Lecture #17: LIMITS AND CONTINUITY OF FUNCTIONS

17.1 Infinitesimals and Infinitely Large Functions

Definition: The function $f(x)$ is called *infinitesimal* (*infinitely small*) as x tends to a, where a is a number or one of the symbols ($\pm\infty$) if

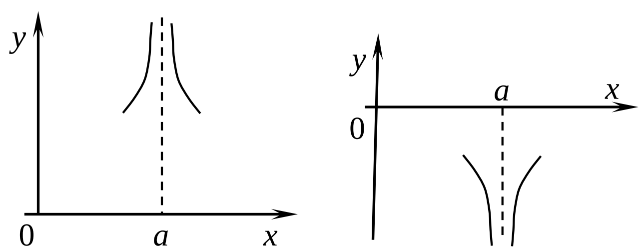
$$\lim_{x\to a} f(x) = 0 \text{ as } x \to a$$

We have the following properties for the infinitesimal functions:

1. The sum of finite number of the infinitesimals is an infinitesimal.
2. The product of an infinitesimal and bounded function is an infinitesimal.
3. The product of infinitesimal functions is an infinitesimal function of the order smaller than the appropriate orders for the each of infinitesimal functions.

Definition: The function $f(x)$ is called *infinitely large* as x tends to a, where a is a number or one of the symbols ($\pm\infty$) if

$$\lim_{x\to a} f(x) = \pm\infty \text{ as } x \to a .$$



Note. In particular cases, the function $f(x)$ can be infinitely large *positive* value or infinitely large *negative* value at the point $x = a$.

We have the following properties for the infinitely large functions:

1. The sum of infinitely large and bounded functions is infinitely large function.
2. The sum of the infinitely large functions of the same sign is infinitely large function of the same sign.
3. The product of infinitely large and bounded functions is an infinitely large function.

4. The product of infinitely large functions is an infinitely large function of the order higher than the appropriate orders for the each of infinitely large functions.

Evaluate: $\lim\limits_{x\to 2}\frac{x-3}{x^2-2x}$

After substituting in $x = 2$, we see that this limit has the form $-\frac{1}{0}$. That is, as x approaches 2, the numerator approaches −1; and the denominator approaches 0. Consequently, the magnitude of $\frac{x-3}{x^2-2x}$ becomes infinite. To get a better idea of what the limit is, we need to factorize the denominator:

$$\lim_{x\to 2}\frac{x-3}{x^2-2x} = \lim_{x\to 2}\frac{x-3}{x(x-2)}.$$

We then separate $\frac{1}{x-2}$ from the rest of the function

$$= \lim_{x\to 2}\frac{x-3}{x}\cdot\frac{1}{x-2}.$$

Here,

$$\lim_{x\to 2}\frac{x-3}{x} = -\frac{1}{2} \text{ (it means the function is bounded)}$$

and

$$\lim_{x\to 2}\frac{1}{x-2} = -\infty \text{ (it means the function is negative infinitely large)}$$

Therefore, the product has the limit

$$\lim_{x\to 2^-}\frac{x-3}{x^2-2x} = +\infty.$$

So, the function $\frac{x-3}{x^2-2x}$ is positive infinitely large function.

Theorem (Connection between infinitesimal and infinitely large functions)

1. If the function $f(x)$ is infinitely large as $x\to a$ and $f(x)$ does not vanish $f(x)\neq 0$ in some neighborhood of $x = a$ then the function $\alpha(x)=\frac{1}{f(x)}$ is an infinitesimal as $x\to a$.

■ Let us prove the theorem 1 taking an arbitrary sequence that converges to

a, i.e. $x_n \underset{n\to\infty}{\to} a$. Then the sequence $\{f(x_n)\}$ is an infinitely large value as $n \to \infty$ and the value $\{\alpha(x)\} = \left\{\dfrac{1}{f(x)}\right\}$ is an infinitesimal, i.e. $\alpha(x_n) \underset{n\to\infty}{\to} 0$, since $\{x_n\}$ is arbitrary one then $\lim\limits_{x\to a} \alpha(x) = 0$.□

2. If the function $\alpha(x)$ is an infinitesimal as $x \to a$ and it does not vanish $\alpha(x) \neq 0$ in some neighborhood of the point a, then the function $f(x) = \dfrac{1}{\alpha(x)}$ is infinitely large as $x \to a$.

The concept of infinitesimal is closely connected with the concept of a limit of function.

Theorem. Let us suppose that $\lim\limits_{x\to a} f(x) = A$. Then $f(x) = A + \alpha(x)$ where $\alpha(x)$ is an infinitesimal function as $x \to a$. Inversely if $f(x) = A + \alpha(x)$ as $x \to a$ then $A = \lim\limits_{x\to a} f(x)$.

■ Let us prove the first statement. Let $A = \lim\limits_{x\to a} f(x)$. It means that if $x_n \underset{n\to\infty}{\to} a$ then $f(x_n) \underset{n\to\infty}{\to} A$, so $f(x_n) - A \underset{n\to\infty}{\to} 0$. Assigning

$$f(x_n) - A = \alpha(x_n),$$

we obtain that $\alpha(x_n) \underset{n\to\infty}{\to} 0$. So $\alpha(x)$ is an infinitesimal function as $x \to a$ so $f(x) = A + \alpha(x)$.□

17.2 The Indeterminate Forms

Consider a limit of the quotient of two functions:

$$\lim_{x\to a} \frac{f(x)}{g(x)}.$$

If $\lim\limits_{x\to a} f(x) = L_1$ and $\lim\limits_{x\to a} g(x) = L_2 \neq 0$ then $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \frac{L_1}{L_2}$.

However, what happens if $\lim\limits_{x\to a} f(x) = 0$ and $\lim\limits_{x\to a} g(x) = 0$? We begin by attempting to apply the theorem and substituting a for x in the quotient.

This gives: $\frac{0}{0}$, i.e. we cannot apply the theorem.

We call this situation as the indeterminate form of type $\left\|\frac{0}{0}\right\|$. This is considered an indeterminate form because we cannot determine the exact behavior of $\frac{f(x)}{g(x)}$ as $x \to a$ without further analysis.

We have seen examples of this earlier in the lectures. For example,

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \frac{2^2 - 4}{2 - 2} = \left\|\frac{0}{0}\right\| =$$

We can evaluate the limit by factoring the numerator and writing

$$= \lim_{x \to 2} \frac{(x+2)\cancel{(x-2)}}{\cancel{x-2}} = \lim_{x \to 2}(x+2) = 2 + 2 = 4.$$

This operation is called the removing of indeterminacy

Problem-Solving Strategy: Calculating a Limit When $\frac{f(x)}{g(x)}$ *has the Indeterminate Form* $\left\|\frac{0}{0}\right\|$.

1. First, we need to make sure that our function has the appropriate form and cannot be evaluated immediately using the limit laws.
2. We then need to find a function that is equal to $h(x) = \frac{f(x)}{g(x)}$ for all $x \neq a$ over some interval containing a. To do this, we may need to try one or more of the following steps:
   (a) If $f(x)$ and $g(x)$ are polynomials, we should factor each function and cancel out any common factors.
   (b) If the numerator or denominator contains a difference involving a square root, we should try multiplying the numerator and denominator by the conjugate of the expression involving the square root.
   (c) If $\frac{f(x)}{g(x)}$ is a complex fraction, we begin by simplifying it.
3. Last, we apply the limit laws.

Examples:

1) Find the limit: $\lim\limits_{x \to 3} \frac{x^3-2x^2-5x+6}{2x^3+3x^2-32x+15}$.

By substituting 3 for x returns the familiar indeterminate form of $||\frac{0}{0}||$.

Since the numerator and denominator are each polynomials, we know that $(x-3)$ is factor of each. Using whatever method is most comfortable to you, factor out $(x-3)$ from each (using polynomial division). We find that

$$\frac{x^3-2x^2-5x+6}{2x^3+3x^2-32x+15} = \frac{(x-3)(x^2+x-2)}{(x-3)(2x^2+9x-5)}.$$

Then, we can cancel the $(x-3)$ terms as long as $x \neq 3$. Using Theorem we conclude:

$$\lim_{x \to 3} \frac{x^3-2x^2-5x+6}{2x^3+3x^2-32x+15} = \lim_{x \to 3} \frac{(x-3)(x^2+x-2)}{(x-3)(2x^2+9x-5)}$$

$$= \lim_{x \to 3} \frac{(x^2+x-2)}{(2x^2+9x-5)} = \frac{10}{40} = \frac{1}{4}$$

2) Evaluate: $\lim\limits_{x \to 3} \frac{x^2-3x}{2x^2-5x-3}$

In fact, if we substitute 3 into the function we get $||\frac{0}{0}||$, which is undefined.

Factoring and canceling is a good strategy to remove this indeterminacy:

$$\lim_{x \to 3} \frac{x^2-3x}{2x^2-5x-3} = \frac{3^2-3\cdot 3}{2\cdot 3^2-5\cdot 3-3} = ||\frac{0}{0}|| = \lim_{x \to 3} \frac{x(x-3)}{(x-3)(2x+1)}$$

Therefore

$$\lim_{x \to 3} \frac{x(x-3)}{(x-3)(2x+1)} = \lim_{x \to 3} \frac{x}{2x+1}.$$

Finally,

$$\lim_{x \to 3} \frac{x}{2x+1} = \frac{3}{2\cdot 3+1} = \frac{3}{7}.$$

Homework: $\lim\limits_{x \to -3} \frac{x^2+4x+3}{x^2-9}$

3) Evaluate: $\lim\limits_{x \to -1} \frac{\sqrt{x+2}-1}{x+1}$

The function has the indeterminate form $||\frac{0}{0}||$at -1.

Let's begin by multiplying by $\sqrt{x+2}+1$, the conjugate of $\sqrt{x+2}-1$, on the numerator and denominator:

$$\lim_{x\to -1}\frac{\sqrt{x+2}-1}{x+1}=\lim_{x\to -1}\frac{\sqrt{x+2}-1}{x+1}\cdot\frac{\sqrt{x+2}+1}{\sqrt{x+2}+1}$$

$$=\lim_{x\to -1}\frac{x+1}{(x+1)(\sqrt{x+2}+1)}=\lim_{x\to -1}\frac{1}{\sqrt{x+2}+1}=\frac{1}{\sqrt{-1+2}+1}=\frac{1}{2}$$

Homework: $\lim_{x\to 5}\frac{\sqrt{x-1}-2}{x-5}$

4) $\lim_{x\to 1}\frac{\frac{1}{x+1}-\frac{1}{2}}{x-1}$

The function has the indeterminate form ||0/0|| at 1

We simplify the algebraic fraction by taking LCD in the numerator and, then, simplify the expression:

$$\lim_{x\to 1}\frac{\frac{1}{x+1}-\frac{1}{2}}{x-1}=\lim_{x\to 1}\frac{2-(x+1)}{2(x-1)(x+1)}=\lim_{x\to 1}\frac{-(x-1)}{2(x-1)(x+1)}$$

$$=\lim_{x\to 1}\frac{-1}{2(x+1)}=\frac{-1}{2(1+1)}=-\frac{1}{4}$$

Homework: $\lim_{x\to -3}\frac{\frac{1}{x+2}+1}{x+3}$

In similar way one can identify indeterminate forms produced by quotients, products, subtractions, and powers:

1. *Indeterminate Form of Type* $\left\|\frac{\infty}{\infty}\right\|$

Let $f(x)$ and $g(x)$ be two functions such that $\lim_{x\to a} f(x)=\pm\infty$ and $\lim_{x\to a} g(x)=\pm\infty$ where a is a real number, or $\pm\infty$. It is said that the function $\frac{f(x)}{g(x)}$ has the indeterminate form $\left\|\frac{\infty}{\infty}\right\|$ at this point.

1) Find the limit: $\lim_{x\to\infty}\frac{3x+5}{2x+1}$

Since $3x+5$ and $2x+1$ are first-degree polynomials with positive leading coefficients, $\lim_{x\to\infty}(3x+5)=\infty$ and $\lim_{x\to\infty}(2x+1)=\infty$. So, we have the form $\left\|\frac{\infty}{\infty}\right\|$

To evaluate this limit, we will divide the numerator and denominator by the **highest power** of x in the numerator and denominator . In doing so, we saw that

$$\lim_{x\to\infty}\frac{3x+5}{2x+1}=\lim_{x\to\infty}\frac{3+5/x}{2+1/x}=\begin{Bmatrix}\lim\limits_{x\to\infty}\frac{5}{x}\to 0\\ \lim\limits_{x\to\infty}\frac{1}{x}\to 0\end{Bmatrix}=\frac{3}{2}$$

2) Find the limit: $\lim_{x\to\infty}\frac{x^3+3x+5}{2x^3-6x+1}$.

This is of the form $\left\|\frac{\infty}{\infty}\right\|$. Divide the numerator and denominator by x^3 since it the highest degree in this expression. Thus, we obtain

$$\lim_{x\to\infty}\frac{x^3+3x+5}{2x^3-6x+1}=[\frac{\infty}{\infty}]=\lim_{x\to\infty}\frac{\frac{x^3+3x+5}{x^3}}{\frac{2x^3-6x+1}{x^3}}=\lim_{x\to\infty}\frac{\frac{x^3}{x^3}+\frac{3x}{x^3}+\frac{5}{x^3}}{\frac{2x^3}{x^3}-\frac{6x}{x^3}+\frac{1}{x^3}}$$

$$=\lim_{x\to\infty}\frac{1+\frac{3}{x^2}+\frac{5}{x^3}}{2-\frac{6}{x^2}+\frac{1}{x^3}}=\frac{\lim\limits_{x\to\infty}(1+\frac{3}{x^2}+\frac{5}{x^3})}{\lim\limits_{x\to\infty}(2-\frac{6}{x^2}+\frac{1}{x^3})}=\frac{\lim\limits_{x\to\infty}1+\lim\limits_{x\to\infty}\frac{3}{x^2}+\lim\limits_{x\to\infty}\frac{5}{x^3}}{\lim\limits_{x\to\infty}2-\lim\limits_{x\to\infty}\frac{6}{x^2}+\lim\limits_{x\to\infty}\frac{1}{x^3}}$$

$$=\frac{1+0+0}{2-0-0}=\frac{1}{2}.$$

3) $\lim_{x\to\infty}\frac{(2x+3)^{10}(3x-2)^{20}}{(x+5)^{30}}$.

We divide both the numerator and denominator by x^{30} the highest power of the fraction:

$$\lim_{x\to\infty}\frac{(2x+3)^{10}(3x-2)^{20}}{(x+5)^{30}}=[\frac{\infty}{\infty}]=\lim_{x\to\infty}\frac{\frac{(2x+3)^{10}(3x-2)^{20}}{x^{30}}}{\frac{(x+5)^{30}}{x^{30}}}$$

$$=\lim_{x\to\infty}\frac{\frac{(2x+3)^{10}}{x^{10}}\cdot\frac{(3x-2)^{20}}{x^{20}}}{\frac{(x+5)^{30}}{x^{30}}}=\lim_{x\to\infty}\frac{(\frac{2x+3}{x})^{10}\cdot(\frac{3x-2}{x})^{20}}{(\frac{x+5}{x})^{30}}$$

$$= \lim_{x\to\infty} \frac{(2+\frac{3}{x})^{10}\cdot(3-\frac{2}{x})^{20}}{(1+\frac{5}{x})^{30}} = \frac{2^{10}\cdot 3^{20}}{1^{30}} = 2^{10}\cdot 3^{10}\cdot 3^{10} = 18^{10}.$$

2. *Indeterminate Form of Type* $\|\infty-\infty\|$

1) Find the limit: $\lim\limits_{x\to 0}\left(\frac{1}{x}+\frac{5}{x(x-5)}\right)$

As $x\to 0$, i.e. after substituting $x=0$ into $\frac{1}{x}$ the magnitude becomes infinite: $f(x)=\frac{1}{x}=\frac{1}{0}\to\infty$ and, also, $g(x)=\frac{5}{x(x-5)}=\frac{1}{0}\to\infty$.

We are interested in $\lim\limits_{x\to 0}(f(x)-g(x))$. Depending on whether $f(x)$ grows faster, or $g(x)$ grows faster, or they grow at the same rate, something can happen in this limit, but we do not say precisely in advance.

Since $f(x)\to\infty$ and $g(x)\to\infty$, we write $\|\infty-\infty\|$ to denote the indeterminate form of this limit. As with our previous indeterminate form, $\|\infty-\infty\|$ has no meaning on its own and we must do more analysis to determine the value of the limit.

To remove this indeterminacy, we will perform addition and then apply our previous strategy. Observe that

$$\frac{1}{x}+\frac{5}{x(x-5)}=\frac{x-5+5}{x(x-5)}=\frac{x}{x(x-5)}.$$

Thus,

$$\lim_{x\to 0}\left(\frac{1}{x}+\frac{5}{x(x-5)}\right)=\|\infty-\infty\|=\lim_{x\to 0}\frac{x}{x(x-5)}=\lim_{x\to 0}\frac{1}{x-5}=-\frac{1}{5}.$$

Homework: $\lim\limits_{x\to 3}\left(\frac{1}{x-3}-\frac{4}{x^2-2x-3}\right)$

2) Evaluate: $\lim\limits_{x\to\infty}\left(\sqrt{x^2+1}-\sqrt{x^2-1}\right)$.

If $x\to\infty$, then

$$\lim_{x\to\infty}\sqrt{x^2+1}=\infty \quad\text{and}\quad \lim_{x\to\infty}\sqrt{x^2-1}=\infty.$$

Thus, we deal here with an indeterminate form of type $\|\infty-\infty\|$ Multiply this expression (both the numerator and the denominator) by the corresponding conjugate expression.

$$\lim_{x\to\infty}(\sqrt{x^2+1}-\sqrt{x^2-1}) = \lim_{x\to\infty}\frac{(\sqrt{x^2+1})^2-(\sqrt{x^2-1})^2}{(\sqrt{x^2+1}+\sqrt{x^2-1})}$$

$$= \lim_{x\to\infty}\frac{x^2+1-(x^2-1)}{(\sqrt{x^2+1}+\sqrt{x^2-1})} = \lim_{x\to\infty}\frac{\not{x^2}+1-\not{x^2}+1}{(\sqrt{x^2+1}+\sqrt{x^2-1})}$$

$$= \lim_{x\to\infty}\frac{2}{(\sqrt{x^2+1}+\sqrt{x^2-1})}.$$

By using the product and the sum rules for limits, we obtain

$$= \lim_{x\to\infty}\frac{2}{(\sqrt{x^2+1}+\sqrt{x^2-1})} = \frac{\lim_{x\to\infty}2}{\lim_{x\to\infty}\sqrt{x^2+1}+\lim_{x\to\infty}\sqrt{x^2-1}} \sim \frac{2}{\infty+\infty} \sim \frac{2}{\infty} = 0.$$

3) $\lim_{t\to+\infty}(\sqrt{t+\sqrt{t+1}}-\sqrt{t})$.

Multiply and divide it by the conjugate expression, we get

$$\lim_{t\to+\infty}(\sqrt{t+\sqrt{t+1}}-\sqrt{t}) = [\infty-\infty] = \lim_{t\to+\infty}\frac{(\sqrt{t+\sqrt{t+1}})^2-(\sqrt{t})^2}{\sqrt{t+\sqrt{t+1}}+\sqrt{t}}$$

$$= \lim_{t\to+\infty}\frac{\not{t}+\sqrt{t+1}-\not{t}}{\sqrt{t+\sqrt{t+1}}+\sqrt{t}} = \lim_{t\to+\infty}\frac{\sqrt{t+1}}{\sqrt{t+\sqrt{t+1}}+\sqrt{t}} = [\frac{\infty}{\infty}].$$

Both the numerator and denominator now approach ∞ as $t\to+\infty$ Hence, we divide numerator and denominator by $t^{1/2}$ the highest power of t in the denominator. Then

$$= \lim_{t\to+\infty}\frac{\sqrt{t+1}}{\sqrt{t+\sqrt{t+1}}+\sqrt{t}} = \lim_{t\to+\infty}\frac{\frac{\sqrt{t+1}}{\sqrt{t}}}{\frac{\sqrt{t+\sqrt{t+1}}+\sqrt{t}}{\sqrt{t}}} = \lim_{t\to+\infty}\frac{\sqrt{\frac{t+1}{t}}}{\sqrt{\frac{t+\sqrt{t+1}}{t}}+1}$$

$$= \lim_{t\to+\infty}\frac{\sqrt{1+\frac{1}{t}}}{\sqrt{1+\sqrt{\frac{1}{t}+\frac{1}{t^2}}}+1} = \frac{\sqrt{1}}{\sqrt{1}+1} = \frac{1}{2}.$$

3. *Indeterminate Form of Type* ||0·∞||

Suppose we want to evaluate $lim_{x\to a}(f(x)\cdot g(x))$, where $f(x)\to 0$ and $g(x)\to\infty$ (or $-\infty$) as $x\to a$. Since one term in the product is approaching zero but the other term is becoming arbitrarily large (in magnitude), anything can happen to the product.

We use the notation ||0·∞|| to denote the form that arises in this situation. The expression ||0·∞|| is considered indeterminate because we cannot determine without further analysis the exact behavior of the product $f(x) \cdot g(x)$ as $x \to a$.

17.3 The First Remarkable Limit

To evaluate limits of various trigonometric functions, first consider a limit:

$$\lim_{x\to 0}\frac{\sin x}{x}$$

Applying our theorems, we attempt to find the limit as

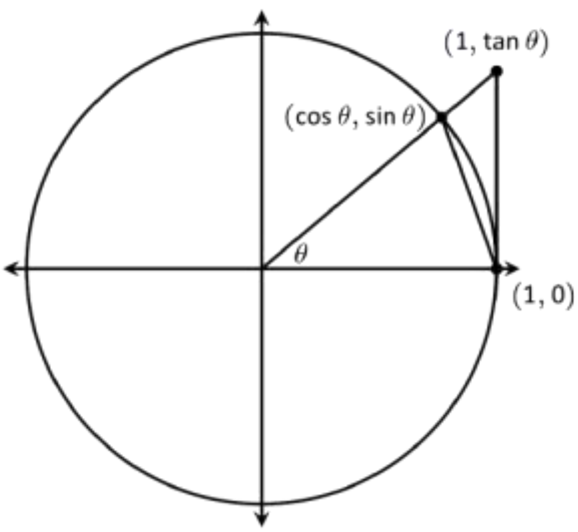
$$\lim_{x\to 0}\frac{\sin x}{x} \to \frac{\sin 0}{0} \to \frac{"0"}{0}.$$

Therefore, we are still unable to evaluate this limit with tools we currently have at hand.

Use the Squeeze Theorem will allow us to show that

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

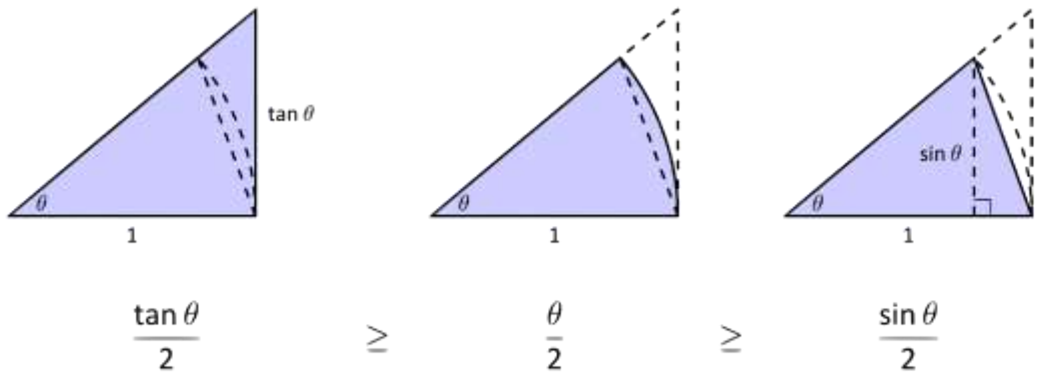
This limit is called the *first remarkable limit.*



To prove it we begin by considering the unit circle. Each point on the unit circle has coordinates $(cos\theta, sin\theta)$ for some angle θ as shown in Figure.

Using similar triangles, we can extend the line from the origin through the point to the point $(1, tan\theta)$, as shown. (Here we are assuming that $0 \leq \theta \leq \pi/2$. Later we will show that we can also consider $\theta \leq 0$)

Figure shows three regions have been constructed in the first quadrant, two triangles and a sector of a circle, which are also drawn below. The area of the large triangle is ½ $tan\theta$; the area of the sector is $\theta/2$; the area of the triangle contained inside the sector is ½ $sin\theta$. It is then clear from the diagram that



Multiply all terms by $\frac{2}{sin\theta}$, giving

$$\frac{1}{\cos\theta} \geq \frac{\theta}{\sin\theta} \geq 1.$$

Taking reciprocals reverses the inequalities, giving

$$\cos\theta \leq \frac{\sin\theta}{\theta} \leq 1.$$

Now take limits.

$$\lim_{\theta\to 0}\cos\theta \leq \lim_{\theta\to 0}\frac{\sin\theta}{\theta} \leq \lim_{\theta\to 0}1$$

$$\cos 0 \leq \lim_{\theta\to 0}\frac{\sin\theta}{\theta} \leq 1$$

$$1 \leq \lim_{\theta\to 0}\frac{\sin\theta}{\theta} \leq 1$$

Clearly this means that

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

Consider x instead of θ, as a result $\lim_{x\to 0}\frac{\sin x}{x} = 1$

Note: this limit tells us more than just that as x approaches 0, $\frac{sin(x)}{x}$ approaches 1. Both x and $sinx$ are approaching 0, but the ratio of x and $sinx$ approaches 1, meaning that they are approaching 0 in essentially the same way.

Another way of viewing this is: for small x , the functions $y = x$ and $y = sinx$ are essentially indistinguishable.

17.4 The Second Remarkable Limit

Consider the limit $\lim_{x\to\infty}(1+\frac{1}{x})^x$

One can see that inside the parentheses we have an expression that is approaching 1 (though never equaling 1), and we know that 1 raised to any power is still 1. At the same time, the power is growing toward infinity. What happens to a number near 1 raised to a very large power? In this case we deal with a new indeterminate forms $\|1^{\infty}\|$

One can prove that in this particular case, the result approaches Euler's number, e , approximately 2.718.

The Number e as a Limit of the Numerical Sequence

Let us consider the numerical sequence

$$x_1=(1+1)^1, x_2=\left(1+\frac{1}{2}\right)^2, x_3=\left(1+\frac{1}{3}\right)^3,\ldots, x_n=\left(1+\frac{1}{n}\right)^n,\ldots$$

and prove that it has a limit as $n\to\infty$.

Taking $n=1,2,3,\ldots,$ we obtain that $x_1=2;\ x_2=2{,}25;\ x_3=2{,}37;\ \ldots,$ so at increasing values n the corresponding value x_n is increasing too. Let us prove that for all the values n it will be $x_{n+1}>x_n$.

Applying Newton's binomial formula one can obtain:

$$\left(1+\frac{1}{n}\right)^n=1+n\frac{1}{n}+\frac{n(n-1)}{2!}\frac{1}{n^2}+\frac{n(n-1)(n-2)}{3!}\frac{1}{n^3}+\ldots$$
$$+\frac{n(n-1)(n-2)\ldots[n-(n-1)]}{n!}\frac{1}{n^n},$$

that is,

$$x_n=2+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\ldots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)$$
$$\ldots\left(1-\frac{n-1}{n}\right). \qquad (*)$$

Substituting instead of n value $n+1$ we can obtain x_{n+1}

$$x_{n+1}=2+\frac{1}{2!}\left(1-\frac{1}{n+1}\right)+\frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)+\ldots$$

$$\ldots+\frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdot\ldots\left(1-\frac{n-1}{n+1}\right).$$

It is easy to note that each term beginning from the second one on the right side is increasing and besides there appears additional positive term, therefore $x_{n+1} > x_n$.

On other hand, due to (*) we have

$$x_n < 2+\frac{1}{2}+\frac{1}{2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\ldots+\frac{1}{2\cdot 3\cdot\ldots\cdot n} < 2+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-1}} <$$

$$< 2+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots = 2+\frac{\frac{1}{2}}{1-\frac{1}{2}} = 3,$$

that is, $x_n < 3,\ \forall n$.

So the sequence $\{x_n\}$ is monotone increasing and is bounded from above by the number 3. Consequently there exists a $\lim\limits_{n\to\infty} x_n$, which is not exceeding the number 3. Besides, since $x_n > 2$, then $\lim\limits_{n\to\infty} x_n > 2$. So,

$$2 < \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \le 3. \Rightarrow \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

The $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ is called the number $e = 2.71828182\ldots$.

Then,

the limit $\lim\limits_{x\to\infty}(1+\frac{1}{x})^x = \|1^\infty\| = e$ is called the *second remarkable limit*

■ Let $x \to +\infty$. Obviously that for every value x that lies between two positive integers number it will be $n \le x < n+1$, whence $\frac{1}{n+1} < \frac{1}{x} \le \frac{1}{n}$. So

$$\left(1+\frac{1}{n+1}\right)^n < \left(1+\frac{1}{x}\right)^x \le \left(1+\frac{1}{n}\right)^{n+1} \qquad (**)$$

Let us calculate the following limits

$$\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{n}=\lim_{n\to\infty}\frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{1+\frac{1}{n+1}}=\frac{\lim\limits_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{n+1}}{\lim\limits_{n\to\infty}\left(1+\frac{1}{n+1}\right)}=\frac{e}{1}=e;$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n+1}\right)^{1}=e\cdot 1=e.$$

Therefore, passing to the limit in relation (**) and using the Squeeze Theorem we obtain

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=e.$$

Let us prove that if $x\to-\infty$ then

$$\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^{x}=e.$$

Let us fulfill the following substitution $t=-(x+1)\Rightarrow x=-(t+1)$

$$\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^{x}=\lim_{t\to\infty}\left(1-\frac{1}{t+1}\right)^{-(t+1)}=\lim_{t\to\infty}\left(\frac{t}{t+1}\right)^{-(t+1)}=$$

$$=\lim_{t\to\infty}\left(\frac{t+1}{x}\right)^{t+1}=\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^{t+1}=\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^{t}\cdot\lim_{t\to\infty}\left(1+\frac{1}{t}\right)=e\cdot 1=e.$$

The another form second remarkable limit is

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}}=\left\|1^{\infty}\right\|=e$$

Note: The second remarkable limit can be used to remove an Indeterminate Form $||\infty^0||$ as well

17.5 Comparison of the Infinitesimals

In order to compare two infinitesimal values it is necessary to calculate a limit of their ratio. Since we consider both the comparison of the infinitesimal of the numerical sequences $\{\alpha_n\},\{\beta_n\}$ and infinitesimal functions $\alpha(x)$ and $\beta(x)$ then instead of record $\lim\limits_{n\to\infty}\frac{\alpha_n}{\beta_n}$ or $\lim\limits_{x\to a}\frac{\alpha(x)}{\beta(x)}$ we will write just $\lim\frac{\alpha}{\beta}$.

Let the values α and β be infinitesimals values.

1. If $\lim\frac{\alpha}{\beta}=0$, then it means that α approaches zero more quickly than β. In this case we will say that α is an infinitesimal value of the higher order than β and write:

$$\alpha=o(\beta).$$

Example 1. Let us compare two infinitesimals values

$$\alpha_n=\frac{1}{n^2+1},\ \beta_n=\frac{1}{n+1}$$

as $n\to\infty$. With this purpose we calculate the following limit

$$\lim_{n\to\infty}\frac{\alpha_n}{\beta_n}=\lim_{n\to\infty}\frac{n+1}{n^2+1}=0.$$

It means that $\alpha=o(\beta)$.

2. If $\lim\frac{\alpha}{\beta}=\infty$, then we will say that an infinitesimal α is a value of lower order than β. Obviously that in this case $\beta=o(\alpha)$.

3. If $\lim\frac{\alpha}{\beta}=c$, where $c<\infty$ and $c\neq 0$, then α and β are called infinitesimals values of the same order. In this case we will write down: $\alpha=O^*(\beta)$, or $\beta=O^*(\alpha)$.

Example 2. Let values $\alpha(x)=\sqrt{1+x}-1,\quad \beta(x)=x$ be given as $x\to 0$. Then

$$\lim_{x\to 0}\frac{\alpha(x)}{\beta(x)}=\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}=\frac{1}{2},$$

i. e., as $x\to 0$ then $\alpha(x)=O^*(\beta(x))$.

Definition. An infinitesimal α is called an infinitesimal of the k-th order relative by to an infinitesimal β, if $\alpha = O^*\left(\beta^k\right)$, that is if $\lim \frac{\alpha}{\beta^k} = c$, where c is an arbitrary finite number which is not equal to zero $(c \neq 0)$.

Example 3. Let us compare two values $\alpha(x) = \sqrt{1+x^3} - 1$, $\beta(x) = x$ as $x \to 0$. Define the order of smallness of the value $\alpha(x)$ about $\beta(x)$. We have

$$\alpha(x) = \left(\sqrt{1+x^3} - 1\right) = \frac{\left(1+x^3\right) - 1}{\sqrt{1+x^3} + 1} = \frac{x^3}{\sqrt{1+x^3} + 1}$$

Consequently

$$\lim_{x\to 0} \frac{\alpha(x)}{(\beta(x))^3} = \lim_{x\to 0} \frac{x^3}{x^3\left(\sqrt{1+x^3} + 1\right)} = \lim_{x\to 0} \frac{1}{\sqrt{1+x^3} + 1} = \frac{1}{2},$$

whence we get that $\alpha(x) = O^*\left\{(\beta(x))^3\right\}$, i. e. $\alpha(x)$ is the infinitesimal value of third order about $\beta(x)$.

It is easy to check the following properties:

1. If $\alpha = O^*(\beta)$ and $\beta = O^*(\gamma)$, then $\alpha = O^*(\gamma)$ (property of transaction).

2. If $\alpha = O^*(\gamma)$ and $\beta = o(\gamma)$, then $\alpha + \beta = O^*(\gamma)$.

3. If $\alpha = O^*(\gamma)$ $\beta = O^*(\gamma)$, then either $\alpha + \beta = O^*(\gamma)$ or $\alpha + \beta = o(\gamma)$.

For example, let us check the property 3. Due to condition we have

$$\lim \frac{\alpha}{\gamma} = c_1, \ \lim \frac{\beta}{\gamma} = c_2,$$

where c_1 and c_2 are finite numbers which are not equal to zero. So

$$\lim \frac{\alpha + \beta}{\gamma} = \lim\left(\frac{\alpha}{\gamma} + \frac{\beta}{\gamma}\right) = \lim \frac{\alpha}{\gamma} + \lim \frac{\beta}{\gamma} = c_1 + c_2.$$

If $c_2 \neq -c_1$, then $c_1 + c_2 \neq 0$, therefore $\alpha + \beta = O^*(\gamma)$. But if $c_2 = -c_1$, then $c_1 + c_2 = 0$, hence $\alpha + \beta = o(\gamma)$. In general case

$$\alpha + \beta = \left(\alpha + \beta = O^*(\gamma)\right) \cup \left(\alpha + \beta = o(\gamma)\right). \square$$

Equivalent Infinitesimals

Definition. Infinitesimals values α and β are called equivalent ones if $\lim\frac{\alpha}{\beta}=1$. The equivalent infinitesimals are denoted by: $\alpha\sim\beta$.

Example. Let us consider the variables $\alpha_n=\frac{1}{n}$, $\beta_n=\frac{1}{n+1}$ as $n\to\infty$. Then

$$\lim_{n\to\infty}\frac{\alpha_n}{\beta_n}=\lim_{n\to\infty}\frac{n+1}{n}=1.$$

Hence $\alpha_n\sim\beta_n$ as $n\to\infty$.

The properties of the equivalent infinitesimals

1. If $(\alpha\sim\beta)\wedge(\beta\sim\gamma)\Rightarrow(\alpha\sim\gamma)$ (Transitivity).

■ Indeed

$$\lim\frac{\alpha}{\gamma}=\lim\left(\frac{\alpha}{\beta}\cdot\frac{\beta}{\gamma}\right)=\lim\frac{\alpha}{\beta}\cdot\lim\frac{\beta}{\gamma}=1\cdot1=1,$$

Which is what had to be proved. □

2. Theorem. In order for two infinitesimals α and β to be equivalent it is necessary and sufficient that their difference be an infinitesimal of the higher order than each of them.

■ Necessary. Let the infinitesimals α and β be equivalent, i.e. $\alpha\sim\beta$. Denote by $\gamma=\alpha-\beta$. Then

$$\lim\frac{\gamma}{\beta}=\lim\frac{\alpha-\beta}{\beta}=\lim\left(\frac{\alpha}{\beta}-1\right)=\lim\frac{\alpha}{\beta}-1=1-1=0,$$

So $\gamma=o(\beta)$, but $\alpha=O^*(\beta)$, then $\gamma=o(\alpha)$.

Sufficiency**.** Let us consider $\alpha-\beta=\gamma$, where $\gamma=O(\beta)$. Then

$$\lim\frac{\alpha}{\beta}=\lim\frac{\beta+\gamma}{\beta}=\lim\left(1+\frac{\gamma}{\beta}\right)=1+\lim\frac{\gamma}{\beta}=1+0=1,$$

hence $\alpha\sim\beta$.□

Theorem. The limit of the ratio of two infinitesimals does not change if this ratio is changed by equivalent values.

■ Let us suppose that $\alpha \sim \alpha_1, \beta \sim \beta_1$. Then

$$\lim\frac{\alpha}{\beta}=\lim\left(\frac{\alpha}{\alpha_1}\cdot\frac{\alpha_1}{\beta_1}\cdot\frac{\beta_1}{\beta}\right)=\lim\frac{\alpha}{\alpha_1}\lim\frac{\alpha_1}{\beta_1}\lim\frac{\beta_1}{\beta}=1\cdot\lim\frac{\alpha_1}{\beta_1}\cdot 1=\lim\frac{\alpha_1}{\beta_1},$$

Which is what had to be proved.□

Example. Calculate $\lim\limits_{x\to 0}\frac{\ln(1+x)}{x}$

Solution.

$$\lim_{x\to 0}\frac{\ln(1+x)}{x}=\lim_{x\to 0}\frac{1}{x}\ln(1+x)=\lim_{x\to 0}\ln(1+x)^{\frac{1}{x}}=\ln\lim_{x\to 0}(1+x)^{\frac{1}{x}}=\ln e=1.$$

So

$$\ln(1+x)\sim x,\ \text{as}\ x\to 0.$$

Obvious that

$$\log_a(1+x)\sim\frac{x}{\ln a},\ x\to 0.$$

Example. Calculate $\lim\limits_{x\to 0}\frac{a^x-1}{x\ln a}$.

Solution.

$$\lim_{x\to 0}\frac{a^x-1}{x\ln a}=\lim_{x\to 0}\frac{\ln(1+a^x-1)}{x\ln a}=\lim_{x\to 0}\frac{\ln a^x}{x\ln a}=\lim_{x\to 0}\frac{x\ln a}{x\ln a}=1.$$

So

$$a^x-1\sim x\ln a,\ x\to 0.$$

Theorem. At adding of two infinitesimal values of the different order it is possible to ignore by infinitesimal of higher order, because remaining term will be equivalent to all sum.

■ Let us suppose that

$$\gamma=\alpha+\beta,\quad \beta=o(\alpha)_1\Rightarrow\lim\frac{\beta}{\alpha}=0\Rightarrow\lim\frac{\alpha+\beta}{\alpha}=1.$$

Which is what had to be proved.□

Consequence. While calculation of limits we can ignore in the numerator and denominator by infinitesimals of higher order because the limit does not change since remaining expressions are equivalent to initial ones. Later we will prove that values presented in Tables 1 and 2 are equivalent.

The consequences of the first remarkable limit

There exist the following consequences from the 1-st remarkable limit.

$\sin x \sim x,\ x \to 0$	$\tan x \sim x,\ x \to 0$
$\arcsin x \sim x,\ x \to 0$	$\arctan x \sim x,\ x \to 0$
$1-\cos x \sim \frac{x^2}{2},\ x \to 0$	$1-\text{ch}x \sim \frac{x^2}{2},\ x \to 0$

The consequences of the second remarkable limit

There exist the following consequences from the 2-nd remarkable limit.

$a^x - 1 \sim x\ln a,\ x \to 0$	$e^x - 1 \sim x,\ x \to 0$
`$\log_a(1+x) \sim \frac{x}{\ln a},$ $x \to 0$	$\ln(1+x) \sim x,\ x \to 0$
$(1+x)^{\alpha} - 1 \sim \alpha x,\ x \to 0$	$(1+x)^{\frac{1}{\alpha}} - 1 \sim \frac{x}{\alpha},\ x \to 0$
$\text{sh}x \sim x,\ x \to 0$	$\text{th}\,x \sim x,\ x \to 0$