which:

a) converges to a, that is, $\lim\limits_{n\to\infty} x_n = a$;

b) its terms $x_n \in D_f\ \forall n$, but $x_n \neq a\ \forall n$.

It is obvious that infinite number of such sequences exists.

We construct numerical sequence corresponding to sequence (2.10)

$$\{f(x_n)\} = f(x_1), f(x_2), \ldots, f(x_n), \ldots \tag{2.11}$$

Definition. If sequence (2.11) corresponding for each sequence (2.10) satisfying the conditions a) and b) converges to some number A then this number is called *a limit of a function* $y = f(x)$ *at the point* $x = a$, and is denoted by

$$\lim_{x\to a} f(x) = A, \tag{2.12}$$

or

$$f(x) \underset{x\to a}{\rightarrow} A.$$

Note that numbers x_n can approach a point a in an arbitrary way but in each case it will be

$$\lim_{n\to\infty} f(x_n) = A.$$

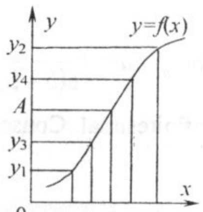

Fig. 2.6

Geometrically it means that if abscises of points x_n approach a then ordinates of these points approach number A.

Let $A = \infty$, then

$$\lim_{x\to a} f(x) = \infty.$$

In this case it is said that function $f(x)$ becomes infinitely large value at the point $x = a$.

Example 1. Let us prove that $\lim\limits_{x\to 3} \dfrac{x}{(x-3)^2} = +\infty$.

Indeed if $x_n \to 3$, then $\dfrac{1}{(x_n-3)^2} x_n$ is infinitely large value (because it is product of infinitely large positive value by value that tends to constant) (Fig. 2.7).

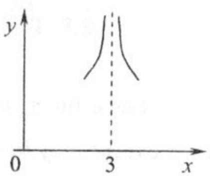

Fig. 2.7

Let us consider the case as $a \to \infty$ and $b \to \infty$ simultaneously, that is,

$$\lim_{x\to\infty} f(x) = \infty.$$

This equality means that if $\{x_n\}$ is some infinitely large sequence then sequence $\{f(x_n)\}$ is infinitely large too.

Example 2. Let us show that

$$\lim_{x\to+\infty} \ln x = +\infty.$$

To prove it let us take any large positive number $M > 0$ and investigate inequality

$$\ln x > M.$$

The inequality is fulfilled for all $x > e^M$. Since $x_n \to +\infty$, then starting with some number n, it will be that $x_n > e^M$. Consequently starting with this number n the inequality $\ln x_n > M$ will be fulfilled as well (Fig. 2.8).

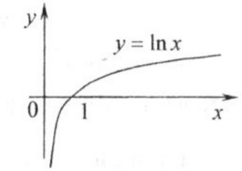

Fig. 2.8

2.9. One-Sided Limits of a Function at a Point

Let us assume that numerical sequence $\{x_n\}$ has a limit a, and $x_n > a$ $\forall n$ (Fig. 2.9). In this case we will say that x_n approaches number a on the right

$$x_n \underset{n\to\infty}{\rightarrow} a + 0.$$

In similar way if $\lim\limits_{n\to\infty} x_n = a$ and at the same time $x_n < a\ \forall n$, then we will say that x_n approaches a on the left:

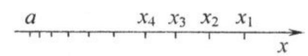

Fig. 2.9

$$x_n \underset{n\to\infty}{\rightarrow} a - 0.$$

Definition. If for each sequence $\{x_n\} \underset{n\to\infty}{\rightarrow} a - 0$ it will be $\{f(x_n)\} \underset{n\to\infty}{\rightarrow} A_1$ then A_1 is called *a limit of the function on the left at the point* $x = a$ (the left limit) (Fig. 2.10):

$$A_1 = \lim_{x\to a-0} f(x).$$

At the same way we can define the *right limit*. The number A_2 is called the right limit of the function at the point $x=a$, if for each convergent sequence $x_1, x_2, ..., x_n, ...$, elements of which convergent to a on the right $x_n \underset{n\to\infty}{\to} a+0$ the corresponding functional sequence converges to $A_2 \{f(x_n)\} \underset{n\to\infty}{\to} A_2$ or

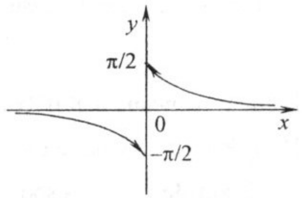

Fig. 2.10

$$A_2 = \lim_{x\to a+0} f(x).$$

In case of an arbitrary function the right-hand and left-hand limits at the point are not always equal to each other.

Example 1. Let us investigate the function $y = \arctan\frac{1}{x}$ at the point $x=0$. If points $x_1, x_2, ..., x_n, ...$ approach zero on the right, that is, $x_n \underset{n\to\infty}{\to} +0$, then $\frac{1}{x_n} \underset{n\to\infty}{\to} +\infty$, and $\arctan\frac{1}{x_n} \underset{n\to\infty}{\to} \frac{\pi}{2}$, so

$$\lim_{x\to+0} \arctan\frac{1}{x} = \frac{\pi}{2}.$$

One can obtain that

$$\lim_{x\to-0} \arctan\frac{1}{x} = -\frac{\pi}{2}.$$

Sometimes in order to define it more exactly they are written as:

$$\lim_{x\to+0} \arctan\frac{1}{x} = \frac{\pi}{2} - 0, \quad \lim_{x\to-0} \arctan\frac{1}{x} = -\frac{\pi}{2} + 0.$$

The sense of such specification is obvious (Fig. 2.10).

So in this case the left-hand and the right-hand limits exist but they are not equal to each other.

Example 2. Let us consider the function $y = 2^{\frac{1}{x}}$. If $x_n \underset{n\to\infty}{\to} +0$, Then $\frac{1}{x_n} \underset{n\to\infty}{\to} +\infty$, it means that $2^{\frac{1}{x_n}} \underset{n\to\infty}{\to} +\infty$. If $x_n \underset{n\to\infty}{\to} -0$, then $\frac{1}{x_n} \underset{n\to\infty}{\to} -\infty$, it means that $2^{\frac{1}{x_n}} \underset{n\to\infty}{\to} +0$. So, $\lim_{x\to-0} 2^{\frac{1}{x}} = +0$, $\lim_{x\to+0} 2^{\frac{1}{x}} = +\infty$.

Thus the left-hand limit at the point $x=0$ exists, but the right-hand limit does not exist. It should be noted that if $x_n \underset{n\to\infty}{\to} +\infty$, then $\frac{1}{x_n} \underset{n\to\infty}{\to} +0$, it means that $2^{\frac{1}{x_n}} \underset{n\to\infty}{\to} 1+0$, so

$$\lim_{x\to+\infty} 2^{\frac{1}{x}} = 1+0.$$

In similar way

$$\lim_{x\to-\infty} 2^{\frac{1}{x}} = 1-0.$$

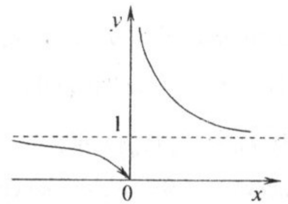

Fig. 2.11

The plot of the graph is presented on the (Fig. 2.11).

Note. It is obvious that **if a function has a limit at a point then it has left-hand and right-hand limits and these limits are equal to each other**

$$\lim_{x\to a} f(x) = A \Rightarrow f(x+0) = f(x-0) = A.$$

It can be proved that contrary is correct as well: if function $y = f(x)$ has a one-sided limits which are equal each to other $f(x+0) = f(x-0) = A$ then a limit of a function exists at this point and is equal to A as well.

Considered definition of a function limit is called Heyne's definition. Later (Fig. 2.11) we will introduce another definition given by Cauchy or on language of "$\varepsilon - \delta$".

2.10. Properties of the Function Limits

Considered above properties of the sequence limit may be easy extended to function of continuous argument.

Function $f(x)$ is called *infinitely large* if $\lim_{x\to a} f(x)=\infty$ as $x\to a$.

Such properties hold true for function of continuous argument:

1. The sum of infinitely large and bounded functions is infinitely large function.
2. The sum of the infinitely large functions of the same sign is infinitely large function of the same sign.

■ Let us prove property 1. Let $\lim_{x\to a} f(x)=\infty$ and $\varphi(x)$ is bounded function as $x\to a$ (it means that there such $M>0$ exists that for all $x\in C_\varepsilon(a)$ where $\varepsilon>0$ the inequality $|\varphi(x)|\le M$ is valid). Then as $x_n \underset{n\to\infty}{\to} a$ we obtain sequence $\{f(x_n)+\varphi(x_n)\}$ which is a sum of infinitely large $\{f(x_n)\}$ and bounded $\{\varphi(x_n)\}$ sequence. Due to of properties of numerical sequence $f(x_n)+\varphi(x_n) \underset{n\to\infty}{\to} \infty$. Since $\{x_n\}$ is an arbitrary sequence then

$$\lim_{x\to a}[f(x)+\varphi(x)]=\infty.\,\square$$

The function $f(x)$ is called *infinitesimal* if $\lim_{x\to a} f(x)=0$ as $x\to a$. We have such properties for infinitesimal fanctions.

1. The sum of finite number of infinitesimals is an infinitesimal.
2. The product of an infinitesimal and bounded function is an infinitesimal.
3. If a function $f(x)$ is infinitely large as $x\to a$ then a function $\alpha(x)=\dfrac{1}{f(x)}$ is an infinitesimal as $x\to a$.
4. If a function $\alpha(x)$ is an infinitesimal as $x\to a$ and if it does not vanish in some neighborhood of the point a (except may be the point a) then a function $f(x)=\dfrac{1}{\alpha(x)}$ is infinitely large as $x\to a$.

■ Let us prove the property 3. Take an arbitrary sequence which

converges to a, i.e. $x_n \underset{n\to\infty}{\to} a$. Then the sequence $\{f(x_n)\}$ is infinitely large value as $n\to\infty$ and the value $\{\alpha(x)\}=\left\{\dfrac{1}{f(x)}\right\}$ is an infinitesimal, i.e. $\alpha(x_n) \underset{n\to\infty}{\to} 0$, since $\{x_n\}$ is arbitrary one then $\lim_{x\to a}\alpha(x)=0$.□

Let us suppose that $\lim_{x\to a} f(x)=A$. Then $f(x)=A+\alpha(x)$ where $\alpha(x)$ is an infinitesimal function as $x\to a$. Inversely if $f(x)=A+\alpha(x)$ where $\alpha(x)\to 0$ as $x\to a$ then $A=\lim_{x\to a} f(x)$.

■ Let us prove the first statement. Let $A=\lim_{x\to a} f(x)$. It means that if $x_n \underset{n\to\infty}{\to} a$ then $f(x_n)\underset{n\to\infty}{\to} A$, so $f(x_n)-A\underset{n\to\infty}{\to}0$. Assigning

$$f(x_n)-A=\alpha(x_n), \tag{2.13}$$

we obtain that $\alpha(x_n)\underset{n\to\infty}{\to}0$. So $\alpha(x)$ is an infinitesimal function as $x\to a$ so $f(x)=A+\alpha(x)$.□

Then following statements are correct.

Theorem 1. Limit of the sum of any finite number of functions, which have a limit, is equal to sum of those limits

$$\lim_{x\to a}\sum_{k=1}^{m} f_k(x)=\sum_{k=1}^{m}\lim_{x\to a} f_k(x).$$

Theorem 2. Limit of product of functions, which have limits, is equal to product of limits

$$\lim_{x\to a}\prod_{k=1}^{m} f_k(x)=\prod_{k=1}^{m}\lim_{x\to a} f_k(x).$$

From this theorem it follows that constant factor can be taken out from sign of limit

$$\lim_{x\to a}[cf(x)]=c\lim_{x\to a} f(x).$$

Theorem 3. Limit of a quotient of functions, which has a limit, is equal to ratio of limits if limit of the denominator does not equal to zero.

$$\lim_{x\to a}\frac{f(x)}{\varphi(x)}=\frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a}\varphi(x)},\ \lim_{x\to a}\varphi(x)\neq 0.$$

■ Let us prove theorem 1. Assume that

$$\lim_{x\to a} f(x)=b,\ \lim_{x\to a}\varphi(x)=c.$$

It means that if $x_n \underset{n\to\infty}{\to} a$ then $\lim\limits_{n\to\infty} f(x_n)=b,\ \lim\limits_{n\to\infty}\varphi(x_n)=c$. But then on the basis of theorem about limit of algebraic sum of the numerical sequence we obtain

$$\lim_{n\to\infty}[f(x_n)+\varphi(x_n)]=\lim_{n\to\infty} f(x_n)+\lim_{n\to\infty}\varphi(x_n),$$

$$\lim_{n\to\infty}[f(x_n)+\varphi(x_n)]=b+c.$$

As $\{x_n\}$ is an arbitrary sequence then

$$\lim_{n\to\infty}[f(x)+\varphi(x)]=b+c.\square$$

Other properties of the function limits:

1. If function $f(x)$ has a limit at the point $x=a$ then it is bounded as $x\to a$.
2. If finite limits

$$\lim_{x\to a} f(x)=b,\ \lim_{x\to a}\varphi(x)=c$$

exist and if in some neighborhood of the point a $f(x)\geq\varphi(x)$ then $b\geq c$.

3. If $\varphi(x)\leq f(x)\leq g(x)$ in some neighborhood of the point a and if $\lim\limits_{x\to a}\varphi(x)=\lim\limits_{x\to a} g(x)=b$ then $\lim\limits_{x\to a} f(x)$ exists and is equal to b.

All these properties follow from corresponding properties of the numerical sequences.

Lemma. If $\lim\limits_{x\to a} f(x)>0$ then there exists such neighborhood $C_\varepsilon(a)$ of the point $x=a$ that $f(x)>0$ $\forall x\in C_\varepsilon(x)$ (Fig. 2.12).

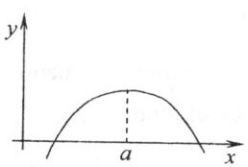

Fig. 2.12

■ Let us assume contrary. It means that in any arbitrary small neighborhood $C_\varepsilon(a)$ there are points x at which $f(x)\leq 0$. Then in any bounded interval containing point a there

exists infinite number of points x for which $f(x)\leq 0$. On the basis of lemma by Boltsano-Cauchy from those points we can take off such subsequence $\{x_n\}$ that $x_n \underset{n\to\infty}{\to} a$. As we assume $f(x_n)\leq 0$ for all points then $\lim\limits_{n\to\infty} f(x_n)\leq 0$ (this limit exists because $\lim\limits_{x\to a} f(x)$ exists). Obtained inequality contradicts to given condition $\lim\limits_{x\to a} f(x)>0.\square$

This lemma is called lemma about preservation of a sign of a function.

Changing $f(x)$ by $-f(x)$ we obtain that if $\lim\limits_{x\to a} f(x)<0$ then there exists such neighbourhood of a point $x=a$ that $f(x)<0$ $\forall x\in C_\varepsilon(x)$.

2.11. The Second Definition of a Function Limit at a Point and on Infinity

Let $\lim\limits_{x\to a} f(x)=A$, where a and A are finite numbers. This equality means that for all sequences $\{x_n\}\underset{n\to\infty}{\to} a$ it will be $\{f(x_n)\}\underset{n\to\infty}{\to} A$. Let us show that the equality $\lim\limits_{x\to a} f(x)=A$ may be treated more simple without connection with sequences. Indeed if x is enough close to a, then $f(x)$ will be however close to A.

Let us take an arbitrary however small positive number $\varepsilon>0$. Since $\lim\limits_{x\to a}|f(x)-A|=0$, then

$$\lim_{x\to a}\left[\varepsilon-|f(x)-A|\right]=\varepsilon,$$

that is,

$$\lim_{x\to a}\left[\varepsilon-|f(x)-A|\right]>0.$$

By theorem of sign preservation of a function there exists some neighbourhood $C_\delta(a)$, Fig. 2.13, in which the inequality

$$\varepsilon-|f(x)-A|>0$$

is fulfilled.

That is,

$$|f(x)-A|<\varepsilon.$$

So if $\lim_{x\to a} f(x)=A$, then $\forall\varepsilon>0$ it may be indicated such number $\delta>0$ (this value of δ, depends on ε), that if the inequality $|x-a|<\delta$ is valid then the inequality $|f(x)-A|<\varepsilon$ is valid as well, i. e.

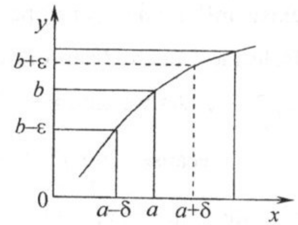

Fig. 2.13

$$(|x-a|<\delta)\Rightarrow(|f(x)-A|<\varepsilon).$$

Obviously that to smaller value ε, there corresponds the smaller value of δ.

Let us prove the contrary assertion. If for any $\varepsilon>0$ there exists such number $\delta(\varepsilon)>0$, that

$$(|x-a|<\delta)\Rightarrow(|f(x)-A|<\varepsilon), \text{ then } A=\lim_{x\to a} f(x).$$

Let $\{x_n\}$ is an arbitrary sequence converging to a, that is, $x_n \underset{n\to\infty}{\to} a$. Let us assign some $\varepsilon>0$. Since $\lim_{n\to\infty} x_n=a$, then starting with some number n, the inequality $|x_n-a|<\delta$ will be fulfilled. But by virtue of initial condition the inequality $|f(x_n)-b|<\varepsilon$ will be fulfilled as well starting with the same number n. Therefore

$$\lim_{n\to\infty} f(x_n)=b.$$

Since the sequence $\{x_n\}$ is an arbitrary then it means that $\lim_{x\to a} f(x)=b$.

So we can formulate the new definition of function limit at a point. It is called the definition on language "ε,δ", or definition by Cauchy: **Number A is called limit of a function $f(x)$ at a point $x=a$, if for any $\varepsilon>0$ it is possible to indicate such number $\delta(\varepsilon)>0$, that**

$$(|x-a|<\delta)\Rightarrow(|f(x)-b|<\varepsilon).$$

Example. Let us show that $\lim_{x\to 0} a^x=1$. For definiteness let us suppose that $a>1$. Assign an arbitrary $\varepsilon>0$ and require that

$$|a^x-1|<\varepsilon,$$

i. e.

$$-\varepsilon<a^x-1<\varepsilon.$$

This inequality is fulfilled if

$$1-\varepsilon<a^x<1+\varepsilon,$$

that is, if

$$\log_a(1-\varepsilon)<x<\log_a(1+\varepsilon).$$

Since $\log_a(1+\varepsilon)<|\log_a(1-\varepsilon)|$, then let us put $\delta=\log_a(1+\varepsilon)$. So we obtain that if $|x|<\delta$, then $|a^x-1|<\varepsilon$, which has to be proved.

Let us consider the inequality

$$\lim_{x\to\infty} f(x)=A.$$

Discussing analogously we can formulate another definition of a function limit on infinity: A number A is called a function limit $f(x)$ on infinity, if for any however small positive number $\varepsilon>0$ it is possible to indicate such number $M(\varepsilon)>0$ (Fig. 2.14), that $\forall x$, satisfying

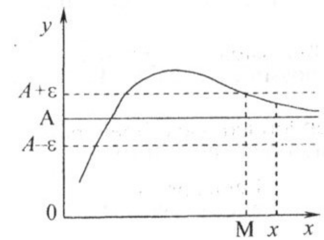

Fig. 2.14

$$(|x|>M)\Rightarrow(|f(x)-A|<\varepsilon).$$

In particular as $x\to+\infty$ or $x\to-\infty$ the condition $|x|>M$ is turned in conditions $x>M$ or $x<-M$.

Suppose that $\lim_{x\to a} f(x)=\infty$. Now we can formulate the following definition: a function $f(x)$ is infinitely large value at a point $x=a$ (Fig. 2.15),

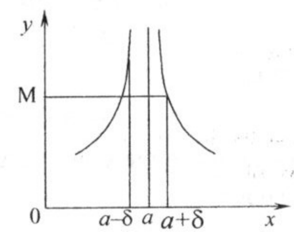

Fig. 2.15

if for any however large number $M>0$ there exists such number $\delta(M)>0$, that $\forall x$, satisfying

$$\forall\left(|x-a|<\delta\right)\Rightarrow\left(|f(x)|>M\right).$$

At last suppose that, $\lim\limits_{x\to\infty} f(x)=\infty$ (Fig. 2.16).

It means that for any however large number $N>0$ it is possible to indicate such number $M(N)>0$, that

$$\left(|x|>M\right)\Rightarrow\left(|f(x)|>N\right).$$

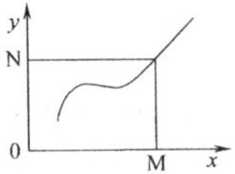

Fig. 2.16

2.12. First Remarkable Limit

Let us consider $\lim\limits_{x\to 0}\frac{\sin x}{x}$, where x is expressed in radians. It is impossible to calculate this limit directly, because the expression $\frac{\sin x}{x}$ is undefined one of the type $\frac{0}{0}$ as $x\to 0$.

Let us prove that

$$\lim_{x\to 0}\frac{\sin x}{x}=1.$$

■ First assume that $x>0$. Obviously that $BC<A\breve{B}<AD$ (Fig. 2.17), whence

$$\frac{BC}{R}<\frac{A\breve{B}}{R}<\frac{AD}{R},$$

that is

$$\sin x<x<\mathrm{tg}x.$$

Let us divide these inequalities by $\sin x$. As result we obtain

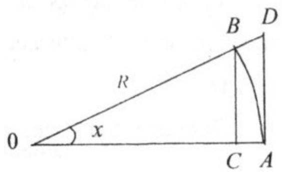

Fig. 2.17

$$1<\frac{x}{\sin x}<\frac{1}{\cos x}.$$

From this it follows that

$$\cos x<\frac{\sin x}{x}<1$$

If $x\to +0$, then $\cos x\to 1$, by virtue of theorem of existence of the limit we obtain that

$$\lim_{x\to +0}\frac{\sin x}{x}=1.$$

Due to even of the function $\frac{\sin x}{x}$ it follows that

$$\lim_{x\to -0}\frac{\sin x}{x}=1.$$

Consequently, indeed

$$\boxed{\lim_{x\to 0}\frac{\sin x}{x}=1}.\ \square \qquad (2.14)$$

There exist following consequences from the 1-st remarkable limit.

$\sin x\sim x,\ x\to 0$	$\tan x\sim x,\ x\to 0$
$\arcsin x\sim x,$ $x\to 0$	$\arctan x\sim x,\ x\to 0$
$\mathrm{sh}x\sim x,\ x\to 0$	$\mathrm{th}\,x\sim x,\ x\to 0$
$1-\cos x\sim\frac{x^2}{2},$ $x\to 0$	$1-\mathrm{ch}x\sim\frac{x^2}{2},\ x\to 0$

2.13. The Number e as Limit of the Numerical Sequence

Let us consider the numerical sequence

$$x_1=(1+1)^1, x_2=\left(1+\frac{1}{2}\right)^2, x_3=\left(1+\frac{1}{3}\right)^3,\dots, x_n=\left(1+\frac{1}{n}\right)^n,\dots$$

and prove that it has limit as $n\to\infty$.

Taking $n=1,2,3,\dots$, we obtain that

$$x_1=2;\ x_2=2{,}25;\ x_3=2{,}37;\ \dots,$$

so at increasing values n corresponding value x_n is an increasing. Let us prove that for all values n it will be $x_{n+1} > x_n$.

Applying Newton's binomial formula one can obtain:

$$\left(1+\frac{1}{n}\right)^n = 1 + n\frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{1}{n^3} + \ldots + \frac{n(n-1)(n-2)\ldots[n-(n-1)]}{n!}\frac{1}{n^n},$$

that is,

$$x_n = 2 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \ldots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{n-1}{n}\right); \qquad (2.15)$$

Substituting instead of n value $n+1$ we can obtain x_{n+1}

$$x_{n+1} = 2 + \frac{1}{2!}\left(1-\frac{1}{n+1}\right) + \frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right) + \ldots + \frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\ldots\left(1-\frac{n-1}{n+1}\right).$$

It is easy to note that every term beginning from the second one on the right side is increasing and besides there appears additional positive term, therefore $x_{n+1} > x_n$.

On other hand, due to (2.15), we have

$$x_n < 2 + \frac{1}{2} + \frac{1}{2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \ldots + \frac{1}{2\cdot 3\cdot\ldots\cdot n} < 2 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}} <$$

$$< 2 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots = 2 + \frac{\frac{1}{2}}{1-\frac{1}{2}} = 3,$$

that is, $x_n < 3$, $\forall n$.

So the sequence $\{x_n\}$ is monotone increasing and is bounded from above by number 3. Consequently there exists $\lim_{n\to\infty} x_n$, which is not exceeding the number 3 as well. Besides, since $x_n > 2$, then $\lim_{n\to\infty} x_n > 2$. So,

$$2 < \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \le 3.$$

$$\boxed{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e}. \qquad (2.16)$$

The $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ is called the number e=2.71828182..., i.e.

2.14. The Second Remarkable Limit

Theorem. The function $\left(1+\frac{1}{x}\right)^x$ approaches the limit e as x approaches infinity, i.e. the following equality

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e. \qquad (2.17)$$

is valid.

■ Let $x\to+\infty$. Obviously that every value x lies between two positive integers number $n \le x < n+1$, whence , $\frac{1}{n+1} < \frac{1}{x} \le \frac{1}{n}$. So

$$\left(1+\frac{1}{n+1}\right)^n < \left(1+\frac{1}{x}\right)^x \le \left(1+\frac{1}{n}\right)^{n+1}. \qquad (2.18)$$

Let us calculate the following limits

$$\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^n = \lim_{n\to\infty}\frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{1+\frac{1}{n+1}} = \frac{\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{n+1}}{\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)} = \frac{e}{1} = e;$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n+1}\right)^{1}=e\times 1=e$$

Therefore passing to limit in relation (2.18) and using the 1-st sign of limit xistence we obtain

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=e, \qquad (2.19)$$

Let us prove that if $x\to-\infty$ then

$$\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^{x}=e. \qquad (2.20)$$

Let us fulfill the following substitution $t=-(x+1)\Rightarrow x=-(t+1)$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=\lim_{t\to\infty}\left(1-\frac{1}{t+1}\right)^{-(t+1)}=\lim_{t\to\infty}\left(\frac{t}{t+1}\right)^{-(t+1)}=$$

$$=\lim_{t\to\infty}\left(\frac{t+1}{x}\right)^{t+1}=\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^{t+1}=\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^{t}\cdot\lim_{t\to\infty}\left(1+\frac{1}{t}\right)=e\cdot 1=e,$$

Put $u=\frac{1}{x}$ then the relation (2.17) takes the form

$$\lim_{u\to 0}(1+u)^{\frac{1}{u}}=e,$$

if we designate u as x again then we obtain

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e.$$

is another form of the second remarkable limit.

There exist following consequences from the 2-nd remarkable limit.

$a^x-1\sim x\ln a$, $x\to 0$	$e^x-1\sim x$, $x\to 0$
$\log_a(1+x)\sim\frac{x}{\ln a}$, $x\to 0$	$\ln(1+x)\sim x$, $x\to 0$
$(1+x)^{\alpha}-1\sim\alpha x$, $x\to 0$	$(1+x)^{\frac{1}{\alpha}}-1\sim\frac{x}{\alpha}$, $x\to 0$

2.15. Comparison of the Infinitesimals

In order to compare two infinitesimals values it is necessary to calculate limit of their ratio. Since we will consider both comparison of the infinitesimal of the numerical sequence $\{\alpha_n\},\{\beta_n\}$ and infinitesimal functions $\alpha(x)$ and $\beta(x)$ then instead of record $\lim\limits_{n\to\infty}\frac{\alpha_n}{\beta_n}$ or $\lim\limits_{x\to a}\frac{\alpha(x)}{\beta(x)}$ we will write just $\lim\frac{\alpha}{\beta}$.

Let the values α and β be infinitesimals values. If $\lim\frac{\alpha}{\beta}=0$, then it means that α approaches zero more quickly than β. In this case we will say that α is an infinitesimal value of the higher order than β and write:

$$\alpha=o(\beta).$$

Example 1. Let us compare two infinitesimals values

$$\alpha_n=\frac{1}{n^2+1},\ \beta_n=\frac{1}{n+1}$$

as $n\to\infty$. With this purpose we calculate the following limit

$$\lim_{n\to\infty}\frac{\alpha_n}{\beta_n}=\lim_{n\to\infty}\frac{n+1}{n^2+1}=0.$$

It means that $\alpha=o(\beta)$.

If $\lim\frac{\alpha}{\beta}=\infty$, then we will say that the infinitesimal α is value of lower order than β. Obviously that in this case $\beta=o(\alpha)$.

If $\lim\frac{\alpha}{\beta}=c$, where $c<\infty$ and $c\neq 0$, then α and β are called infinitesimals values of the same order. In this case we will write: $\alpha=O^*(\beta)$, or $\beta=O^*(\alpha)$.

Example 2. Let values $\alpha(x)=\sqrt{1+x}-1$, $\beta(x)=x$ be given as $x\to 0$. Then

$$\lim_{x\to 0}\frac{\alpha(x)}{\beta(x)}=\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}=\frac{1}{2},$$

i. e., as $x\to 0$, then $\alpha(x)=O^*(\beta(x))$.

Definition. An infinitesimal α is called an infinitesimal of the k-th order relative to an infinitesimal β, if $\alpha=O^*(\beta^k)$, that is, if $\lim\frac{\alpha}{\beta^k}=c$, where c is an arbitrary finite number which is not equal to zero.

Example 3. Let us compare two values $\alpha(x)=\sqrt{1+x^3}-1$, $\beta(x)=x$ as $x\to 0$. Define the order of smallness of the value $\alpha(x)$ about $\beta(x)$. We have

$$\alpha(x)=\left(\sqrt{1+x^3}-1\right)=\frac{\left(1+x^3\right)-1}{\sqrt{1+x^3}+1}=\frac{x^3}{\sqrt{1+x^3}+1}$$

Consequently

$$\lim_{x\to 0}\frac{\alpha(x)}{(\beta(x))^3}=\lim_{x\to 0}\frac{x^3}{x^3\left(\sqrt{1+x^3}+1\right)}=\lim_{x\to 0}\frac{1}{\sqrt{1+x^3}+1}=\frac{1}{2},$$

whence we get that $\alpha(x)=O^*\{(\beta(x))^3\}$, i. e. $\alpha(x)$ is the infinitesimal value of third order about $\beta(x)$.

It is easy to check the following properties:

1. If $\alpha=O^*(\beta)$ and $\beta=O^*(\gamma)$, then $\alpha=O^*(\gamma)$ (property of transaction).
2. If $\alpha=O^*(\gamma)$ and $\beta=o(\gamma)$, then $\alpha+\beta=O^*(\gamma)$.
3. If $\alpha=O^*(\gamma)$ $\beta=O^*(\gamma)$, then either $\alpha+\beta=O^*(\gamma)$, or $\alpha+\beta=O(\gamma)$.

For example let us check the property 3. Due to condition we have

$$\lim\frac{\alpha}{\gamma}=c_1,\ \lim\frac{\beta}{\gamma}=c_2,$$

where c_1 and c_2 are finite numbers which are not equal to zero. So

$$\lim\frac{\alpha+\beta}{\gamma}=\lim\left(\frac{\alpha}{\gamma}+\frac{\beta}{\gamma}\right)=\lim\frac{\alpha}{\gamma}+\lim\frac{\beta}{\gamma}=c_1+c_2.$$

If $c_2\neq -c_1$, then $c_1+c_2\neq 0$, therefore $\alpha+\beta=O^*(\gamma)$. But if $c_2=-c_1$, then $c_1+c_2=0$, hence $\alpha+\beta=O(\gamma)$. In general case

$$\alpha+\beta=\left(\alpha+\beta=O^*(\gamma)\right)\cup\left(\alpha+\beta=o(\gamma)\right).\square$$

2.16. Equivalent Infinitesimal Values

Definition. Infinitesimals values α and β are called equivalent ones if $\lim\frac{\alpha}{\beta}=1$. The equivalent infinitesimals are denoted by: $\alpha\sim\beta$.

Example. Let us consider the variables $\alpha_n=\frac{1}{n}$, $\beta_n=\frac{1}{n+1}$ as $n\to\infty$. Then

$$\lim_{n\to\infty}\frac{\alpha_n}{\beta_n}=\lim_{n\to\infty}\frac{n+1}{n}=1.$$

Hence $\alpha_n\sim\beta_n$ as $n\to\infty$.

The properties of the equivalent infinitesimal values

1. If $(\alpha\sim\beta)\wedge(\beta\sim\gamma)\Rightarrow(\alpha\sim\gamma)$ (Transitivity).

■ Indeed

$$\lim\frac{\alpha}{\gamma}=\lim\left(\frac{\alpha}{\beta}\cdot\frac{\beta}{\gamma}\right)=\lim\frac{\alpha}{\beta}\cdot\lim\frac{\beta}{\gamma}=1\cdot 1=1,$$

Which is what had to be proved. □

2. Theorem. In order for two infinitesimals α and β to be equivalent it is necessary and sufficient that their difference be an infinitesimal of the higher order than every of them.

■ **Necessary**. Let an infinitesimals α and β be equivalent, i.e. $\alpha\sim\beta$. Denote by $\gamma=\alpha-\beta$. Then

$$\lim\frac{\gamma}{\beta}=\lim\frac{\alpha-\beta}{\beta}=\lim\left(\frac{\alpha}{\beta}-1\right)=\lim\frac{\alpha}{\beta}-1=1-1=0,$$

So $\gamma=O(\beta)$, but $\alpha=O^*(\beta)$, then $\gamma=O(\alpha)$.

Sufficiency. Let us consider $\alpha-\beta=\gamma$, where $\gamma=O(\beta)$. Then

$$\lim\frac{\alpha}{\beta}=\lim\frac{\beta+\gamma}{\beta}=\lim\left(1+\frac{\gamma}{\beta}\right)=1+\lim\frac{\gamma}{\beta}=1+0=1,$$

hence $\alpha\sim\beta$.□

Theorem. The limit of the ratio of two infinitesimals does not change if this ratio is changed by equivalent values.

■ Let us suppose that $\alpha \sim \alpha_1, \beta \sim \beta_1$. Then

$$\lim \frac{\alpha}{\beta} = \lim\left(\frac{\alpha}{\alpha_1} \cdot \frac{\alpha_1}{\beta_1} \cdot \frac{\beta_1}{\beta}\right) = \lim \frac{\alpha}{\alpha_1} \lim \frac{\alpha_1}{\beta_1} \lim \frac{\beta_1}{\beta} = 1 \cdot \lim \frac{\alpha_1}{\beta_1} \cdot 1 = \lim \frac{\alpha_1}{\beta_1},$$

Which is what had to be proved.□

Theorem. At adding of two infinitesimal values of the different order it is possible to ignore by infinitesimal of higher order, because remain term will be equivalent to all sum.

■ Let us suppose that $\gamma = \alpha + \beta, \quad \beta = o(\alpha)_1 \Rightarrow \lim \frac{\beta}{\alpha} = 0 \Rightarrow \lim \frac{\alpha + \beta}{\alpha} = 1$.

Which is what had to be proved.□

Consequence. While calculation of limits we can ignored in numerator and denominator by infinitesimals of higher order because limit does not change since remain expressions be equivalent to initial ones.