MATHEMATICAL ANALYSIS
Lecture #14: Preliminary Information

14.1 Basic notations

Mathematical Analysis (just Analysis) is the branch of mathematics dealing with limits and related theories, such as differentiation, integration, measure, sequences, series, and analytic functions

The following symbols will be further used:

– $\forall x$ – instead of words “for all *x*”, “for each *x*”, “for any *x*”, “for every *x*”.
– $\exists x$ means that “there exists such *x* that” or “we can take such *x* that” or “at least for one *x*”.
– $\exists! x$ means “there exists only *x*”

We shall also use the following notations:

– $\Rightarrow$ is used instead of a word "follows". For example, the record $A \Rightarrow B$ means, that "B follows from A", "A results B".
– $\Leftrightarrow$ is used instead of word "equivalency". For example, the record $A \Leftrightarrow B$ means, that "B follows from A" and vice versa "A follows from B", so *A* and *B* are equivalent.

14.2 Sets

Definition. In general, a *set* is defined as totality of some objects (points, numbers, vectors, functions etc.), united by some common feature. The objects composing the set are called its *elements*.

If the given element x belongs to a set A then we will write down so: $x \in A$. Otherwise, if an element x doesn’t belong to set *A*, then will write: $x \notin A$.

Suppose that all elements of the set *A* have some common property. Then we write

$$A = \{x | ......\},$$

where instead of dots the mentioned property of all elements of set can be written.

Example. Let A be set of all x, for which the value of $\arcsin x$, then we should write, that

$$A = \{x | -1 \leq x \leq 1\}.$$

Definition. A set, elements of which are numbers, is called *numerical set*. There exist the standard numerical sets:

The set of natural numbers denoted as N or $N = \{1,2,3,\dots,n\}$;

The set of integers denoted as Z or $Z = \{\dots,-n,\dots,-1,0,1,2,3,\dots,n,\dots\}$;

The set of rational numbers denoted as Q or $Q = \{x|\ \exists m \in Z$ and $\exists\ n \in N :$ $x = m/n\}$;

The set of real numbers denoted as R or $R = \{-\infty,+\infty\}$;

The set of real numbers can be associated with points located on the coordinate line. Then, the following notation is introduced:

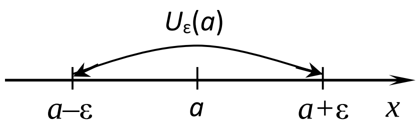
Definition. The numerical set is called *closed interval or segment* if its elements satisfy the condition:

$$[a,b] = \{x \mid a \le x \le b\}.$$

Definition. The numerical set is called *open interval or interval* if its elements satisfy the condition: $(a,b) = \{x \mid a < x < b\}$

Note. There exist also semi intervals $[a,b) = \{x \mid a \le x < b\}$ and $(a,b] = \{x \mid a < x \le b\}$. And semi axes $[a,+\infty) = \{x \mid a \le x < \infty\}$ or $(-\infty,b] = \{x \mid -\infty < x \le b\}$.

Definition. The interval $(a-\varepsilon,\ a+\varepsilon)$ is called ε-neighborhood of the point $x = a$ (Fig.) and denoted by $U_\varepsilon(a)$. Here ε is radius of the neighborhood. Thereby,



Figure

$$U_\varepsilon(a) = \{x \mid |x-a| < \varepsilon\}.$$

Let us introduce the basic operations on sets. The operations of sets can be illustrated by means of Boolean algebra in a graphic way. Such representations are called Euler-Venn's diagrams.

Definition. The *sum of sets A* and *B* is a set $C=A+B$ {or *union of sets* $C=A \cup B$} consisting of elements of either *A* or *B* or both of them.

Definition. *The product of sets A* and *B* is a set $C = A \cdot B$ {or *intersection of sets* $C=A \cap B$} consisting of elements of *A* and *B*.

Definition. *The difference of sets A* and *B* is a set $C = A - B$ (or $C=A\backslash B$) consisting of elements of *A* and not happening of *B*.

Note. Any complementary *set* $\overline{A}$ to a set *A* may be found as $\overline{A} = \Omega - A$.

Definition. It is said that a set A implies the set B and is denoted as $A \subset B$ if from the elements of A it follows the elements of B.

Definition. Two sets A and B are called equal $A=B$ if $A \subset B$ and $B \subset A$.

The Euler-Venn's diagrams corresponding to the given above definitions are shown on Figure 2.

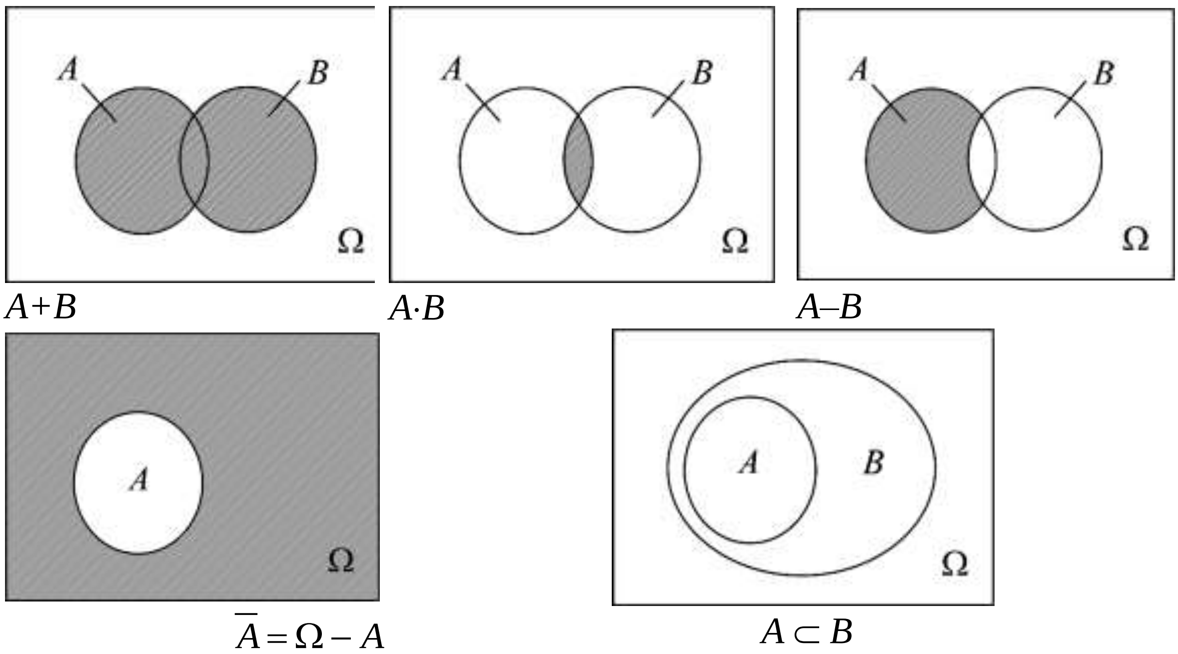


Figure 2

Properties of operations on sets are the following:

1) commutativity of sum and product sets: $A+B= B+ A$ and $A{\cdot}B =B{\cdot}A$;
2) associativity: $(A+B)+C= A+(B+C)$ and $A{\cdot}(B{\cdot}C)=(A{\cdot}B){\cdot}C$;
3) distributivity: $(A+B){\cdot}C= A{\cdot}C +B{\cdot}C$ and $A+(B{\cdot}C)=(A+B){\cdot}(A+C)$;
4) $A+A= A$ and $A{\cdot}A = A$;
5) $A+\Omega= \Omega$ and $A{\cdot}\Omega = A$;
6) $A+ \varnothing = A$ and $A\cdot \varnothing = \varnothing$;
7) $A+ \overline{A} = \Omega$ and $A\cdot \overline{A} = \varnothing$;
8) $\overline{\overline{A}}=A$ and $\overline{\varnothing}=\Omega$ and $\overline{\Omega} = \varnothing$;
9) $A\text{-}B= A{\cdot} \overline{B}$;
10) $A+B=A+ \overline{A}{\cdot}B$
11) $\overline{A+B} = \overline{A} \cdot \overline{B}$ and $\overline{A \cdot B} = \overline{A} + \overline{B}$.

14.3 Constants and Variables Quantities

Definition. A *constant* is a quantity whose numerical values remain fixed in some considered process. In this case, we will write that $x = const$.

Otherwise, a quantity x taking on various numerical values is called *variable*.

The variables are divided into *discrete* and *continuous*.

Definition. A variable is called *discrete* if it accepts only separate, isolated values.

Definition. A variable is called *continuous* if it accepts all intermediate values when it passes from one value to another one.

Definition. The set of all numerical values of a variable is called the *range of the variable*.

14.4 Functions of a single variable

Definition. A *function* is a rule that produces a correspondence between one set of elements, called as the domain, and a second set of elements, called the range, such that to each element in the domain corresponds one definite element in the range.

Because domains and ranges are sets of numbers, i.e. variables, and the rules matching their values are equations, then the equation $y = f(x)$ specifies a *function* y *of variable* x if to each value of the variable *x* there corresponds the one value of variable y.

Here, the variable x is called an *independent variable* or *argument* and the variable y is called a *dependent variable* or *function*. The relation between the variables x and y is called the *functional relation*. The letter f in the functional relation indicates that some kind of operations must be performed on the value of *x* in order to obtain the value of *y*.

Definition. The set of x values for which the values of the function y are determined by the rule $f(x)$ is called the *domain of the function definition* and is denoted by D_f.

Definition. If the function $y = f(x)$ is such that to a greater value of the x argument there corresponds a greater value of the function then the function $y = f(x)$ is called an *increasing*. If to a greater value of the x argument there corresponds a less value of the function then the function $y = f(x)$ is called *decreasing*.

Definition. A function $y = f(x)$ is called *bounded*, if there exists such number $M > 0$ that for each value of its argument *x* from the domain of the function definition the inequality $|f(x)| \leq M$ always is true. Otherwise the function is called *unbounded*.

Definition. A function $y = f(x)$ is called *bounded from above*, if there is such number Q, that the inequality $f(x) \leq Q$ for each value of its argument *x* from the domain of the function definition.

Similarly,

Definition. A function $y = f(x)$ is called *bounded from below*, if there is such number q, that the inequality $f(x) \geq q$ for each value of its argument *x* from the domain of the function definition.

Functions may be given by three basic ways.

1. Tabular representation of a function.

Here the values of the arguments $x: x_1, x_2, \ldots, x_n$ and the appropriate values of the function $y: y_1, y_2, \ldots, y_n$ are written out in a definite order in the table

x	x_1	x_2	$\ldots$	x_n
$y = f(x)$	y_1	y_2	$\ldots$	y_n

2. Graphical representation of a function

The collection of points in the xy-plane whose abscissas are the values of the independent variable and whose ordinates are the corresponding values of the function is called a graph (plot) of the given function.

The function graph is its image.

3. Analytical representation of a function

By *analytical expression* we will understand a series of symbols denoting certain mathematical operations that should be performed in a definite order:

$2^x + \sin x - \log 5x$, $\sqrt[3]{x^4 + 3x^2}$.

If the functional relation $y = f(x)$ is such that *f* denotes an analytical expression then it is said that the function is represented or defined analytically.

For a function represented analytically, the set of independent values *x* at which the analytical expression produces real values (or has a sense) is called a *domain of function definition*.

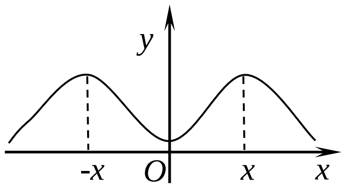
Definition. It is said that a function is given *explicitly* if the equation matching variables *x* and *y* is solved for *y* with respect to *x*.

Example.

$$y = \frac{x^3 + \lg x}{3 - \sin x}.$$

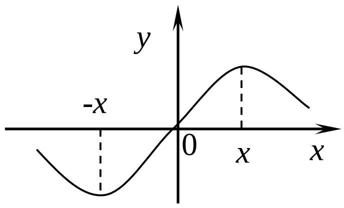
Definition. If a functional relation is not solved for *y*, then it is said that the function is given *implicitly*, e.g. $x10^y - \tan x = 1$,

Definition. A function $f(x)$ given on symmetric interval is called *even* if the following equality $f(-x) = f(x)$ is valid for all $x \in D_f$.



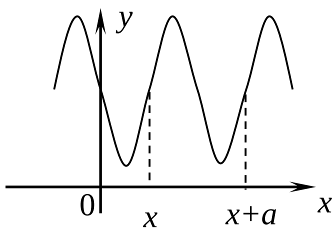
Obvious that the graph of the even function is symmetric about Oy axis.

Definition. A function $f(x)$ given on symmetric interval is called *odd* if for all the following equality $f(-x) = -f(x), \ \forall x \in D_f$ holds true for all $x \in D_f$.



It is easy to see that a graph of the odd function is symmetric about the origin.

Definition. A function $f(x)$ is said to be *periodic* with period *a* $(a \neq 0)$ (*a* being a nonzero constant) if the following equality $f(x+a) = f(x), \ \forall x \in D_f$, for all values of *x* in the domain D_f is valid.



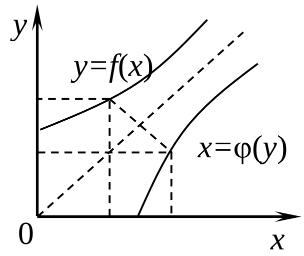
Geometrically, a periodic function can be defined as a function whose graph exhibits translational symmetry, i.e. the function repeats itself on intervals of length *a*.

Definition. An *inverse function* is a function that "reverses" another function.

Let us consider the function $y = f(x)$. Changing the values of x, involves the changing the variable y. On the contrary, any changing the value y results in the changing variable x. Consequently the relation $y = f(x)$ defines not only y as function of x, but and vice versa, it defines the value x as function of y. Thereby, from equality $y = f(x)$ it follows that there is reversal equality $x = \varphi(y)$. Here the function $x=\varphi(y)$ is called an *inverse* function to the function $y=f(x)$.

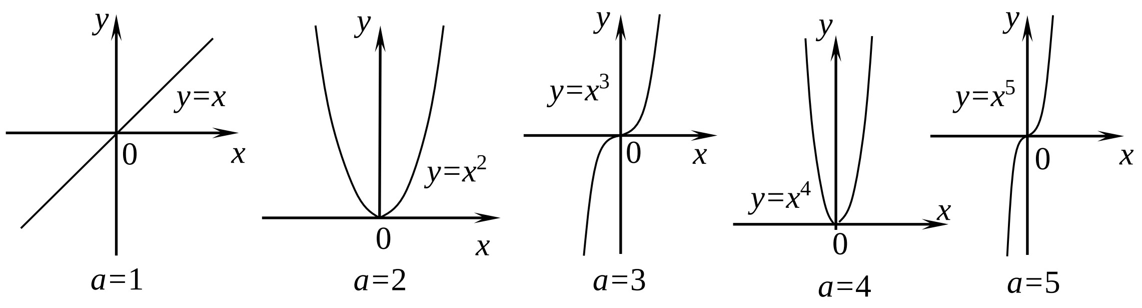
It is obvious that functions $y=f(x)$ and $x=\varphi(y)$ have the same graph, since the both equalities describe the same relation between x and y.

In the relation $x=\varphi(y)$ let us denote the function by y and the argument by x. Then we obtain the function $y=\varphi(x)$, which is also called inverse to given function $y=f(x)$. However, the graphs of such functions will be different. They represent two symmetric curves about straight line $y=x$.

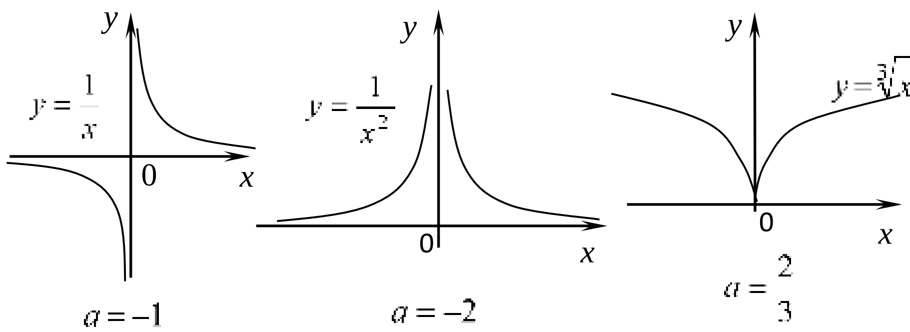


14.5 Basic Elementary Functions

1. Power function. $y=x^a$,where a is any real number. The graph of the power function depends on value of a.



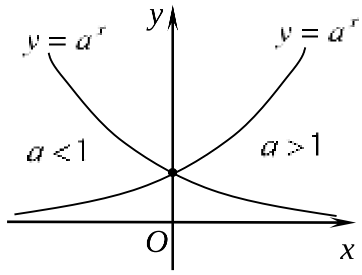
Let us note that a curve $y=x^3$ is called cubic parabola, a curve $y=x^4$ is called parabola of the forth power.

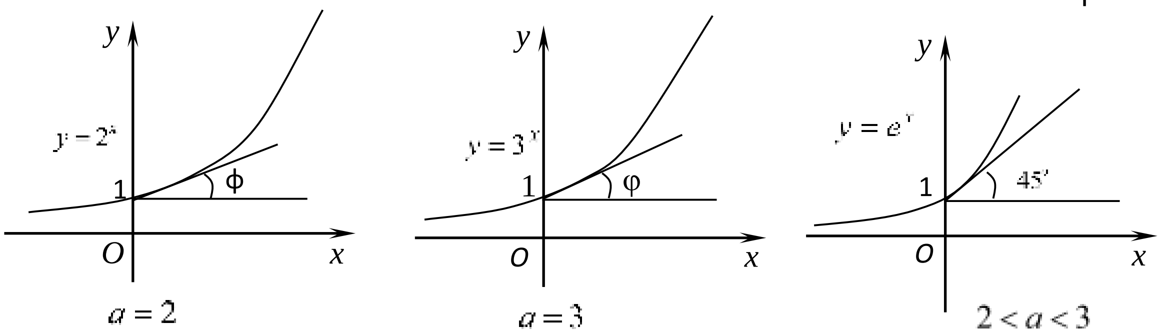


A curve $y=\sqrt[3]{x^2}$ is called semi-cubic parabola. The definition of the power function $y=x^a$ for any real power will be given later.

2. General exponential function. $y = a^x$, where $a > 0$ and $a \neq 1$.

The increasing number *a* involves increasing the angle φ between Ox axis and tangent line to the curve $y = a^x$ at the point $(0,1)$. If $a = 2$ then $\varphi \approx 34^\circ$, if $a = 3$ then $\varphi \approx 47^\circ$.

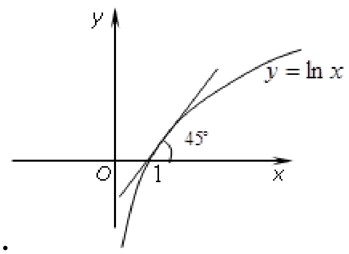




Suppose that the angle $\varphi \approx 45^\circ$ will correspond to this value. This number is denoted by e; it is irrational number equal to 2,71828. The function $y = e^x$ is called *exponential function*.

3. Logarithmic function. $y = \log_a x \;\; (a > 0,\ a \neq 1)$ is inverse to the function $y = a^x$.

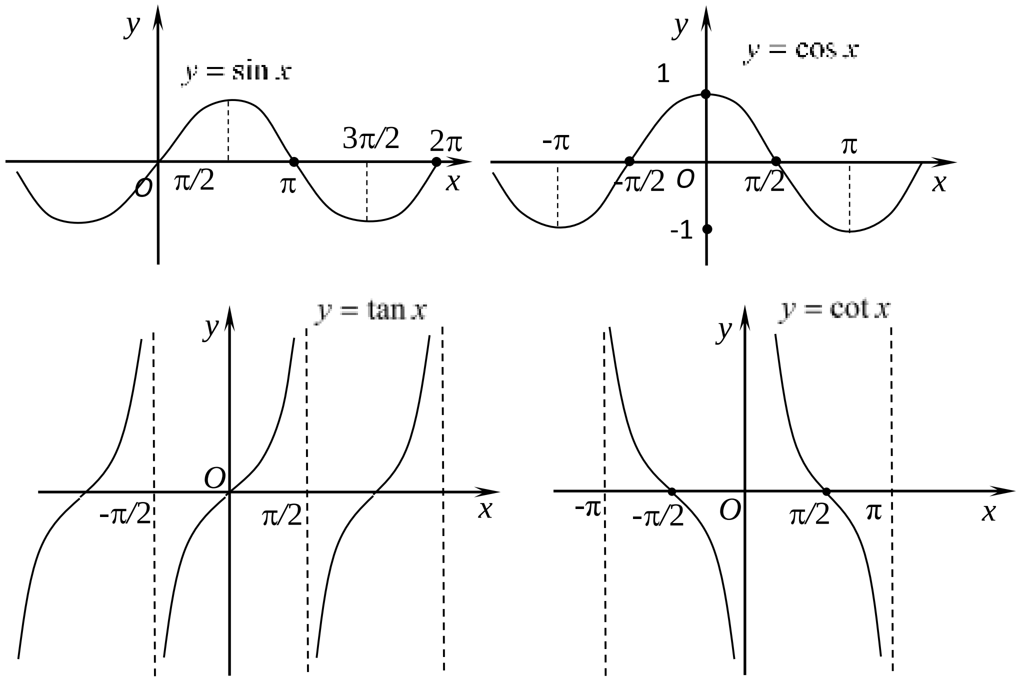
In practice it is often considered $a = e$ and $a = 10$ called the *natural logarithmic* $y = \ln x$ and *decimal logarithm* $y = \lg x$.



4. **Trigonometrical functions**. $y = \sin x$, $y = \cos x$, $y = \tan x$ and $y = \cot x$ are trigonometrical functions. These functions have period 2π and π relatively.

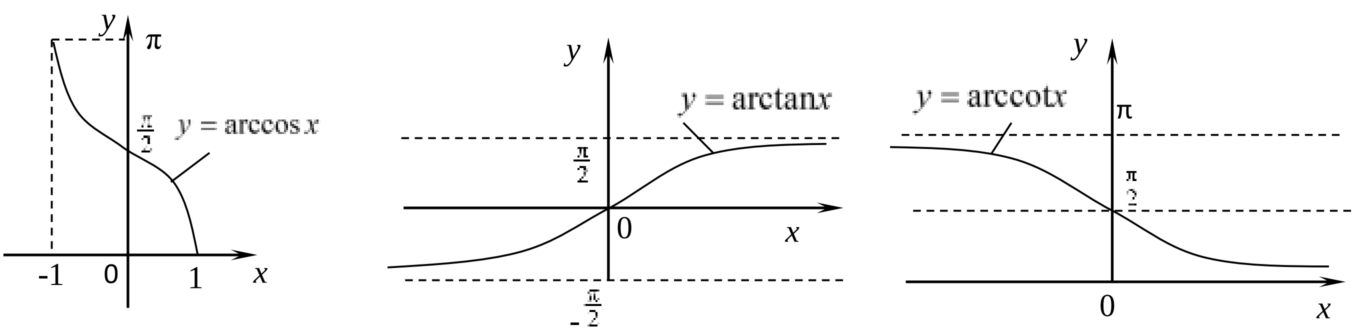
The functions $y = \sin x$, $y = \cos x$ are defined for all values of *x*, the functions $y = \tan x$ and $y = \sec x, \quad (\sec x = \dfrac{1}{\cos x})$ are defined everywhere except at the points

$x = (2k+1)\frac{\pi}{2}$, $(k = 0, \pm 1, \pm 2, ...)$.



The functions $y = \cot x$ and $y = \csc x, \left(\csc x = \frac{1}{\sin x}\right)$ are defined for all values of x except at points $x = k\pi$, $(k = 0, \pm 1, \pm 2, ...)$.

5. Inverse trigonometrical function. The functions $y = \arcsin x$, $y = \arccos x$ (Fig. 1.26), $y = \arctan x$ and $y = \mathrm{arc}\cot x$ (Fig. 1.27-1.28) are main branches of the functions multiple-valued $y = \mathrm{Arcsin}\ x$, $y = \mathrm{Arccos}\ x$, $y = \mathrm{Arc}\tan x$ и $y = \mathrm{Arc}\cot x$, which are inverse to the functions $y = \sin x$, $y = \cos x$, $y = \tan x$ и $y = \cot x$.



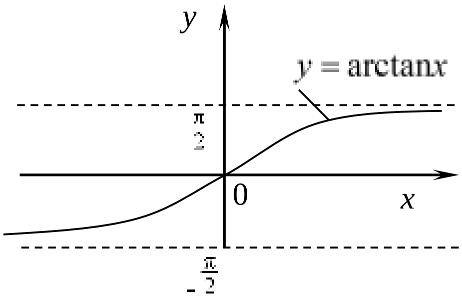


Fig. 1.27

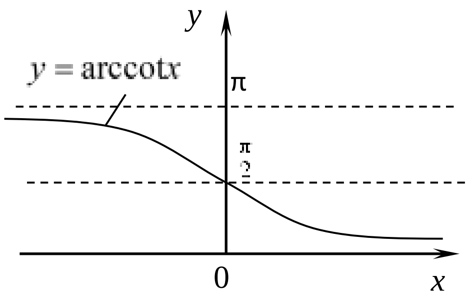
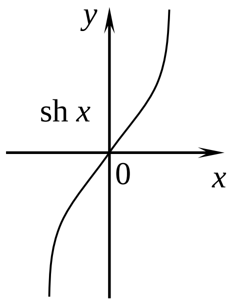
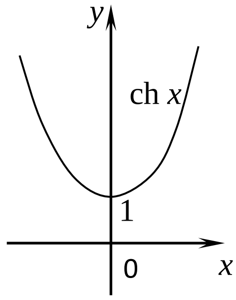


Fig. 1.28

6. Hyperbolic function. A function $y=\frac{e^x+e^{-x}}{2}$ is called *hyperbolic cosine* and is denoted by $y=\text{ch}\,x$. A function $y=\frac{e^x-e^{-x}}{2}$ is called *hyperbolic sine* and is denoted by $y=\text{sh}\,x$. So we have that

$$\text{ch}x=\frac{e^x+e^{-x}}{2},\quad \text{sh}x=\frac{e^x-e^{-x}}{2}.$$





Using these function let us introduce hyperbolic tangent and cotangent

$$\tanh\ x=\frac{\text{sh}\ x}{\text{ch}\ x}=\frac{e^x-e^{-x}}{e^x+e^{-x}},$$

$$\coth\ x=\frac{\text{ch}\ x}{\text{sh}\ x}=\frac{e^x+e^{-x}}{e^x-e^{-x}}.$$

Obviously that the function $y=\text{sh}\,x$ is odd and function $y=\text{ch}\,x$ is even.

$$\text{ch}^2 x - sh^2 x = 1,\quad sh2x = 2shx\text{ch}\,x$$

Def: A function is called *elementary* if it may be given by single formula of the type $y=f(x)$, where the expression $f(x)$ is constructed from the basic elementary functions and constants by using the finite number of the arithmetic operations.

14.7 Composite Function

Let a variable y be function of some variable u, which is function of the variable x, that is

$$y=f(u),\ u=\varphi(x).$$

Then

$$y = f[\varphi(x)],$$

or

$$y = F(x).$$

In this case the variable y is called a *composite function* of the x variable (or function of a function), and the variable u is called *intermediate argument*.

Example. Let us consider the function $y = \sin^3 x$. This function may be written as

$$y = u^3,\ u = \sin x.$$