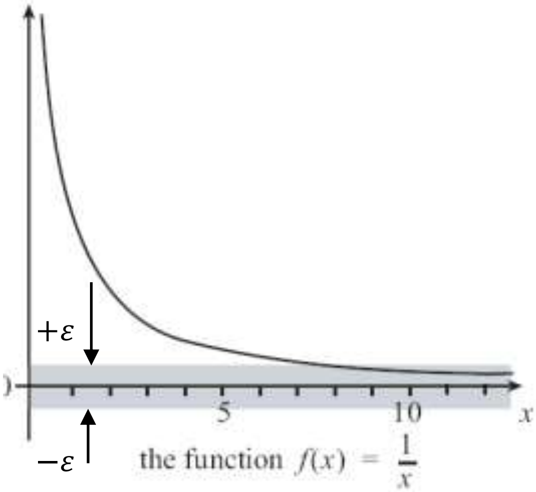
Lecture #16: LIMITS AND CONTINUITY OF FUNCTIONS

16.1 Intuitive Definition of the limit of a single-valued function

The limit of a function of variable x which is a real number can be defined in the same way as the numeric sequence was.

First, we could consider an intuitive limit definition.



Let's consider a function of x whose form is similar to the numerical sequence mentioned above. i.e. $y = \frac{1}{x}$ for $x > 0$.

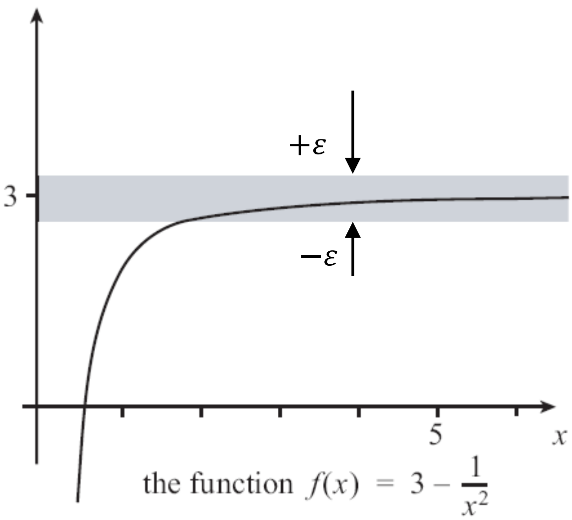
As x gets larger, $f(x)$ gets closer and closer to zero as shown in Fig.

In fact, $f(x)$ will get closer to zero than any small distance ε from $y = 0$ we choose, and will stay closer.

Finally, we can say that $f(x)$ has limit zero as x tends to infinity, and we could write:

$$f(x) = \frac{1}{x} \to 0 \text{ as } x \to \infty \text{ or } \lim_{x\to\infty}\left(\frac{1}{x}\right) = 0.$$

Another intuitive example of a function that has a limit as x tends to infinity is the function $f(x) = 3 - \frac{1}{x^2}$ for $x > 0$.



As x gets larger, $f(x)$ gets closer and closer to 3.

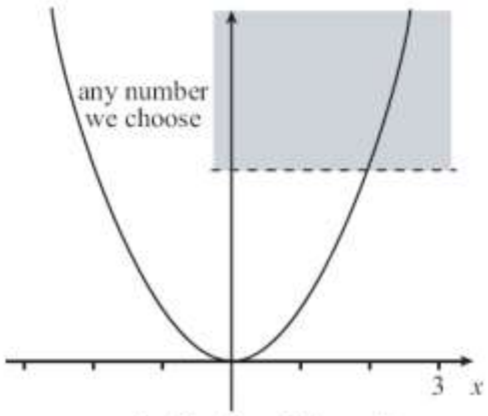
For any small distance ε from $y = 3$, $f(x)$ eventually gets closer to 3 than that distance $y = 3 \pm \varepsilon$, and stays closer.

So we say that $f(x)$ has limit 3 as x tends to infinity, and we could write

$$f(x) = 3 - \frac{1}{x^2} \to 0 \text{ as } x \to \infty \text{ or } \lim_{x\to\infty}\left(3 - \frac{1}{x^2}\right) = 0.$$

In general, we say that $f(x)$ tends to a real limit A as x tends to infinity if, however small a distance ε from $f(x) = A$ we choose, $f(x)$ gets closer than that distance to A and stays closer as x increases.

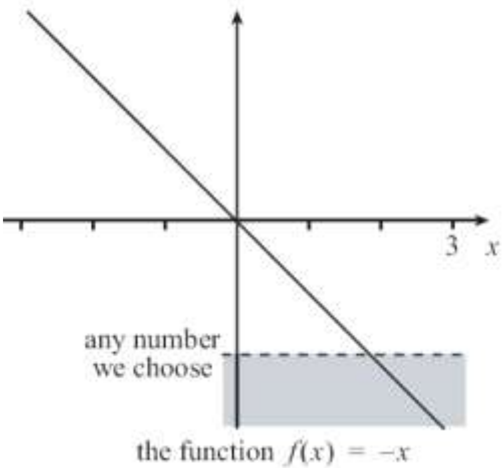
Of course, not all functions have real limits as x tends to infinity. Let us look at some other types of behavior:



the function $f(x) = x^2$

1) If we take the function $f(x) = x^2$, we see that $f(x)$ does not get closer to any particular number as x increases.
Instead, $f(x)$ just gets larger and larger. At some point, $f(x)$ will get larger than any number we choose, and will stay larger.
In this case, we say that $f(x)$ tends to infinity as x tends to infinity, and we could write

$$f(x) = x^2 \to \infty \text{ as } x \to \infty \text{ or } \lim_{x\to\infty} x^2 = \infty.$$



the function $f(x) = -x$

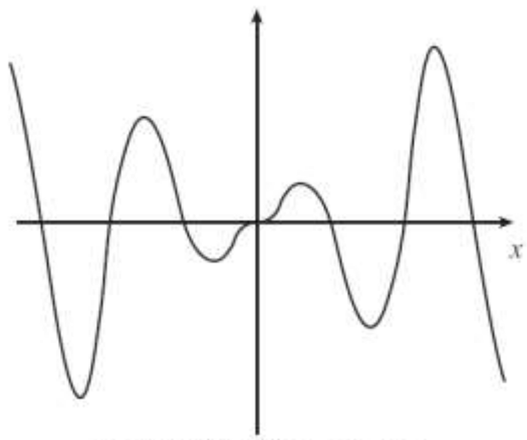
2) The function $f(x) = -x$ does not have a real limit as x tends to infinity as well.
As x gets larger, this function eventually gets more negative than any number we can choose, and it will stay more negative.
In this case, we say that $f(x)$ tends to minus infinity as x tends to infinity, and we could write

$$f(x) = -x \to -\infty \text{ as } x \to \infty \text{ or } \lim_{x\to\infty}(-x) = -\infty.$$

Some functions do not have any kind of limit as x tends to infinity.

3) For example, consider the function $f(x) = x \sin x$.

This function does not get close to any particular real number as x gets large, because we can always choose a value of x to make $f(x)$ larger than



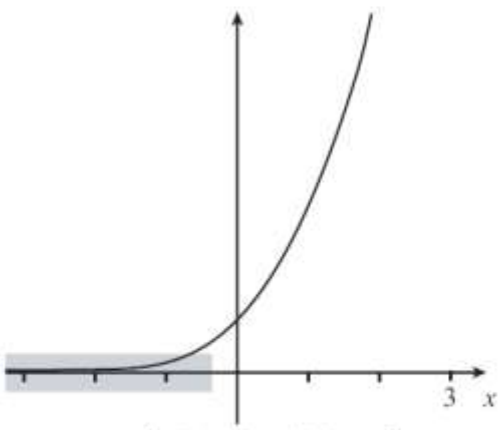
the function $f(x) = x \sin x$

any number we choose.

However, $f(x)$ does not tend to infinity, because it does not stay larger than the number we have chosen, but instead returns to zero. For a similar reason, $f(x)$ does not tend to minus infinity.

So we cannot talk about the limit of this function as x tends to infinity.

As well as defining the limit of a function as x tends to infinity, we can also define the limit as x tends to *minus infinity*.



the function $f(x) = e^x$

Consider the function $f(x) = e^x$. As x becomes more and more negative, $f(x)$ gets closer and closer to zero.

However small a distance ε we choose, $f(x)$ gets closer than that distance to zero, and it stays closer as x becomes more negative.

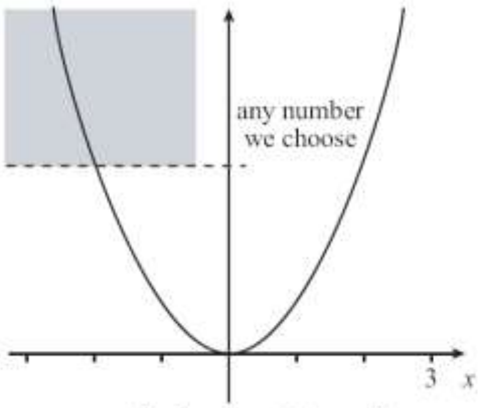
We can say that $f(x)$ has limit zero as x tends to minus infinity, and we could write:

$$f(x) = e^x \to 0 \text{ as } x \to -\infty \text{ or } \lim_{x\to-\infty}\left(e^x\right) = 0.$$

In general, we can write $f(x) \to A$ as $x \to -\infty$ or $\lim_{x\to-\infty} f(x) = A$ if, however small a distance we choose, $f(x)$ eventually gets closer to A than that distance, and stays closer, as x becomes large and negative.

If, as x gets more negative, a function gets larger and stays larger than any number we can choose, we write: $f(x) \to \infty$ as $x \to -\infty$ or $\lim_{x\to-\infty} f(x) = \infty$

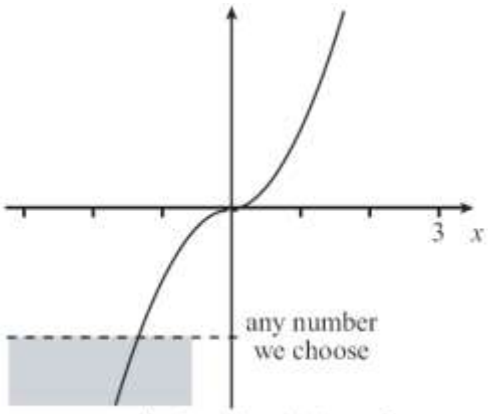
For example, take the function $f(x) = x^2$ again.



We have already seen that it tends to infinity as x tends to infinity.
But it also tends to infinity as x tends to minus infinity.
As x gets large and negative, the function gets larger than any number we can choose, and stays larger. So, we could write:

$f(x) = x^2 \to \infty$ as $x \to -\infty$ or $\lim_{x \to -\infty} x^2 = \infty$

If, instead, as x gets more negative, a function gets more negative and stays more negative than any number we can choose, we could write: $f(x) \to -\infty$ as $x \to -\infty$ or $\lim_{x \to -\infty} f(x) = -\infty$.



As an example, consider the function $f(x) = x^3$. You can see that, as x gets more and more negative, x^3 becomes more negative than any number we can choose, and stays more negative.
So $f(x)$ tends to minus infinity as x tends to minus infinity. So, we could write:

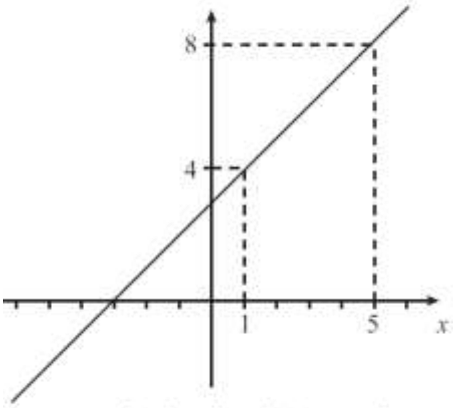
$f(x) = x^3 \to -\infty$ as $x \to -\infty$ or $\lim_{x \to -\infty} x^3 = -\infty$.

Some functions do not have any kind of limit as x tends to minus infinity. For example, consider the function $f(x) = x \sin x$ that we saw earlier.

Thereby, the *key points* are the following:

– The function $f(x)$ has a real limit A as x tends to infinity if, however small a distance ε we choose, $f(x)$ gets closer than this distance to A and stays closer, no matter how large x becomes.
– The function $f(x)$ tends to infinity as x tends to infinity if, however large a number we choose, $f(x)$ gets larger than this number and stays larger, no matter how large x becomes.
– The function $f(x)$ tends to minus infinity as x tends to infinity if, however large and negative a number we choose, $f(x)$ gets more negative than this number and stays more negative, no matter how large x becomes.

There is one more type of limit that we can define for functions.



the function $f(x) = x + 3$

Let us consider the function $f(x) = x + 3$.

If we choose a number, such as 1, then as x gets closer and closer to that number, $f(x)$ also gets closer and closer to a number, in this case 4.

We could write:

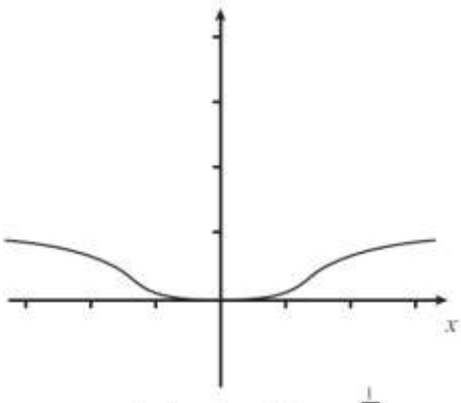
$$f(x) \to 4 \text{ as } x \to 1 \text{ or } \lim_{x \to 1}(x+3) = 4.$$

Similarly, $f(x)$ gets closer and closer to 8 as x gets closer and closer to 5. So we could write:

$$f(x) \to 8 \text{ as } x \to 5 \text{ or } \lim_{x \to 5}(x+3) = 8.$$

Now this definition of a limit might not look very useful. We know that when $x = 1$ then the value of $f(x)$ is 4. And again, when $x = 5$ then the value of $f(x)$ is 8. Why would we want to bother looking at what happens when x gets closer and closer to these numbers?

The reason is that we might sometimes have a function that is not defined at a point.



the function $f(x) = e^{-\frac{1}{x^2}}$

For example, consider the graph of the function $f(x) = e^{-\frac{1}{x^2}}$.

This function is defined for every value apart from zero, because at $x = 0$ we have a fraction with a zero denominator inside the exponent function.

But if we look at the rest of the graph, we can see that $f(x)$ gets closer and closer to zero as x gets closer and closer to zero. So we could write

$$f(x) = e^{-\frac{1}{x^2}} \to 0 \text{ as } x \to 0 \text{ or } \lim_{x \to 0} e^{-\frac{1}{x^2}} = 0.$$

The limit of $f(x)$ as x tends to a real number, is the value $f(x)$ that it approaches as x gets closer to that real number.

Summarizing abovementioned examples, we can give a rigorous definition of the limit of a function.

16.2 Rigorous Definition of a Function Limit at a Point and on Infinity

Let's consider a function $y = f(x)$ defined in some neighbourhood of the point $x=a$, except for maybe the point a itself. The definition domain of the function D_f can be presented as an infinity sequence of numbers $\{x_n\} = x_1, x_2, ..., x_n, ...$ for which we can define that

a) its terms $x_n \in D_f \ \forall n$, but $x_n \neq a \ \forall n$;

b) it converges to a, that is, $\lim_{n\to\infty} x_n = a$.

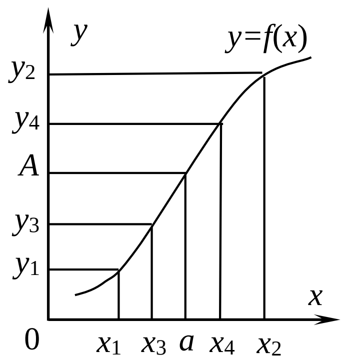
Because between the definition domain of the function and the range of function values there is one-to-one correspondence, then numerical sequence of the function values corresponding to sequence of $\{x_n\}$ can be constructed as the following $\{f(x_n)\} = f(x_1), f(x_2), ..., f(x_n), ...$

Definition (by Heine**).** If the sequence of the function values $\{f(x_n)\}$ corresponding to the sequence of its arguments, for which the mentioned conditions a) and b) are satisfied, converges to some number A, then this number is called *a limit of function* $y = f(x)$ *as x tends to a* (*at the point* $x = a$), and is denoted by

$$\lim_{x\to a} f(x) = A,$$

or

$$f(x) \underset{x\to a}{\to} A.$$



Note that numbers x_n can approach a point a in an arbitrary way but in each case it will be

$$\lim_{n\to\infty} f(x_n) = A.$$

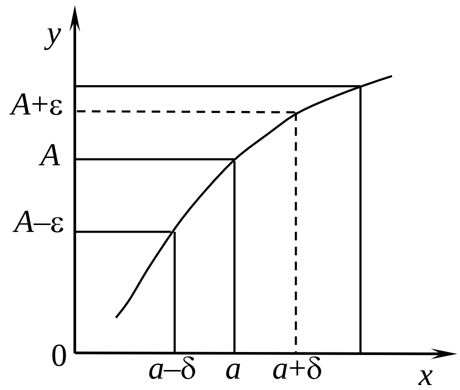
Geometrically it means that if abscises of points x_n approach a then ordinates of these points approach number A.

The definition of the limit, $\lim_{x\to a} f(x) = A$ may be formulated in the other way without connection with sequences. The idea that if x is enough close to a, then $f(x)$ will be close to A can be stated by

Definition (by Cauchy**).** The number A is called *a limit of a function* $y = f(x)$ *as x tends to a* (*at the point* $x = a$), i.e.

$\lim\limits_{x \to a} f(x) = A$, if for any small positive number $\forall \varepsilon > 0$ there exists such number $\delta(\varepsilon) > 0$, the value of which δ depends on ε, then if the inequality $|x - a| < \delta$ is valid, the inequality $|f(x) - A| < \varepsilon$ is valid as well, i.e.

$$(|x - a| < \delta) \Rightarrow (|f(x) - A| < \varepsilon).$$



Obviously, that to smaller value ε, there corresponds the smaller value of δ.

This definition is known as $\varepsilon - \delta$ or Cauchy definition for limit.

Example. Using $\varepsilon - \delta$ definition of limit, show that

$$\lim_{x \to 3} (3x - 2) = 7.$$

Let $\varepsilon > 0$ be an arbitrary positive number. Choose $\delta = \frac{\varepsilon}{3}$. We see that if

$$0 < |x - 3| < \delta,$$

then

$$|f(x) - A| = |(3x - 2) - 7| = |3x - 9| = 3|x - 3| < 3\delta = 3 \cdot \frac{\varepsilon}{3} = \varepsilon.$$

Thus, by Cauchy definition, the limit is proved.

Example. Using $\varepsilon - \delta$ definition of limit, show that $\lim\limits_{x \to 0} a^x = 1$. For definiteness let us suppose that $a > 1$.

Assign an arbitrary $\varepsilon > 0$ and require that

$$|a^x - 1| < \varepsilon, \text{ i. e. } -\varepsilon < a^x - 1 < \varepsilon \Leftrightarrow 1 - \varepsilon < a^x < 1 + \varepsilon,$$

Then

$$\log_a(1 - \varepsilon) < x < \log_a(1 + \varepsilon).$$

Since $\log_a(1 + \varepsilon) < |\log_a(1 - \varepsilon)|$, then let us put $\delta = \log_a(1 + \varepsilon)$.

So we can see that if $|x| < \delta$, then $|a^x - 1| < \varepsilon$, what has to be proved.

Analogously, we can make a statement of the limit of a function at infinity.

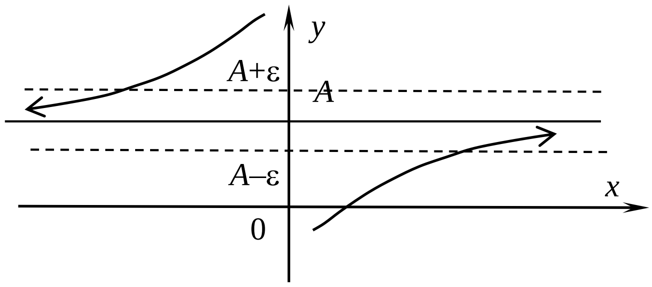
Definition. The number A is a limit of a function $y = f(x)$ at infinity, i.e. $\lim_{x\to\infty} f(x) = A$, if for any small positive number $\forall\varepsilon > 0$ there exists such number $M > 0$, the value of which M depends on ε, then if the inequality $|x| > M$ is valid, the inequality $|f(x) - A| < \varepsilon$ is valid as well, i.e.

$$|x| > M \Rightarrow |f(x) - A| < \varepsilon .$$

We can distinguish the following limits at infinity:

1) If $x \to +\infty$ then a limit $\lim_{x\to+\infty} f(x) = A$ exists if for any small positive number $\forall\varepsilon > 0$ there exists such number $M > 0$, the value of which depends on ε, then if the inequality $x > M$ is valid, the inequality $|f(x) - A| < \varepsilon$ is valid as well, i.e.

$$x > M \Rightarrow |f(x) - A| < \varepsilon$$



2) If $x \to -\infty$ then a limit $\lim_{x\to-\infty} f(x) = A$ exists if for any small positive number $\forall\varepsilon > 0$ there exists such number $N < 0$, the value of which depends on ε, then if the inequality $x < N$ is valid, the inequality $|f(x) - A| < \varepsilon$ is valid as well, i.e.

$$x < N \Rightarrow |f(x) - A| < \varepsilon .$$

A graphical presentation of these two limit statements is presented in the Figure. It is important to notice that the line $y = A$ is called a horizontal asymptote for the graph of $f(x)$.

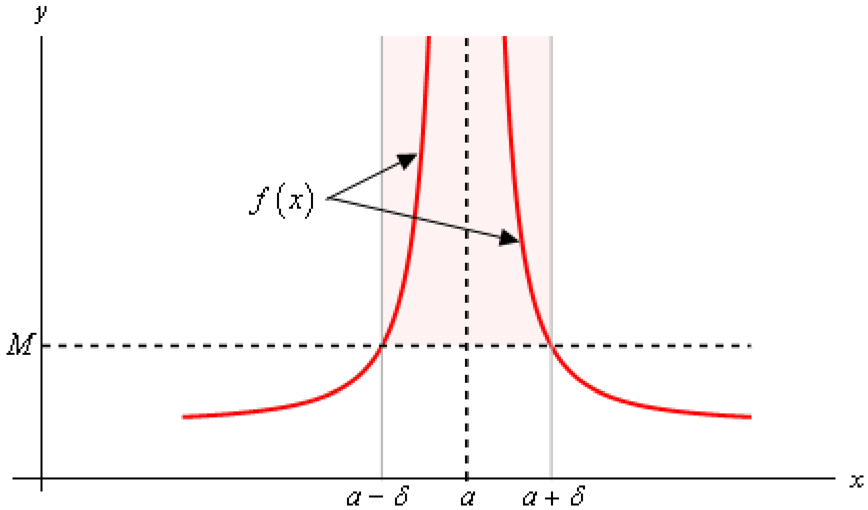
In such way we can give the definition of infinite limits at a point. Here are the two definitions that we need to cover both possibilities, limits that are positive infinity and limits that are negative infinity.

Definition. Let $f(x)$ be a function defined on an interval that contains $x = a$, except possibly at $x = a$. Then we say that,

$$\lim_{x \to a} f(x) = \infty$$

if for every number $M > 0$ there is some number $\delta > 0$ such that $f(x) > M$ whenever $0 < |x - a| < \delta$

Here is a quick sketch illustrating Definition.



What Definition is telling us is that no matter how large we choose M to be we can always find an interval around $x = a$, given by $0 < |x - a| < \delta$ for some number δ, so that as long as we stay within that interval the graph of the function will be above the line $y = M$ as shown in the graph above. Also note that we don't need the function to actually exist at $x = a$ in order for the definition to hold. This is also illustrated in the graph above.

Analogously,

Definition. Let $f(x)$ be a function defined on an interval that contains $x = a$, except possibly at $x = a$. Then we say that,

$$\lim_{x \to a} f(x) = -\infty$$

if for every number $N < 0$ there is some number $\delta > 0$ such that $f(x) < N$ whenever $0 < |x - a| < \delta$

In these two definitions note that M must be a positive number and that N must be a negative number. That's an easy distinction to miss if you aren't paying close attention.

For our final limit definition let's look at limits at infinity that are also infinite in value. There are four possible limits to define here. We'll do one of them and leave the other three to you to write down if you'd like to.

Definition. Let $f(x)$ be a function defined on $x > K$ for some K. Then we say that,

$$\lim_{x \to \infty} f(x) = \infty$$

if for every number $N > 0$ there is some number $M > 0$ such that

$$f(x) > N$$

whenever

$$x > M$$

The other three definitions are almost identical. The only differences are the signs of M and/or N and the corresponding inequality directions.

Next, let's give the precise definitions for the *right- and left-handed limits* (One Sided Limits).

Definition For the right-hand limit we say that,

$$\lim_{x \to a^+} f(x) = A$$

if for every number $\varepsilon > 0$ there is some number $\delta > 0$ such that

$$|f(x) - A| < \varepsilon$$

whenever $0 < x - a < \delta$(or $a < x < a + \delta$)

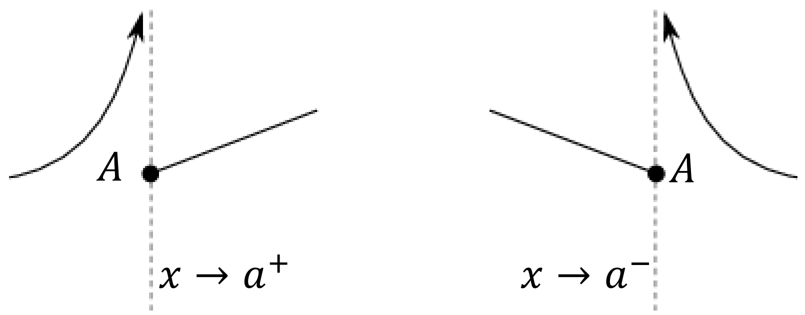
Definition For the left-hand limit we say that,

$$\lim_{x \to a^-} f(x) = A$$

if for every number $\varepsilon > 0$ there is some number $\delta > 0$ such that

$|f(x) - A| < \varepsilon$

whenever $-\delta < x - a < 0$(or $a - \delta < x < a$)



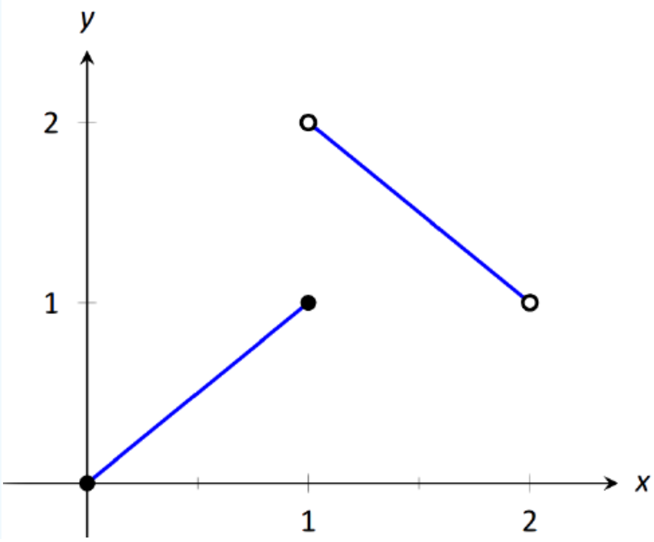
Note Practically speaking, when evaluating a left-hand limit, we consider only values of x "to the left of a," i.e., where $x < a$. The admittedly imperfect notation $x \to a^-$ is used to imply that we look at values of x to the left of a. The notation has nothing to do with positive or negative values of either x or a.

A similar statement holds for evaluating right-hand limits; there we consider only values of x to the right of a, i.e., $x > a$. We can use the theorems from previous sections to help us evaluate these limits; we just restrict our view to one side of a.

Example: Let $f(x) = \begin{cases} x & 0 \le x \le 1 \\ 3 - x & 1 < x < 2 \end{cases}$, as shown in Figure.

Find each of the following:

1. $\lim_{x \to 1^-} f(x)$
2. $\lim_{x \to 1^+} f(x)$
3. $\lim_{x \to 1} f(x)$
4. $f(1)$
5. $\lim_{x \to 0^+} f(x)$
6. $f(0)$
7. $\lim_{x \to 2^-} f(x)$
8. $f(2)$



Solution

For these problems, the visual aid of the graph is likely more effective in evaluating the limits than using f itself. Therefore we will refer often to the graph.

1. As x goes to 1 *from the left*, we see that $f(x)$ is approaching the value of 1. Therefore $\lim_{x \to 1^-} f(x) = 1$.
2. As x goes to 1 *from the right*, we see that $f(x)$ is approaching the value of 2. Recall that it does not matter that there is an "open circle" there; we are evaluating a limit, not the value of the function. Therefore $\lim_{x \to 1^+} f(x) = 2$.
3. *The* limit of f as x approaches 1 does not exist, as discussed in the first section. The function does not approach one particular value, but two different values from the left and the right.
4. Using the definition and by looking at the graph we see that $f(1) = 1$.
5. As x goes to 0 from the right, we see that $f(x)$ is also approaching 0. Therefore $\lim_{x \to 0^+} f(x) = 0$. Note we cannot consider a left-hand limit at 0 as f is not defined for values of $x < 0$.
6. Using the definition and the graph, $f(0) = 0$.
7. As x goes to 2 from the left, we see that $f(x)$ is approaching the value of 1. Therefore $\lim_{x \to 2^-} f(x) = 1$.
8. The graph and the definition of the function show that $f(2)$ is not defined.

16.3 Finding Limits Analytically

In Subsection 1 we explored the concept of the limit without a strict definition, meaning we could only make some approximations.

In the previous subsection we gave the strict definition of the limit and demonstrated how to use it to verify our approximations by using 1) the Heine definition and 2) the Cauchy definition or a $\epsilon - \delta$ proof or

However, this process has its shortcomings, not the least of which is the fact that they are cumbersome. This section gives a series of theorems which allow us to find limits much more quickly and analytically.

The following theorem states that already established limits do behave in a well-known manner.

Basic Limit Properties:

Let b, a, L and K be real numbers, let n be a positive integer, and let $f(x)$ and $g(x)$ be functions with the following limits:

$$\lim_{x\to a} f(x) = L \quad \text{and} \quad \lim_{x\to a} g(x) = K.$$

The following limits hold:

1. Constants: $\lim_{x\to a} b = b$
   The limit of a constant function is the constant
2. Identity: $\lim_{x\to a} x = a$
3. Sums/Differences: $\lim_{x\to a}(f(x) \pm g(x)) = L \pm K$
   Note: This rule states that the limit of the sum of two functions is equal to the sum of their limits: $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$.
   Extended Sum Rule:
   $$\lim_{x\to a}[f_1(x) + \cdots + f_n(x)] = \lim_{x\to a} f_1(x) + \cdots + \lim_{x\to a} f_n(x).$$
4. Scalar Multiples: $\lim_{x\to a} b \cdot f(x) = bL$
   Note: This rule states: $\lim_{x\to a} f(x) = b \lim_{x\to a} f(x)$

5. Products: $\lim_{x\to a} f(x)\cdot g(x) = LK$

   Note: This rule says that the limit of the product of two functions is the product of their limits (if they exist):

$$\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x).$$

   Extended Product Rule:

$$\lim_{x\to a}[f_1(x)f_2(x)\cdots f_n(x)] = \lim_{x\to a} f_1(x)\cdot \lim_{x\to a} f_2(x)\cdots \lim_{x\to a} f_n(x).$$

6. Quotients: $\frac{\lim_{x\to a} f(x)}{g(x)} = \frac{L}{K}$ for $K \neq 0$

   Note: The limit of quotient of two functions is the quotient of their limits, provided that the limit in the denominator function is not zero:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)},\text{ if } \lim_{x\to a} g(x) \neq 0.$$

7. Powers: $\lim_{x\to a} f(x)^n = L^n$

   Note: $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n,$

8. Roots: $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{L}$

   Note: $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}.$

We make a note about Property #8: when n is even, L must be greater than 0. If n is odd, then the statement is true for all L .

9. Limit of an Exponential Function:

$$\lim_{x\to a} b^{f(x)} = b^{\lim_{x\to a} f(x)},$$

   where the base $b > 0$

10. Limit of a Logarithm of a Function:

$$\lim_{x\to a}[\log_b f(x)] = \log_b[\lim_{x\to a} f(x)],$$

   where the base $b > 0$

11. Compositions: Adjust our previously given limit situation to:

$$\lim_{x\to c} f(x) = L,\ \lim_{x\to L} g(x) = K \text{ and } \ g(L) = K.$$

   Then

$$\lim_{x\to c} g(f(x)) = K.$$

12. **The Squeeze Theorem**:

   Suppose that $g(x) \leq f(x) \leq h(x)$ for all x close to a except perhaps for $x = a$. If

$$\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L, \text{ then}$$

$$\lim_{x\to a} f(x) = L.$$

The idea here is that the function $f(x)$ is squeezed between two other functions having the same limit L

We apply the theorem to an example.

Let

$$\lim_{x\to 2} f(x) = 2, \lim_{x\to 2} g(x) = 3 \text{ and } p(x) = 3x^2 - 5x + 7.$$

Find the limits:

1. $\lim_{x\to 2}\big(f(x) + g(x)\big) = \lim_{x\to 2} f(x) + \lim_{x\to 2} g(x) = 2 + 3 = 5$

Note: Use the Sum/Difference rule

2. $\lim_{x\to 2}(5f(x) + g(x)^2) = \lim_{x\to 2} 5f(x) + \lim_{x\to 2}(g(x)^2) = 5 \cdot \lim_{x\to 2} f(x) + (\lim_{x\to 2} g(x))^2 = 5 \cdot 2 + 3^2 = 19.$

Use the Scalar Multiple and Sum/Difference rules

3. $\lim_{x\to 2} p(x) = \lim_{x\to 2}(3x^2 - 5x + 7) = \lim_{x\to 2} 3x^2 - \lim_{x\to 2} 5x + \lim_{x\to 2} 7 = 3 \cdot 2^2 - 5 \cdot 2 + 7 = 9$

Note: Combine the Power, Scalar Multiple, Sum/Difference and Constant Rules

4. Find the limit: $\lim_{x\to 9} \frac{4x^2}{1+\sqrt{x}}$.

Using the properties of limits (the sum rule, the power rule, and the quotient rule), we get

$$\lim_{x\to 9} \frac{4x^2}{1+\sqrt{x}} = \frac{\lim_{x\to 9} 4x^2}{\lim_{x\to 9}(1+\sqrt{x})} = \frac{4\lim_{x\to 9} x^2}{\lim_{x\to 9} 1 + \lim_{x\to 9}\sqrt{x}} = \frac{4 \cdot 9^2}{1+\sqrt{9}} = 81.$$

Note that in task 3 the limit at $x = 2$ is found just by plugging $x = 2$ into the function $p(x)$. This holds true for all polynomials, and also for rational functions (which are quotients of polynomials), as stated in the following theorem.

Theorem. Limits of Polynomial and Rational Functions

Let $p(x)$ and $q(x)$ be polynomials and c a real number. Then:

1. $\lim_{x\to c} p(x) = p(c)$
2. $\lim_{x\to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}, q(c) \neq 0$

Example Evaluate $\lim_{x\to -1} \frac{3x^2-5x+1}{x^4-x^2+3}$.

Using condition 2 of Theorem, find

$$\lim_{x\to -1} \frac{3x^2-5x+1}{x^4-x^2+3} = \frac{3(-1)^2-5(-1)+1}{(-1)^4-(-1)^2+3} = \frac{9}{3} = 3$$

Evaluating a Limit by Factoring and Canceling

Finding the limit of a function expressed as a quotient can be more complicated. We often need to rewrite the function algebraically before applying the properties of a limit. If the denominator evaluates to 0 when we apply the properties of a limit directly, we must rewrite the quotient in a different form. One approach is to write the quotient in factored form and simplify.

Example: Evaluate $\lim_{x\to 2}(\frac{x^2-6x+8}{x-2})$.

Factor where possible, and simplify.

$$\lim_{x\to 2}(\frac{x^2-6x+8}{x-2}) = \lim_{x\to 2}(\frac{(x-2)(x-4)}{x-2}) \quad \text{Factor the numerator.}$$

$$= \lim_{x\to 2}(\frac{\cancel{(x-2)}(x-4)}{\cancel{x-2}}) \quad \text{Cancel the common factors.}$$

$$= \lim_{x\to 2}(x-4) \quad \text{Evaluate.}$$

$$= 2-4 = -2$$

Note: When the limit of a rational function cannot be evaluated directly, factored forms of the numerator and denominator may simplify to a result that can be evaluated.

Notice, the function $f(x) = \frac{x^2-6x+8}{x-2}$ is equivalent to the function $f(x) = x-4, x \neq 2$, but it does not the same as the function $f(x) = x-4$

Notice that the limit exists even though the function is not defined at $x = 2$.

Homework: $\lim_{x\to 7}(\frac{x^2-11x+28}{7-x})$.

Evaluating the Limit of a Quotient by Finding the least common denominator (LCD)

Evaluate: $\lim\limits_{x\to 5}\left(\frac{\frac{1}{x}-\frac{1}{5}}{x-5}\right)$

Find the LCD for the denominators of the two terms in the numerator, and convert both fractions to have the LCD as their denominator.

$$\lim_{x\to 5}\left(\frac{\frac{1}{x}-\frac{1}{5}}{x-5}\right)=\lim_{x\to 5}\left(\frac{\frac{5-x}{5x}}{x-5}\right)=\lim_{x\to 5}\left(\frac{-\cancel{(x-5)}}{5x\cancel{(x-5)}}\right)=-\lim_{x\to 5}\left(\frac{1}{5x}\right)=-\frac{1}{25}$$

When determining the limit of a rational function that has terms added or subtracted in either the numerator or denominator, the first step is to find the common denominator of the added or subtracted terms; then, convert both terms to have that denominator, or simplify the rational function by multiplying numerator and denominator by the least common denominator. Then check to see if the resulting numerator and denominator have any common factors.

Homework: (a) $\lim\limits_{x\to -5}\left(\frac{\frac{1}{5}+\frac{1}{x}}{10+2x}\right)$; (b) $\lim\limits_{x\to 1}\frac{\frac{1}{x+1}-\frac{1}{2}}{x-1}$;

Evaluating a Limit Containing a Root Using a Conjugate (Recall that $a\pm\sqrt{b}$ are conjugates)

Evaluate: $\lim\limits_{x\to 0}(\frac{\sqrt{25-x}-5}{x})$.

1 Multiply numerator and denominator by the conjugate

$$\lim_{x\to 0}\left(\frac{\sqrt{25-x}-5}{x}\right)=\lim_{x\to 0}\left(\frac{\left(\sqrt{25-x}-5\right)}{x}\cdot\frac{\left(\sqrt{25-x}+5\right)}{\left(\sqrt{25-x}+5\right)}\right)$$

2 Multiply: $\sqrt{(25-x)}-5)\cdot(\sqrt{(25-x)}+5)=(25-x)-25$

3 Combine like terms and simplify:

$$=\lim_{x\to 0}\left(\frac{(25-x)-25}{x\left(\sqrt{25-x}+5\right)}\right)=\lim_{x\to 0}\left(\frac{-x}{x\left(\sqrt{25-x}+5\right)}\right)=\frac{-1}{\sqrt{25-0}+5}=\frac{-1}{5+5}=-\frac{1}{10}$$

When determining a limit of a function with a root as one of two terms where we cannot evaluate directly, think about multiplying the numerator and denominator by the conjugate of the terms.

Homework: $\lim\limits_{x\to 5}\frac{\sqrt{x-1}-2}{x-5}$

Evaluating the Limit of a Quotient of a Function by Factoring

Evaluate $\lim_{x\to 4}\left(\frac{4-x}{\sqrt{x}-2}\right)$

1 Factorize the numerator

$$\lim_{x\to 4}\left(\frac{4-x}{\sqrt{x}-2}\right)=\lim_{x\to 4}\left(\frac{(2+\sqrt{x})(2-\sqrt{x})}{\sqrt{x}-2}\right)$$

2 Factor -1 out of the denominator. Simplify

$$=\lim_{x\to 4}\left(\frac{(2+\sqrt{x})\cancel{(2-\sqrt{x})}}{-\cancel{(2-\sqrt{x})}}\right)=\lim_{x\to 4}-(2+x)=-(2+\sqrt{4})=-4$$

Multiplying by a conjugate would expand the numerator; look instead for factors in the numerator. Four is a perfect square so that the numerator is in the form

$$a^2-b^2$$

and may be factored as

$$(a+b)(a-b).$$

Homework: $\lim_{x\to 3}\left(\frac{x-3}{\sqrt{x}-\sqrt{3}}\right)$

Key Concepts

1. The properties of limits can be used to perform operations on the limits of functions rather than the functions themselves.
2. The limit of a polynomial function can be found by finding the sum of the limits of the individual terms.
3. The limit of a function that has been raised to a power equals the same power of the limit of the function. Another method is direct substitution.
4. The limit of the root of a function equals the corresponding root of the limit of the function.
5. One way to find the limit of a function expressed as a quotient is to write the quotient in factored form and simplify.
6. Another method of finding the limit of a complex fraction is to find the LCD.
7. A limit containing a function containing a root may be evaluated using a conjugate.
8. The limits of some functions expressed as quotients can be found by factoring.