CHAPTER II
LIMIT THEORY

2.1. Limit of Numerical Sequence

If to every natural number there corresponds the definite real number x_n then it is said that numerical sequence $\{x_n\} = x_1, x_2, ..., x_n, ...,$ is given. Numbers $x_1, x_2, x_3...$ are called *terms* of this sequence. If n is an arbitrary (current) *natural number*, then expression $x_n = f(n)$ is called *general term* of the numerical sequence. So the general term x_n is a function of the natural number $x_n = f(n)$.

Definition. Ordered infinite set of numbers

$$\{x_n\} = x_1, x_2, ..., x_n, ... \tag{2.1}$$

is called *numerical sequence* if each number x_i $(i = 1, 2, ...)$ is a function of some natural number, that is $x_n = f(n)$.

Example 1. In numerical sequence

$$-1, 1, -1, 1, -1, 1, ...$$

common term is equal to $(-1)^n$.

Example 2. Let us consider the sequence

$$1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, ...$$

Obviously that the general term is $x_n = \frac{1}{n^2}$.

So the numerical sequence (2.1) is particular case of the ordered discrete variable taking values $x_1, x_2, x_3...$.

Definition. Constant number a is called *limit of numerical sequence* (2.1) if for every arbitrarily small positive number $\varepsilon > 0$ it is possible to indicate such number $N(\varepsilon)$ that the following inequality

$$|x_n - a| < \varepsilon \tag{2.2}$$

is fulfilled for all $n \geq N$.

In this case we write that

$$a = \lim_{n\to\infty} x_n . \tag{2.3}$$

So

$$\left(a = \lim_{n\to\infty} x_n\right) \Leftrightarrow \left(\forall\varepsilon > 0\ \exists N(\varepsilon)\ \forall n \geq N \Rightarrow |x_n - a| < \varepsilon\right).$$

Equality (2.3) based on expression (2.2) means geometrically that starting with $n = N + 1$ all points x_n belong to ε-neighborhood of the point $x = a$. But we can take ε arbitrary small it means that at $n \to \infty$ points x_n will be thicken around point $x = a$.

So each an arbitrary small neighborhood of the point a contains infinite set of points x_n: x_{N+1}, x_{N+2}, $x_{N+3}, \ldots$, and in the same time only finite number of

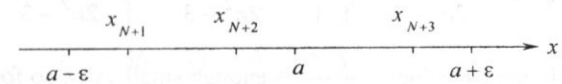

Fig. 2.1

points x_n: $x_1, x_2, ..., x_N$ may be outside of each ε-neighborhood (Fig. 2.1).

Example 3. Let us take sequence

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, ... = \left\{\frac{n}{n+1}\right\}.$$

Let us prove that $\lim_{n\to\infty} x_n = 1$.

We have, $|x_n - 1| = \left|\frac{n}{n+1} - 1\right| = \left|\frac{-1}{n+1}\right| = \frac{1}{n+1}$.

Hence, inequality $|x_n - 1| < \varepsilon$ may be written as:

$$\frac{1}{n+1} < \varepsilon,$$

$$n + 1 > \frac{1}{\varepsilon},$$

that is,

$$n > \frac{1}{\varepsilon} - 1. \tag{2.4}$$

If we take, for example, $\varepsilon = 0{,}001$ we obtain $n > 999$, so it follows that $N(0{,}001) = 999 + 1 = 1000$. So starting with $N = 1000$ it will obtain $|x_n - 1| < 0{,}001$. If we take smaller ε we will obtain bigger N starting with which the in equality $|x_n - 1| < \varepsilon$ holds true.

Example 4. Let $x_n = \dfrac{4n^2 + 1}{2n^2 - 3}$.

If n is enough large number then $x_n \approx \dfrac{4n^2}{2n^2} = 2$. So we can suppose that $\lim\limits_{n\to\infty} x_n = 2$. Let us prove this fact. We have

$$|x_n - 2| = \left|\frac{4n^2 + 1}{2n^2 - 3} - 2\right| = \left|\frac{4n^2 + 1 - 4n^2 + 6}{2n^2 - 3}\right| = \frac{7}{2n^2 - 3}.$$

At enough large n value $\dfrac{7}{2n^2 - 3}$ is enough small value so for an arbitrary small $\varepsilon > 0$ it is

$$|x_n - 2| < \varepsilon,$$

from here it follows that $\lim\limits_{n\to\infty} x_n = 2$.

Let us take for example $\varepsilon = 0{,}001$. Then inequality $|x_n - 2| < \varepsilon$ makes sense

$$\frac{7}{2n^2 - 3} < 0{,}001,$$

$$2n^2 > 7003,$$

that is, $n^2 > 3501{,}5$,

$$n > \sqrt{3501{,}5}.$$

But integer part of number $\sqrt{3501{,}5}$ is equal to $\left[\sqrt{3501{,}5}\right] = 59$, so $N(0{,}001) = 60$. Starting from $N = 60$, inequality $|x_n - 2| < 0{,}001$ holds true.

Example 5. Let us take the sequence

$$\left\{\frac{1 + (-1)^n}{2}\right\} = 0, 1, 0, 1, \ldots$$

It is obvious that it has no limit.

2.2. The Simplest Properties of the Limits

1. Any numerical sequence has only one limit.

■ **Proof.** Let us suppose that sequence has two limits a and b, $a \neq b$, for example $b > a$. By definition limit it follows that

$$\forall \varepsilon > 0,\ \exists N_1,\ \forall (n \geq N_1) \Rightarrow |x_n - a| < \frac{\varepsilon}{2}.$$

And

$$\exists N_2,\ \forall (n \geq N_2) \Rightarrow |x_n - b| < \frac{\varepsilon}{2}.$$

Let $N = \max(N_1, N_2)$. Then

$$\forall x_n,\ n > N,\ \Rightarrow \left(|x_n - a| < \frac{\varepsilon}{2}\right) \wedge \left(|x_n - b| < \frac{\varepsilon}{2}\right).$$

Let us choose $\varepsilon < \dfrac{b - a}{2}$. The last assertion geometrically means that starting with some N, all points $x_n (n > N)$ of the given numerical sequence belong to the ε-neighborhoods of the points a and b simultaneously, that is, $(x_n \in C_\varepsilon(a)) \wedge (x_n \in C_\varepsilon(b))$. But the intersection of this sets is empty set that is,

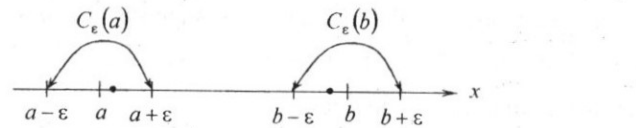

Fig. 2.2

$C_\varepsilon(a) \cap C_\varepsilon(b) = \varnothing$ for such ε (Fig. 2.2), it is impossible because $a \neq b$. Obtained contradiction proves property 1 .□

2. If for each n $(\forall n)$ $x_n = a$, $(a = const)$, that is,

$$\{x_n\} = a, a, a, .., a, ...,$$

then $\lim_{n\to\infty} x_n = a$. In other words *a limit of the constant is equal the same constant*, that is, $\lim_{n\to\infty} a = a$.

■ In this case $|x_n - a| = 0 \quad \forall n$, that is,

$$\forall\varepsilon > 0, \ |x_n - a| < \varepsilon \quad \forall n. \square$$

3. Theorem. If a sequence has a limit equal to a ($\lim_{n\to\infty} x_n = a$) and $a < b$ $(a > c)$ then there exists such number N $(\exists N)$ that $\forall n > N \Rightarrow x_n < b\,(x_n > c)$.

■ Since $\lim_{n\to\infty} x_n = a$ then by definition of the limit $\forall\varepsilon > 0$ there $\exists N, \forall n > N \Rightarrow |x_n - a| < \varepsilon$. Let us choose $\varepsilon > 0$ in such way that $\varepsilon = b - a > 0$. Since $|x_n - a| < \varepsilon$ or $a - \varepsilon < x_n < a + \varepsilon = b \Rightarrow x_n < a + \varepsilon = b \Rightarrow x_n < b$. Which is what had been proved.□

Note. In particular if $b = 0$ and $\lim_{n\to\infty} x_n = a$, $a > 0$, $(a < 0)$ then beginning with some number N all values of the variable x_n, $(n > N)$ will be positive (negative) as well, that is, $x_n > 0$ $(x_n < 0)$. Analogously the case $(a > c)$ may be proved. But in this case as ε it should be taken $\varepsilon = a - c$.

4. Theorem. If a sequence has limit $\left(\lim_{n\to\infty} x_n = a\right)$ and all values of the variable $x_n \le b$ $(x_n \ge c)$, $(\forall n > N)$ starting with some N then $a \le b$ $(a \ge b)$ as well.

■ Let us assume the contrary. We suppose that $a > b$ $(a < c)$ then by virtue of theorem 3 there $\exists N$, $\Rightarrow$ $x_x > b$ $(x_n < c)$, $\forall n > N$, which is contradiction to theorem condition. □

5. Theorem. If $\lim_{n\to\infty} x_n = a$, $\lim_{n\to\infty} y_n = b$ and $x_n \le y_n$ then $a \le b$.

■ Indeed if we have $a > b$ then starting with some n we would have $x_n > y_n$ (Fig. 2.3),. that is contradiction to given conditions.□

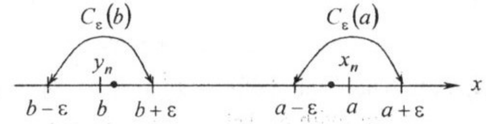

Fig. 2.3

Consequence. If $\lim_{n\to\infty} x_n = a$ and $x_n \ge 0 \ \forall n$, then $a \ge 0$, that is, non-negative sequence does not have negative limit. Similarly no positive sequence does not have positive limit.

Note. If $\lim_{n\to\infty} x_n = a$, $\lim_{n\to\infty} y_n = b$ and $x_n < y_n \ \forall n$ then it is not obligatory that $a < b$ as long as $a = b$. For example, for each n it will be

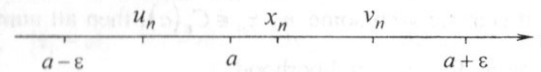

Fig. 2.4

$\frac{1}{n^2} < \frac{1}{n}$, but sequences

$$\left\{\frac{1}{(n+1)^2}\right\} = \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, ...$$

and

$$\left\{\frac{1}{n+1}\right\} = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, ...$$

have the same limit, equal to 0.

6. Theorem. (The first sign of limit existence). If $u_n \le x_n \le v_n \quad \forall n$ and $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = a$ then sequence $\{x_n\}$ has also limit equal to a.

■ Indeed by definition of limit it follows that

$$\forall\varepsilon > 0,\ \exists N_1,\ \forall(n \ge N_1) \Rightarrow |u_n - a| < \varepsilon,$$

or

$$a - \varepsilon < u_n < a + \varepsilon, \ \forall n > N_1 \qquad (2.5)$$

and $\exists N_2$, $\forall(n \ge N_2) \Rightarrow |v_n - a| < \varepsilon$, or

$$a-\varepsilon < v_n < a+\varepsilon, \ \forall n > N_2. \qquad (2.6)$$

Let $N = \max(N_1, N_2)$. Then the both inequality (2.5) and (2.6) would be fulfilled simultaneously starting with N. Taking into account the initial condition it may be written

$$a-\varepsilon < u_n < x_n < v_n < a+\varepsilon \Rightarrow a-\varepsilon < x_n < a+\varepsilon \Rightarrow |x_n - a| < \varepsilon .$$

It means that $\lim_{n\to\infty} x_n = a$.□

Sometime property 6 is called the theorem about "2 policemen".

7. Theorem. If sequence $\{x_n\}$ has limit a then each subsequence of this sequence has limit equal to a.

■ Indeed, if starting with some n, $x_n \in C_\varepsilon(a)$, then all numbers x_{n_k} that follow after x_n belong for this neighborhood.□

Note. If sequence $\{x_n\}$ has limit equal to a, then it is called *convergent* to the number a. Obviously that addition and subtraction finite number of the terms do not influent on convergence of this sequence.

2.3. Limit of Monotonic Sequence

Definition. The numerical sequence $\{x_n\}$ is called *monotone increasing (decreasing)* if the following inequality

$$x_n < x_{n+1} \qquad (x_n > x_{n+1})$$

is valid $\forall n \in N$.

Definition. The numerical sequence is called *bounded from above* if there is such number Q, that

$$x_n < Q \text{ (or } x_n \le Q) \ \forall n \in N.$$

Definition. The numerical sequence is called *bounded from bellow* if there is such number P that

$$x_n > P \text{ (or } x_n \ge P) \ \forall n \in N.$$

Definition. The numerical sequence bounded both from above and from bellow is called *bounded.*

It is clear that the numerical sequence is bounded if and only if when $\exists M \mid |x_n| \le M \ \forall n \in N$.

8. Theorem. If a numerical sequence has a limit $\lim_{n\to\infty} x_n = a$ then it is bounded, that is, there exists such number M that $|x_n| < M \ \forall n \in N$ (N – is set of natural numbers).

■ Let $\lim_{n\to\infty} x_n = a$. Let us take as ε the number 1, that is, $\varepsilon = 1$. By definition of limit there $\exists N$, $\Rightarrow |x_n - a| < 1 \ \forall n > N$. Then only finite number of points $x_1, x_2, \ldots, x_N$ is outside interval $C_\varepsilon(a)$. Let us maxim number among numbers 1, $|x_1 - a|$, $|x_2 - a|$, $\ldots, |x_N - a|$, designate by M, that is, $M = \max(1, |x_1 - a|, |x_2 - a|, \ldots, |x_N - a|)$. Then $|x_n - a| < M$, $\forall n \in N$. So the

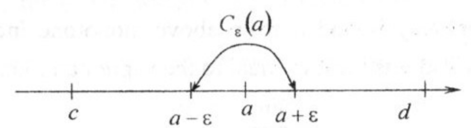

Fig. 2.5

segment $[c, d]$ which contains all points $x_1, x_2, \ldots, x_n, \ldots$, (Fig. 2.5) exists and has the following form

$$c = a - M \le x_n \le a + M = d.□$$

Note. Inverse statement to the theorem 3 is not true that is, sequence can be bounded but has no limit. For example, sequence 0, 1, 0, 1, ...

Let we have some sequence $\{x_n\} = x_1, x_2, x_3, x_4, x_5, \ldots$. Choose some terms $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$ from this sequence by some way, at that if in initial sequence the number x_{n_l} stands before number x_{n_k}, then it takes place in new sequence too. Sequence $\{x_{n_k}\} \subset \{x_n\}$ is called *subsequence of initial sequence.*

Obviously if a numerical sequence is bounded from above (bellow) then there exist infinite set of numbers such that $x_n < Q_1 < Q_2 < \ldots$ ($x_n > P_1 > P_2 > \ldots$).

Definition. The least value of all upper bounds is called *supremum* (exact upper bound). This definition may be formulated in another way.

Definition. A number a is called *supremum* of numerical sequence $\{x_n\}$ if:

1) $x_n \le a \ \forall n \in N$;

2) $\forall \varepsilon > 0 \ \exists N \mid a - \varepsilon < x_N \le a$.

The supremum is designated by $a = \sup\limits_n x_n$.

Definition. The greatest value of all below bounds is called *infimum*. That is, a is infimum if:

1) $x_n \ge a \ \forall n \in N$;

2) $\forall \varepsilon > 0 \ \exists N \mid a < x_N \le a + \varepsilon$.

Theorem *(without proof)*. Any non-empty bounded from above (below) set of real numbers has a finite supremum (infimum).

9. Theorem. Any bounded from above monotone increasing numerical sequence has limit and this limit is equal to the supremum, i.e.

$$\lim_{n\to\infty} x_n = a,$$

where $a = \sup\limits_n x_n$, $x_n < x_{n+1}$, $\forall n$.

■ Let the numerical sequence be monotone increasing and be bounded from above. Then it has supremum $a = \sup\{x_n\}$. Let us show that the limit of the sequence exists and equal to *a*, that is, $a = \lim x_n$. Take an arbitrary $\varepsilon > 0$. Since $a = \sup\{x_n\} \Rightarrow x_n \le a \ \forall n = 1, 2, 3 \cdots$, there exists such number $\exists N$ that $x_n > a - \varepsilon$. By virtue of monotonicity of the given sequence the following inequality

$$a - \varepsilon < x_N \le x_n \le a$$

is fulfilled $\forall n \ge N$. So $|a - x_n| < \varepsilon \ \forall n \ge N$. But it means that $a = \lim x_n$.

Similarly we can prove the following theorem

10. Theorem. Any bounded from below monotonic decreasing numerical sequence has limit and this limit is equal to infimum, i.e.

$$\lim_{n\to\infty} x_n = a,$$

where $a = \inf\limits_n x_n$, $x_n > x_{n+1}$, $\forall n$.

You can see that if numerical sequence is convergent then it is bounded. From this it follows that if monotone increasing sequence is convergent then it is bounded from above on the other hand if monotone increasing (decreasing) sequence is bounded from above then it is convergent (that is, it has a limit). Thus the following statement is valid: **in order the monotone increasing sequence to be convergent it is necessary and sufficiently that it is bounded from above (bellow).**

2.4. Infinitesimals and their Main Properties

Definition. A sequence $\{\alpha_n\} = \alpha_1, \alpha_2, \alpha_3, ...$ is called *infinitesimal* if $\lim\limits_{n\to\infty} \alpha_n = 0$, i.e., for every however small positive number $\varepsilon > 0$ there exists such number *N* that starting with this number the following inequality

$$|\alpha_n| < \varepsilon$$

holds true $\forall n > N$. In other words sequence $\{\alpha_n\}$ is called infinitesimal if its absolute value becomes and remains smaller than any however small positive value, as $n \to \infty$.

Example. Let $\alpha_n = \dfrac{(-1)^{n-1}}{2^n}$, then

$$\alpha_1 = \frac{1}{2}, \alpha_2 = -\frac{1}{4}, \alpha_3 = \frac{1}{8}, \alpha_4 = -\frac{1}{16}, ...$$

If we take enough large n we make value $|\alpha_n|$ by an arbitrary small, so $\lim\limits_{n\to\infty} \alpha_n = 0$.

Let us take $\varepsilon = 0{,}001$ and solve inequality $|\alpha_n| < 0{,}001$. We obtain

$$\frac{1}{2^n} < 0{,}001,$$

i.e.

$$2^n > 1000,$$

so $n > 10$. Starting with $n = 10$ we have $|\alpha_n| < 0{,}001$.

From definition of infinitesimal it follows that if sequence $\{\alpha_n\}$ is infinitesimal then sequence $\{-\alpha_n\}$ is infinitesimal as well.

Some Properties of Infinitesimals:

1) The algebraic sum of infinitesimals is also an infinitesimal.

■ Let us take two infinitesimal sequences $\{\alpha_n\}$ and $\{\beta_n\}$. By definition of infinitesimals for every however small positive number $\varepsilon > 0$ it is possible to indicate a number N such that the inequalities

$$|\alpha_n| < \frac{\varepsilon}{2}, \ |\beta_n| < \frac{\varepsilon}{2}$$

hold true simultaneously for all $n \geq N$.

Since

$$|\alpha_n + \beta_n| \leq |\alpha_n| + |\beta_n|$$

and

$$|\alpha_n| + |\beta_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

we get that $|\alpha_n + \beta_n| < \varepsilon$. From this it follows that $\{\alpha_n + \beta_n\}$ is an infinitesimal sequence.

In the case of three sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ we suppose $\alpha_n + \beta_n + \gamma_n = (\alpha_n + \beta_n) + \gamma_n$ and use just proved statement. So property 1 holds true for any finite number of terms.□

Consequence. The sum of infinite number of infinitesimal sequences may not be an infinitesimal sequence. For example,

$$\underbrace{\frac{1}{n^2} + \frac{1}{n^2} + \ldots + \frac{1}{n^2}}_{n \text{ items}} = \frac{1}{n} \underset{n \to \infty}{\to} 0; \quad \frac{1}{n} + \frac{1}{n} + \ldots + \frac{1}{n} = 1 = const;$$

$$\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \ldots + \frac{1}{\sqrt{n}} = \sqrt{n} \underset{n \to \infty}{\to} +\infty.$$

2) The product of infinitesimal and bounded sequences is an infinitesimal sequence

■ Let $\{x_n\}$ be a bounded sequence and $\{\alpha_n\}$ be an infinitesimal one. Then there exists such $M > 0$ that $|x_n| < M \ \forall n$. Let us take however small positive $\varepsilon > 0$. Then starting with some N, $|\alpha_n| < \frac{\varepsilon}{M} \ \forall n \geq N$. Hence starting with this N the inequality

$$|\alpha_n x_n| < \frac{\varepsilon}{M} \cdot M = \varepsilon, \ \forall n \geq N$$

holds true.

So $\{\alpha_n x_n\}$ is an infinitesimal sequence.□

Consequence. Product of infinitesimal and unbounded sequences may not be infinitesimal sequence. For example,

$$\frac{1}{n^2} \cdot n = \frac{1}{n} \underset{n \to \infty}{\to} 0; \quad \frac{1}{n} \cdot n = 1 = const; \quad \frac{1}{\sqrt{n}} \cdot n = \sqrt{n} \underset{n \to \infty}{\to} +\infty.$$

2.5. Infinitely Large Values and Their Main Properties

Definition. Numerical sequence $\{x_n\}$ is called *infinitely large*, if for each arbitrary large fixed number $M > 0$ starting with some N it will be $|x_n| > M$ for all $\forall n \geq N$. This fact is denoted by $x_n \underset{n \to \infty}{\to} \infty$ or $\lim\limits_{n \to \infty} x_n = \infty$. From this definition it follows that if $\{x_n\}$ is infinitely large value then sequence $\{-x_n\}$ is infinitely large too.

Example 1. Let $x_n = \sqrt{n-1}$. Let us show that $\lim\limits_{n \to \infty} x_n = \infty$. Let us take some arbitrary however large number $M > 0$. Then inequality $|x_n| > M$ means that $\sqrt{n-1} > M$. Whence $n > M^2 + 1$. So for each $M = const$ there exists such number $N = M^2 + 1$ starting with which it will be $|x_n| > M \ \ \forall n > N$. Given fact is proved.

If $x_n \underset{n\to\infty}{\to} \infty$ and if starting with some N it will be $x_n > 0$ $\forall n > N$ then we will write $x_n \underset{n\to\infty}{\to} +\infty$ or $\lim\limits_{n\to\infty} x_n = +\infty$. In the same way such equality

$$\lim_{n\to\infty} x_n = -\infty$$

is defined.

Example 2. Let us consider the sequence $x_n = \sqrt{n^2 - 1}$. It is obviously that $\lim\limits_{n\to\infty} x_n = +\infty$.

Example 3. Let consider the sequence $x_n = (-1)^n n^2$, i.e.

$$\{x_n\} = -1, 4, -9, 16, -25, \dots .$$

It is obviously that $\lim\limits_{n\to\infty} x_n = +\infty$ or $\lim\limits_{n\to\infty} x_n = -\infty$ but we should write $\lim\limits_{n\to\infty} x_n = \infty$.

Some properties of infinitely large values:

1. Sum of an infinitely large value and bounded value is an infinitely large value.
2. Sum of infinitely large values of the same sign is an infinitely large value of the same sign.
3. Product of infinitely large sequence and sequence which does not turn into zero is infinitely large sequence.

It is easy to see that infinitely large sequence is unbounded. Contrary assertion is not true, i.e. unbounded sequence may not be infinitely large. For example, sequence

$$1, 0, -2, 0, 3, 0, -4, 0, 5, \dots$$

In this case for each large $M > 0$ we have at some n it will be $|x_n| > M$ but for the next value x_{n+1} it will be $x_{n+1} = 0$ so $|x_{n+1}| < M$.

2.6. The Connection Between Infinitely Large and Infinitesimals

Theorem. If $\{x_n\}$ is an infinitely large value then $\{\alpha_n\} = \left\{\frac{1}{x_n}\right\}$ is an infinitesimal sequence.

■ Let us assign some number $M > 0$. By definition of the infinitely large value there exist such number N starting of which the inequality $|x_n| > M$ is true for all $\forall n > N$. So $|\alpha_n| = \frac{1}{|x_n|} < \frac{1}{M}$. Since M may be taken an arbitrary large then $\frac{1}{M} = \varepsilon$ may be an arbitrary small. Hence $|\alpha_n| < \varepsilon$, it means that $\{\alpha_n\} = \left\{\frac{1}{x_n}\right\}$ is an infinitesimal sequence.□

Theorem. If $\{\alpha_n\}$ is infinitesimal sequence and $\alpha_n \neq 0$ at each n then $\{x_n\} = \left\{\frac{1}{\alpha_n}\right\}$ is infinite large sequence.

This property is proved quite the same.

Concept of an infinitesimal is closely connected with conception of a limit. Let

$$\lim_{n\to\infty} x_n = a \text{ then } |x_n - a| < \varepsilon. \qquad (2.7)$$

The sequence $\{x_n - a\}$ is an infinitesimal and vice versa if $\{x_n - a\}$ is an infinitesimal then $\lim\limits_{n\to\infty} x_n = a$. So the following assertion is true.

Theorem. A number a is a limit of sequence $\{x_n\}$, if and only if the difference $\{x_n - a\}$ is an infinitesimal. So any numerical sequence, which has a limit, differs from it on the infinitesimal. If $a = \lim\limits_{n\to\infty} x_n$, then

$$x_n = a + \alpha_n, \qquad (2.8)$$

where $\{\alpha_n\}$ is infinitesimal.

2.7. Properties of Limits Connected with Arithmetic Operations

1. Theorem. The limit of algebraic sum of convergent sequences is equal to the algebraic sum of those limits.

■ Let $\lim\limits_{n\to\infty} x_n = a$, $\lim\limits_{n\to\infty} y_n = b$. Then accordingly to (2.8),

$$x_n = a + \alpha_n,\ y_n = b + \beta_n,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are infinitesimals. From this it follows that

$$x_n \pm y_n = (a \pm b) + (\alpha_n \pm \beta). \qquad (2.9)$$

Due to property 1 sequence $\{\alpha_n + \beta_n\}$ is an infinitesimal. So from (2.9) it follows that sequence $\{x_n + y_n\}$ differs from number $a+b$ on infinitesimal. It means that

$$\lim_{n\to\infty}(x_n + y_n) = a + b,$$

i. e.

$$\lim_{n\to\infty}(x_n \pm y_n) = \lim_{n\to\infty} x_n \pm \lim_{n\to\infty} y_n. \square$$

This theorem holds true for any finite number of terms.

2. Theorem. The limit of product of convergent sequences is equal to product of those limits.

■ Let $\lim\limits_{n\to\infty} x_n = a$, $\lim\limits_{n\to\infty} y_n = b$. Then

$$x_n y_n = (a + \alpha_n)(b + \beta_n) = ab + (\alpha_n b + a\beta_n + \alpha_n \beta_n).$$

Since constant is particular case of bounded sequence then $\{\alpha_n b\}$ and $\{a\beta_n\}$are infinitesimal. The value $\{\alpha_n \beta_n\}$ is an infinitesimal too. Then $\{\alpha_n b + a\beta_n + \alpha_n\beta_n\}$ is an infinitesimal. Now value $x_n y_n$ differs from number ab on infinitesimal so

$$\lim_{n\to\infty} x_n y_n = ab,$$

or

$$\lim_{n\to\infty} x_n y_n = \lim_{n\to\infty} x_n \cdot \lim_{n\to\infty} y_n. \square$$

Consequence. Constant factor may be taken out of the limit sign.

■ Let $c = const$. Then accordingly to the theorem 2, we have

$$\lim_{n\to\infty} cx_n = \lim_{n\to\infty} c \cdot \lim_{n\to\infty} x_n = c \cdot \lim_{n\to\infty} x_n. \square$$

3. Theorem. The limit of quotient of two convergent sequences is equal to quotient of limits of the numerator and the denominator provided the limit of the denominator does not vanish.

■ Indeed let $\lim\limits_{n\to\infty} x_n = a$, $\lim\limits_{n\to\infty} y_n = b$ and $b \neq 0$. Then

$$\frac{x_n}{y_n} = \frac{a+\alpha_n}{b+\beta_n} = \frac{a}{b} + \left(\frac{a+\alpha_n}{b+\beta_n} - \frac{a}{b}\right) = \frac{a}{b} + \frac{b\alpha_n - a\beta_n}{b(b+\beta_n)}.$$

The variable $b\alpha_n - a\beta_n$ is the infinitesimal. Since $b \neq 0$ value $b + \beta_n$ is not the infinitesimal therefore the sequence $\left\{\frac{1}{b(b+\beta_n)}\right\}$ is bounded. Then $(b\alpha_n - a\beta_n)\frac{1}{b(b+\beta_n)}$ is the infinitesimal, i. e. value $\frac{x_n}{y_n}$ differs from $\frac{a}{b}$ on the infinitesimal. Consequently

$$\lim_{n\to\infty}\frac{x_n}{y_n} = \frac{a}{b} = \frac{\lim\limits_{n\to\infty} x_n}{\lim\limits_{n\to\infty} y_n}. \square$$

Note. If $\lim\limits_{n\to\infty} y_n = 0$ and $\lim\limits_{n\to\infty} x_n \neq 0$, then $\lim\limits_{n\to\infty}\frac{x_n}{y_n} = \infty$. The case when

$$\left(\lim_{n\to\infty} x_n = 0\right) \wedge \left(\lim_{n\to\infty} y_n = 0\right)$$

will be considered below.

2.8. The Limit of a Function at a Point and on Infinity

Let a function $y = f(x)$ be defined in some neighbourhood of the point $x=a$, except may be the point a itself. Let us denote the domain of definition by D_f. Take some sequence

$$\{x_n\} = x_1, x_2, ..., x_n, ... \qquad (2.10)$$