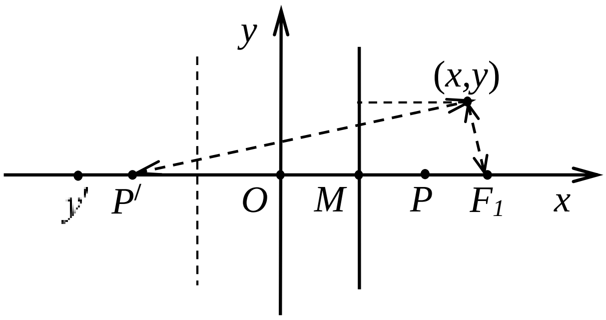

Figure 43

$$4a^2 = (x+c)^2 + y^2 + (x-c)^2 + y^2 - 2\sqrt{\left((x+c)^2 + y^2\right)\left((x-c)^2 + y^2\right)},$$

$$4a^2 = 2x^2 + 2y^2 + 2c^2 - 2\sqrt{\left((x+c)^2 + y^2\right)\left((x-c)^2 + y^2\right)},$$

$$-2a^2 + \left(x^2 + y^2 + c^2\right) = \sqrt{\left((x+c)^2 + y^2\right)\left((x-c)^2 + y^2\right)},$$

$$-2a^2 + \left(x^2 + y^2 + c^2\right) = \sqrt{\left(x^2 + c^2 + y^2 - 2cx\right)\left(x^2 + c^2 + y^2 + 2cx\right)},$$

$$4a^4 + \left(x^2 + y^2 + c^2\right)^2 - 4a^2\left(x^2 + y^2 + c^2\right) = \left(x^2 + y^2 + c^2\right)^2 - 4c^2x^2,$$

$$4a^4 - 4a^2x^2 - 4a^2y^2 - 4a^2c^2 + 4x^2c^2 = 0,$$

$$a^4 - a^2x^2 - a^2y^2 - a^2c^2 + x^2c^2 = 0,$$

$$x^2\left(c^2 - a^2\right) - a^2y^2 = a^2\left(c^2 - a^2\right).$$

Dividing both sides of this equation by the right-hand member gives

$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1.$$

Note. From Fig. 43 It follows that for any triangle with vertices F_1, F_2 and any point of hyperbola we have

$$\begin{cases} r_1 < r_2 + 2c \\ r_2 < r_1 + 2c \end{cases} \Leftrightarrow \begin{cases} r_1 - r_2 < 2c \\ r_2 - r_1 < 2c \end{cases} \Leftrightarrow |r_1 - r_2| = 2a < 2c.$$

Thus, $c>a$ and therefore $c^2 - a^2 > 0$ and the equation becomes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where $b^2 = c^2 - a^2$. This equation is called *the canonical equation of hyperbola*.

Note 1. The equation of hyperbola could be found directly from initial definition of hyperbola as we got it for ellipse. Since for hyperbola $\varepsilon > 1$, we have

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-\varepsilon^2)} = 1,$$

$$\frac{x^2}{a^2} - \frac{y^2}{a^2(\varepsilon^2-1)} = 1,$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where $b^2 = a^2(\varepsilon^2 - 1)$.

Note 2. From the formulas obtained above, namely

$$c = a\varepsilon,\ d_0 = \frac{a}{\varepsilon},$$

it follows that directrices are situated closer to the origin than foci.

Equations of the directrices are

$$x = \pm\frac{a}{\varepsilon},$$

where

$$\varepsilon = \frac{a}{c}.$$

Properties of hyperbola graph:

1) Axes Ox and Oy are axes of hyperbola symmetry. The origin $(0,0)$ is *a center* of hyperbola.

2) $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1 \Leftrightarrow \dfrac{x^2}{a^2}=1+\dfrac{y^2}{b^2} \Rightarrow \dfrac{x^2}{a^2}\geq 1 \Rightarrow |x|>|a|$, i.e. the graph of hyperbola is situated outside the band of width 2*a* unbounded in vertical direction.

3) If $y=0$ then $x=\pm a$. Obtained points $V_1(a,0), V_2(-a,0)$ of intersection with the axis O*x* are called *the vertices* of this hyperbola. The nomenclature of the hyperbola is slightly different from that of an ellipse. *The transverse axis* is of length 2*a* and is the distance between the vertices of the hyperbola. *The conjugate axis* is of length 2*b* and is perpendicular to the transverse axis.

4) For any point of hyperbola from the right semi-plane (*x*>0) we have

$$y_h=\pm b\sqrt{\frac{x^2}{a^2}-1}=\pm\frac{b}{a}x\pm b\sqrt{\frac{x^2}{a^2}-1}\mp\frac{b}{a}x=$$

$$=\pm\frac{b}{a}x\pm\left(b\sqrt{\frac{x^2}{a^2}-1}-\frac{b}{a}x\right)=\pm\frac{b}{a}x\pm\frac{b}{a}\left(\sqrt{x^2-a^2}-x\right)=$$

$$=\pm\frac{b}{a}x\pm\frac{b}{a}\left(\frac{x^2-a^2-x^2}{\sqrt{x^2-a^2}+x}\right)=\pm\frac{b}{a}x\pm\frac{b}{a}\left(\frac{-a^2}{\sqrt{x^2-a^2}+x}\right).$$

Thus

$$y_h-\left(\pm\frac{b}{a}\right)=\frac{b}{a}\left(\frac{-a^2}{\sqrt{x^2-a^2}+x}\right)\xrightarrow[x\to+\infty]{}0.$$

It means that while $x\to+\infty$ the branch of hyperbola tends (becomes extremely close) to the straight lines $y=\pm\dfrac{b}{a}x$. These straight lines

$$y=\pm\frac{b}{a}x$$

are called *the asymptotes* of the hyperbola.

Result of the analysis made above is presented on Fig.44.

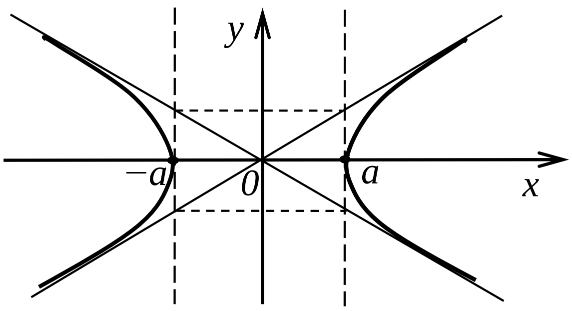

Figure 44

Whenever the foci are on the O*y* axis and the directrices are straight lines of the form $y=\pm d_0$, the equation of the hyperbola takes form

$$-\frac{x^2}{b^2(\varepsilon^2-1)}+\frac{y^2}{b^2}=1,$$

$$-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

where $F_1(0,c)=F_1(0,b\varepsilon)$, $F_2(0,-c)=F_2(0,-b\varepsilon)$, $|r_1-r_2|=2b$, $a^2=b^2(\varepsilon^2-1)$.

This equation represents a hyperbola with its transverse axis on the axis Oy called *the Conjugate Hyperbola*.

Note 1. Graph of the conjugate hyperbola possesses the same properties as that of common hyperbola, except properties 2) and 3). Vertices of the conjugate hyperbola are points $V_1(0,b)$, $V_2(0,-b)$. Moreover,

$$-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{y^2}{b^2}=1+\frac{x^2}{a^2}\geq 1 \Rightarrow |y|>b.$$

Graph of conjugate hyperbola is presented by Fig. 45.

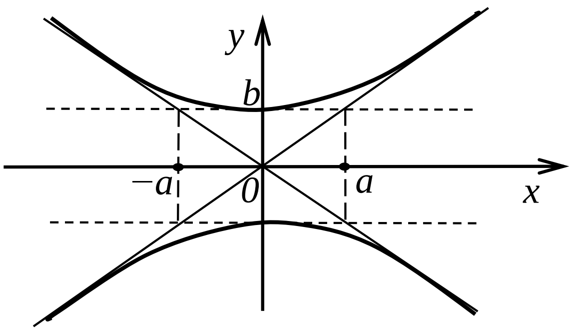

Figure 45

Note 2. If the center of hyperbola is in the point (x_0, y_0) but hyperbola axes are parallel to coordinate axes, then the canonical equation of this hyperbola has the following form

$$\frac{(x-x_0)^2}{a^2}-\frac{(y-y_0)^2}{b^2}=1$$

or

$$-\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}=1.$$

Example. Plot the graph of the following hyperbola:

$$x^2-4x-4y^2-8y-36=0.$$

To plot this hyperbola we should first reduce its equation to the canonical form. Let us complete the squares in x and y in the equation:

$$x^2-4x-4y^2-8y-36=x^2-4x-4\left(y^2+2y\right)-36=$$
$$=x^2-4x+4-4-4\left(y^2+2y+1-1\right)-36=$$

$$= (x-2)^2 - 4 - 4(y+1)^2 + 4 - 36 =$$

$$= (x-2)^2 - 4(y+1)^2 - 36 = 0.$$

After transposing free member **36** to the right-hand side and dividing the equation by it we get:

$$\frac{(x-2)^2}{36} - \frac{(y+1)^2}{9} = 1.$$

That is a hyperbola with semi-axes

$$a = \sqrt{36} = 6,\ b = \sqrt{9} = 3$$

and the vertex (2;–1).

Its graph is presented on Fig.46.

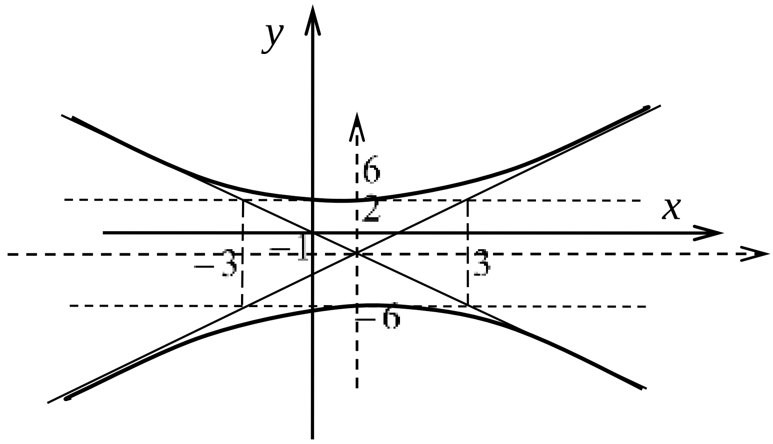

Figure 46