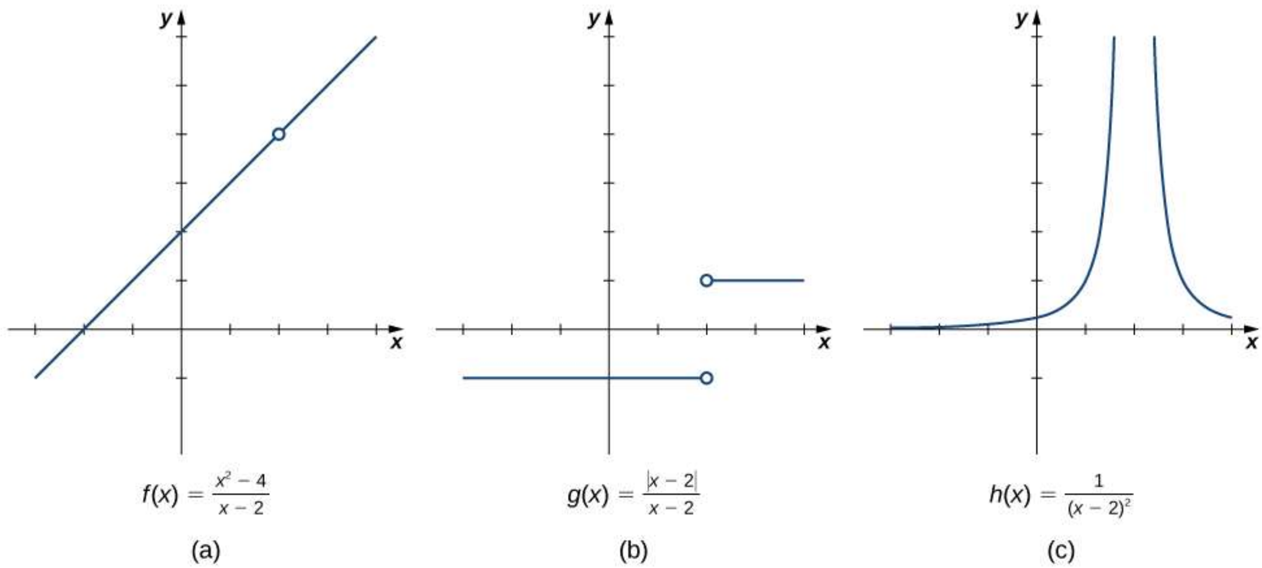
We are studying the concept of a limit or limiting process, which is essential to the understanding of calculus.
We began understanding limits by using an intuitive approach. At the end, armed with a conceptual understanding of limits, we examined the formal definition of a limit.

Recall the intuitive understanding limits. Let's take a look at the graphs of the functions:
(a) $f(x) = \frac{x^2-4}{x-2}$ (b) $g(x) = \frac{|x-2|}{x-2}$ (c) $h(x) = \frac{1}{(x-2)^2}$



Let's focus our attention on the behavior of the graphs at and around $x = 2$. As we move from left to right along the graph of functions, we see that the functions are undefined at $x = 2$, but if we make this statement and no other, we get a very incomplete picture of how each function behaves in the vicinity of $x = 2$. To express the behavior of each graph in the vicinity of 2 more completely, we need to introduce the concept of a limit.

(a) Let's first take a closer look at how the function $f(x) = \frac{x^2-4}{x-2}$ behaves around $x = 2$. As the values of x approach 2 from either side of 2, the values of $y = f(x)$ approach 4. Mathematically, we say that the limit of $f(x)$ as x approaches 2 is 4. Symbolically, we express this limit as

$$\lim_{x\to 2} f(x) = 4$$

We can estimate limits by constructing tables of functional values and by looking at their graphs. This process is described in the following Problem-Solving Strategy:
To evaluate $\lim_{x\to a} f(x)$, we begin by completing a table of functional values. We should choose two sets of x-values—one set of values approaching a and less than a, and another

set of values approaching a and greater than a . Table demonstrates what your tables might look like:

x	$f(x)$	x	$f(x)$
$a-0.1$	$f(a-0.1)$	$a+0.1$	$f(a+0.1)$
$a-0.01$	$f(a-0.01)$	$a+0.001$	$f(a+0.001)$
$a-0.001$	$f(a-0.001)$	$a+0.0001$	$f(a+0.001)$
$a-0.0001$	$f(a-0.0001)$	$a+0.00001$	$f(a+0.0001)$

Calculating the functional values of $f(x)$ for x-values near a with a selected accuracy, we can watch the y-value as the x-values approach a. If the y-values approach L as our x-values approach a from both directions, then $\lim_{x\to a} f(x) = L$

$x = 2$	$f(x) = \dfrac{x^2-4}{x-2}$	x=2	$f(x) = \dfrac{x^2-4}{x-2}$
$2-0.1 = 1.9$	$f(1.9) = 3.9$	$2+0.1 = 2.1$	$f(2.1)$ =4.1
$2-0.01 = 1.99$	$f(1.99)$ =3.99	2+0.01 = 2.01	$f(2.01) = 4.01$
$2-0.001 = 1.999$	$f(1.999) = 3.999$	$2+0.001 = 2.001$	$f(2.001) = 4.001$
$2-0.0001 = 1.9999$	$f(1.9999) = 3.9999$	$2+0.0001 = 2.0001$	$f(2.0001) = 4.0001$

As a result, we can say that $\lim_{x\to 2} \frac{x^2-4}{x-2} = 4 \pm \varepsilon$ with accuracy $\varepsilon = 0.0001$ selected in advance (or within a certain tolerance level of 4).

(b) Let's consider how the function $g(x) = \frac{|x-2|}{x-2}$ behaves around $x = 2$.

As the values of x approach 2 from the left side of 2, the values of $y = g(x)$ approach -1, whereas as the values of x approach 2 from the right side of 2, the values of $y = g(x)$ approach +1

$x = 2$	$g(x) = \dfrac{\lvert x-2\rvert}{x-2}$	$x = 2$	$g(x) = \dfrac{\lvert x-2\rvert}{x-2}$
$2-0.1 = 1.9$	$f(1.9) = -1$	$2+0.1 = 2.1$	$f(2.1) = 1$
$2-0.01 = 1.99$	$f(1.99) = -1$	2+0.01 = 2.01	$f(2.01) = 1$
$2-0.001 = 1.999$	$f(1.999) = -1$	$2+0.001 = 2.001$	$f(2.001) = 1$
$2-0.0001 = 1.9999$	$f(1.9999) = -1$	$2+0.0001 = 2.0001$	$f(2.0001) = 1$

(c) Let's consider how the function $h(x) = \frac{1}{(x-2)^2}$ behaves around $x = 2$.

As the values of x approach 2 from the either side of 2, the values of $y = h(x)$ tend to positive infinity.

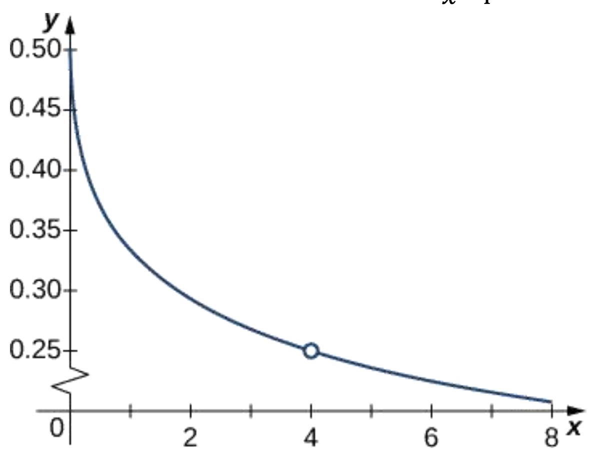
$x=2$	$h(x)=\frac{1}{(x-2)^2}$	$x=2$	$h(x)=\frac{1}{(x-2)^2}$
$2-0.1=1.9$	$f(1.9)=100$	$2+0.1=2.1$	$f(2.1)=100$
$2-0.01=1.99$	$f(1.99)=10000$	$2+0.01=2.01$	$f(2.01)=10000$
$2-0.001=1.999$	$f(1.999)=10^6$	$2+0.001=2.001$	$f(2.001)=10^6$
$2-0.0001=1.9999$	$f(1.9999)=10^8$	$2+0.0001=2.0001$	$f(2.0001)=10^8$

2) Evaluate $\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}$ using a table of functional values.

x	$\frac{\sqrt{x}-2}{x-4}$	x	$\frac{\sqrt{x}-2}{x-4}$
3.9	0.251582341869	4.1	0.248456731317
3.99	0.25015644562	4.01	0.24984394501
3.999	0.250015627	4.001	0.249984377
3.9999	0.250001563	4.0001	0.249998438
3.99999	0.25000016	4.00001	0.24999984

We see that the functional values less than 4 appear to be decreasing toward 0.25 whereas the functional values greater than 4 appear to be increasing toward 0.25. We conclude that $\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}=\frac{1}{4}\pm\varepsilon$, with accuracy $\varepsilon=0.00001$ (or within a certain tolerance level of 1/4)

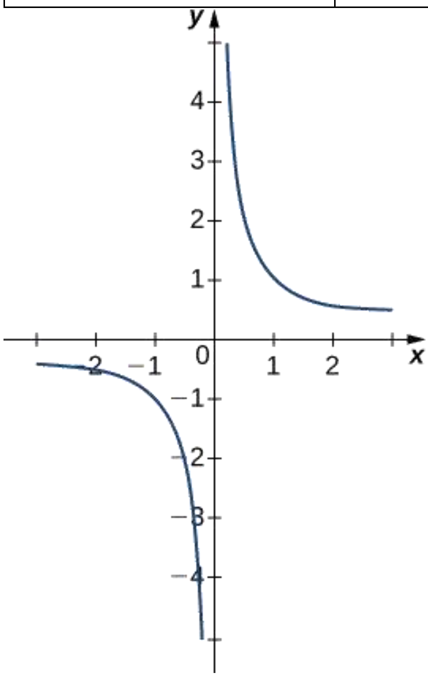
We confirm this estimate using the graph of $f(x)=\frac{\sqrt{x}-2}{x-4}$



3) Evaluate $\lim_{x\to 0}\frac{1}{x}$

x	$1/x$	x	$1/x$
-0.1	-10	0.1	10
-0.01	-100	0.01	100
-0.001	-1000	0.001	1000
-0.0001	-10,000	0.0001	10,000
-0.00001	-100,000	0.00001	100,000

-0.000001	-1,000,000	0.000001	1,000,000



We see that the functional values less than 0 appear to be decreasing toward -10^6 whereas the functional values greater than 0 appear to be increasing toward 10^6. We conclude that

if x approaches 0 from the left: $\lim\limits_{x \to 0^-} \frac{1}{x} = -\infty$;

if x approaches 0 from the right: $\lim\limits_{x \to 0^+} \frac{1}{x} = +\infty$;

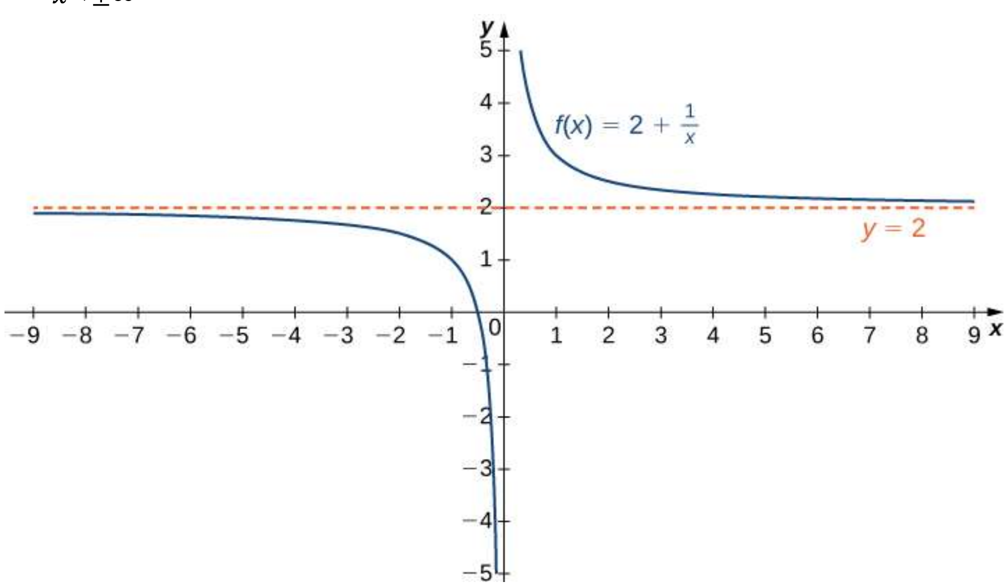
Since the limits have different values, we conclude that

$$\lim_{x \to 0} \frac{1}{x} \text{ does not exit}$$

4) Evaluate: $\lim\limits_{x \to \pm\infty} \left(2 + \frac{1}{x}\right)$

x	10	100	1,000	10,000
$2 + \frac{1}{x}$	2.1	2.01	2.001	2.0001
x	−10	−100	−1000	−10,000
$2 + \frac{1}{x}$	1.9	1.99	1.999	1.9999

We see that as x becomes positive larger and larger, the functional values approach 2 whereas as x becomes negative larger and larger, the functional values get near 2. We conclude that $\lim\limits_{x \to \pm\infty} (2 + \frac{1}{x}) = 2$ (within a certain tolerance level of 2)



Given a function $y = f(x)$ and an x -value, c , we say that "the limit of the function f, as x approaches c , is a value L " or by the other words

if " y tends to L " as " x tends to c ."

if " y approaches L " as " x approaches c ."

if " y is near L " whenever " x is near c ."

In fact, the function may not even exist at c or may equal some value different than L at c.

The problem with these definitions is that the words "tends," "approach," and especially "near" are not exact. In what way does the variable x tend to, or approach, c ? How near do x and y have to be to c and L , respectively? So, more rigorous definition is required.

We introduce the formal definition of a limit known as "the epsilon--delta," definition:

If x is within a certain tolerance level of c , then the corresponding value $y = f(x)$ is within a certain tolerance level of L .

The traditional notation for the x -tolerance is the *lowercase Greek letter delta*, or δ , and the y -tolerance is denoted by *lowercase Greek letter epsilon*, or ε . One more rephrasing the previous definition:

If x is within δ units of c , then the corresponding value of y is within ϵ units of L . We can write " x is within δ units of c " mathematically as

$$|x - c| < \delta, \text{which is equivalent to } c - \delta < x < c + \delta.$$

Similarly, " y is within ϵ units of L " is

$$|y - L| < \varepsilon, \text{which is equivalent to } L - \varepsilon < y < L + \varepsilon.$$

Finally, the wordless definition of the limit:

$$\lim_{x \to c} f(x) = L \Leftrightarrow \forall \epsilon > 0, \exists \delta > 0 s.t. 0 < |x - c| < \delta \longrightarrow |f(x) - L| < \epsilon.$$

An example will help us understand this definition.

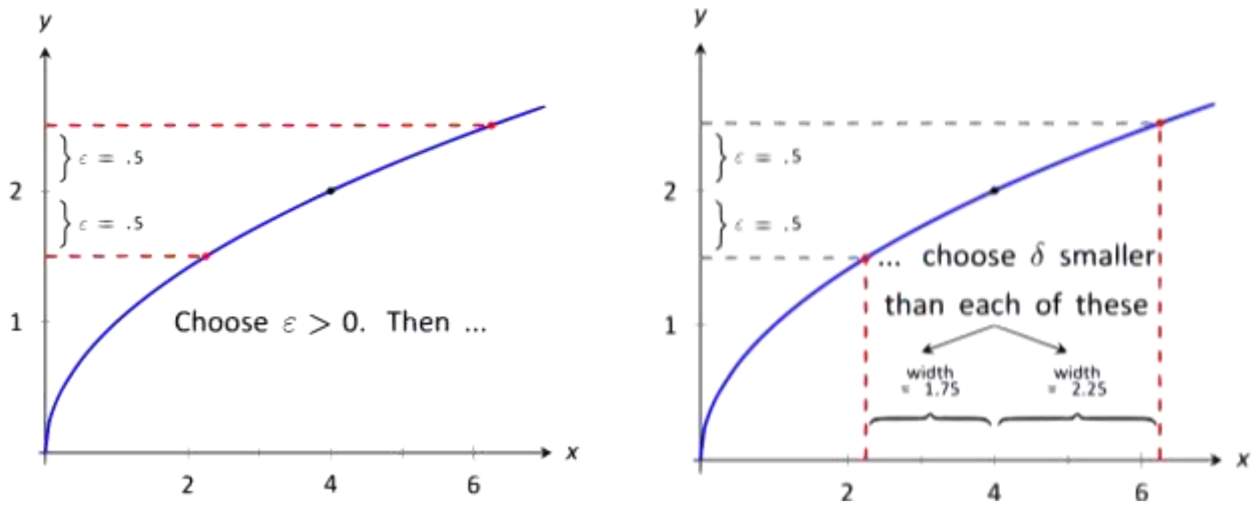
Show that $\lim_{x \to 4} \sqrt{x} = 2.$

Before we use the formal definition, let's try some numerical tolerances. What if the y tolerance is 0.5, or $\epsilon = 0.5$? How close to 4 does x have to be so that y is within 0.5 units of 2, i.e., $1.5 < y < 2.5$? In this case, we can proceed as follows:

$$
\begin{aligned}
1.5 &< y < 2.5 \\
1.5 &< \sqrt{x} < 2.5 \\
1.5^2 &< x < 2.5^2 \\
2.25 &< x < 6.25.
\end{aligned}
$$

So, what is the desired x tolerance?

Remember, we want to find a symmetric interval of x values, namely $4 - \delta < x < 4 + \delta$. The lower bound of 2.25 is 1.75 units from 4 whereas the upper bound of 6.25 is 2.25 units from 4. So, we need the smaller of these two distances; we must have $\delta \leq 1.75$. See Figure



With $\epsilon = 0.5$ we pick $\delta \leq 1.75$

Given the y tolerance $\epsilon = 0.5$, we have found an x tolerance, $\delta \leq 1.75$, such that whenever x is within δ units of 4, then y is within ϵ units of 2. That's what we were trying to find.

One can try to take the other y tolerance, e.g. $\epsilon = 0.01$. Again how close to 4 does x have to be in order for y to be within 0.01 units of 2 or $1.99 < y < 2.01$?

So, one more we just square these values to get

$$
1.99^2 < x < 2.01^2
$$

$$
3.9601 < x < 4.0401.
$$

In this case we must have $\delta \leq 0.0399$, which is the minimum distance from 4 of the two bounds given above, i.e.

Further, so we switch to general ϵ try to determine δ symbolically. We start by assuming $y = \sqrt{x}$ is within ϵ units of 2:

$$
\begin{aligned}
|y-2| &< \epsilon \\
-\epsilon < y-2 &< \epsilon && \text{(Definition of absolute value)} \\
-\epsilon < \sqrt{x}-2 &< \epsilon && (y=\sqrt{x}) \\
2-\epsilon < \sqrt{x} &< 2+\epsilon && \text{(Add 2)} \\
(2-\epsilon)^2 < x &< (2+\epsilon)^2 && \text{(Square all)} \\
4-4\epsilon+\epsilon^2 < x &< 4+4\epsilon+\epsilon^2 && \text{(Expand)} \\
4-(4\epsilon-\epsilon^2) < x &< 4+(4\epsilon+\epsilon^2). && \text{(Rewrite in the desired form)}
\end{aligned}
$$

The "desired form" in the last step is " $4 - something < x < 4 + something$." Since we want this last interval to describe an *x* tolerance around 4, we have to choose:

$$\delta \le min\{4\epsilon - \epsilon^2, 4\epsilon + \epsilon^2\}.$$

Since $\epsilon > 0$, the minimum is $\delta \le 4\epsilon - \epsilon^2$

Then, If $\epsilon = 0.5$, the formula gives $\delta \le 4(0.5) - (0.5)^2 = 1.75$, and when $\epsilon = 0.01$, the formula gives $\delta \le 4(0.01) - (0.01)^2 = 0.399$

So given any $\epsilon > 0$, set $\delta \le 4\epsilon - \epsilon^2$. Then if $|x-4| < \delta$ (and $x \ne 4$), then $|f(x) - 2| < \epsilon$, satisfying the definition of the limit.

We have shown formally (and finally!) that $\lim_{x\to 4}\sqrt{x} = 2$.

This formal definition of the limit is not an easy concept to understand. Our example is actually "easy" example, using "simple" function. However, it is very difficult to prove, using the techniques given above for more complex functions. So, we need another tool more powerful than this considered.

Fortunately, it is possible to achieve it by using a series of theorems which allow us to find limits much more quickly and analytically.