Lecture #18: LIMITS AND CONTINUITY OF FUNCTIONS

17.6 Continuity Function at a Point

As we have studied limits, we have gained the intuition that limits measure “where a function is heading" That is, if

$$\lim_{x\to 1} f(x) = 3$$

then as x is close to 1, $f(x)$ is close to 3. We have seen, though, that this is not necessarily a good indicator of what $f(1)$ actually is. This can be problematic; functions can tend to one value but attain another. This section focuses on functions that do not exhibit such behavior, we consider Continuous Functions.

Definition: The function $f(x)$ is called *continuous at the point* x_0 if it is defined in some neighborhood of the point x_0 and at the point x_0 itself and the following equality is valid:

$$\lim_{x\to x_0} f(x) = f(x_0) \text{ as } x \to x_0$$

A useful way to establish whether or not a function $f(x)$ is continuous at x_0 is to verify the following three things:

a) $f(x)$ is defined at this point x_0 and some its neighborhood (that is, there exists such $\delta > 0$, that $U_\delta(x_0) \subset D_f$ belongs to the domain of function);

b) there are exist the right and left limits of the function $f(x)$ at the point x_0, i.e. the limit $\lim_{x\to x_0} f(x)$ exists;

c) these one-sided limits are equal to each other;

d) the limits are equal to the value of the function $f(x)$ at the point x_0.

Another Definition of the Function Continuity

Continuity is inherently tied to the properties of limits. Because of this, the definition of limits the continuity of function at a point can be also formulated by “ε-δ language”.

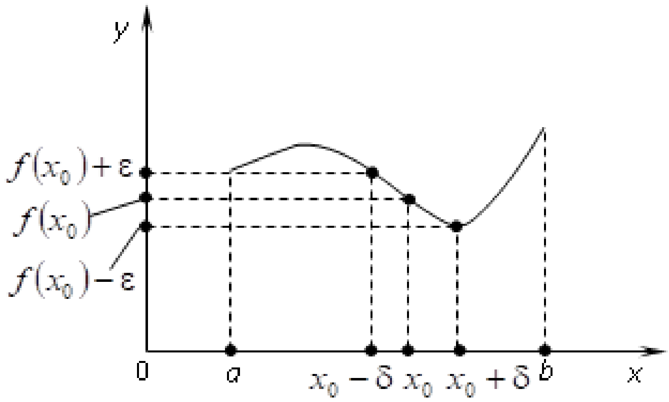
Definition: A function $f(x)$ is called *continuous at a point* x_0 if for any however small positive value $\varepsilon > 0$ it is possible to find a value $\delta(\varepsilon) > 0$, that from the inequality $|x - x_0| < \delta$ for all *x*, the following inequality

$$|f(x) - f(x_0)| < \varepsilon$$

is fulfilled.

This definition in symbol form may be written as follows:

$$\forall\varepsilon > 0\ \exists\delta(\varepsilon) > 0 \,|\, (|x - x_0| < \delta) \Rightarrow (|f(x) - f(x_0)| < \varepsilon)$$



With the general definition, the continuity of function at a point x_0 can be rewritten in the form:

$$\lim_{\Delta x \to 0} \Delta y = 0$$

Here, $x - x_0 = \Delta x,\ f(x) - f(x_0) = \Delta y$. That is to investigate the function behavior at some point x_0, we need to give an increment at the given point such that $x = x_0 + \Delta x \in D_f$, belongs to the domain of function. Then, the difference of the function at both the points $f(x_0 + \Delta x) - f(x_0)$ is called *increment of the function* $y = f(x)$.

Thereby, if a function is continuous at the point then an infinitesimal increment of the function at the infinitesimal increment of the argument occurs.

Note. The equality $\lim\limits_{\Delta x \to 0} \Delta y = 0$ is a necessary and sufficient condition for the function to be continuous at the given point.

Note. Let the function $f(x)$ be continuous at the point x_0. Then from the equality $\lim\limits_{x\to x_0} f(x)=f(x_0)$ follows the equality $\lim\limits_{x\to x_0} f(x)=f\left(\lim\limits_{x\to x_0} x\right)$.

It means that to calculate the limit of continuous function it is enough to replace its argument by its limiting value.

Arithmetic Operations on Continuous Functions

Theorem 1. The algebraic sum of the continuous functions at the given point is continuous function at this point.

■ Indeed, let the functions $f(x)$ and $\varphi(x)$ be continuous at the point x_0. Then

$$\lim_{x\to x_0} f(x)=f(x_0),\ \lim_{x\to x_0} \varphi(x)=\varphi(x_0).$$

Designate $F(x)=f(x)+\varphi(x)$. Then

$$\lim_{x\to x_0} F(x)=\lim_{x\to x_0}[f(x)+\varphi(x)]=$$

$$=\lim_{x\to x_0} f(x)+\lim_{x\to x_0}\varphi(x)=f(x_0)+\varphi(x_0)=F(x_0),$$

which is what had to be proved. □

Theorem 2. The product of two continuous functions at the given point is continuous function at this point as well.

Theorem 3. The quotient of two continuous functions is a continuous function if the denominator does not vanish at the point under consideration. The latter theorems can be proved in similar way.

Continuity of the Composite Function

Theorem. (Continuity of a composite function). Let two functions $y=f(u)$ and $u=\varphi(x)$ be given. If the function $\varphi(x)$ is continuous at the point x_0 and $\varphi(x_0)=u_0$, and the function $f(u)$ is continuous at the point u_0, then the composite function $F(x)=f[\varphi(x)]$ is continuous function as well at the point x_0 under consideration.

Theorem about Continuous Function

Theorem. If the function $f(x)$ is continuous at the point $x=a$ and if $f(a)\neq 0$, then there exists such δ-neighborhood of the point a that for all x of this δ-neighborhood, the function $f(x)$ does not vanish and preserves its sign in the neighborhood of this point.

17.7 Classification of Discontinuity Points

As mentioned above, a function $f(x)$ is continuous at a point x_0 if the following five conditions hold true:

1. The function $f(x)$ is defined at the point $x=x_0$ and in its neighborhood.

2. There exists the right-sided limit of the function

$$\lim_{x\to x_0+0} f(x)=f(x_0+0)=b_1.$$

3. There exists the left-sided limit of the function

$$\lim_{x\to x_0-0} f(x)=f(x_0-0)=b_2.$$

4. One-sided limits are equal and their general value coincides with the limit of the function at the point $x=x_0$.

5. Limit of the function is equal to value of the function at the point $x=x_0$, i.e. $\lim\limits_{x\to x_0} f(x)=f(x_0)$.

Definition: If at least one of these conditions is false then it is said that the function has *discontinuity* at the point $x=x_0$ (a brake point).

We need to distinguish points of discontinuity of the *I-st and II-nd* kinds.

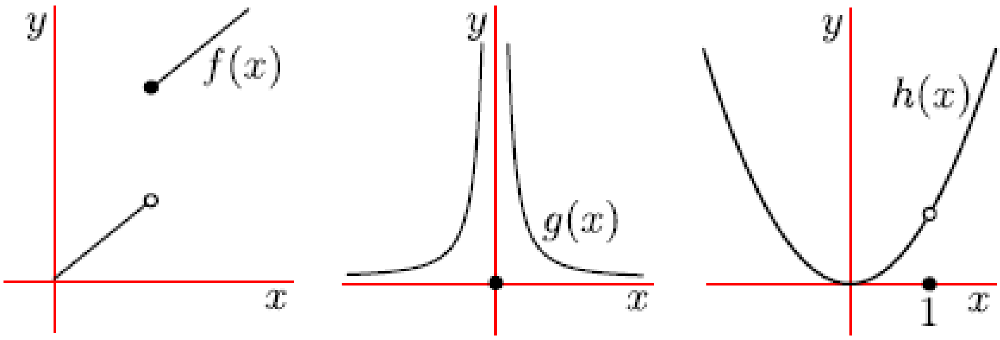
Definition: *If a function has finite, but not equal to each other one-sided limits at a point $x=x_0$ then this points is a I-st kind discontinuity point.*

Definition: *If at least one of one-sided limits tends to infinity or does not exist at a point $x=x_0$ then this point is called a II-nd kind discontinuity point.*

In particular, if the properties 1 and 5 are false then the point $x=x_0$ is called the point of *removable discontinuity* (the I-st kind). In this case it is possible to determine the value function at the point of discontinuity additionally and obtain the continuous function.

If there exist *finite one-sided limits* but they are *not equal to* each other, i.e. $b_1 \neq b_2$, then the point $x = x_0$ is called the *point of discontinuity of the I-st kind such as «jump»* ($|b_2 - b_1|$ is value of the function jump).

Consider the functions drawn below



These are

$$f(x) = \begin{cases} x & x < 1 \\ x+2 & x \geq 1 \end{cases}$$

$$g(x) = \begin{cases} 1/x^2 & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$h(x) = \begin{cases} \dfrac{x^3 - x^2}{x-1} & x \neq 1 \\ 0 & x = 1 \end{cases}$$

This example illustrates different sorts of discontinuities:

- The function $f(x)$ has a "*jump discontinuity*" because the function "jumps" from one finite value on the left to another value on the right.
- The second function, $g(x)$, has an "*infinite discontinuity*" since $lim f(x) = +\infty$.
- The third function, $h(x)$, has a "*removable discontinuity*" because we could make the function continuous at that point by redefining the function at that point. i.e. setting $h(1) = 1$. That is

$$\text{new function } h(x) = \begin{cases} \dfrac{x^3 - x^2}{x-1} & x \neq 1 \\ 1 & x = 1 \end{cases}$$

So to investigate the nature of discontinuity point of the function $f(x)$ it is necessary:

1. Find points at which the function might have indeterminacy.

2. Calculate the one-sided limits:

$$\lim_{x\to x_0-0} f(x)=b_1 \text{ and } \lim_{x\to x_0+0} f(x)=b_2.$$

3. To define a kind of the discontinuity taking into account the obtained values of these limits.

Example 1. Let us consider the function $y=\frac{\sin x}{x}$. Obviously this function is not defined at the point $x=0$, but you know that the function has limit at the point $x=0$ and this limit is equal to 1, i.e. $\lim\limits_{x\to 0}\frac{\sin x}{x}=1$. Hence the given function has break at the point $x=0$. But this break is removable (I-st kind). Indeed, if we define the function in the form

$$y(x)=\begin{cases}\frac{\sin x}{x}, & \text{if} \quad x\neq 0,\\ 1, & \text{if} \quad x=0,\end{cases}$$

then such function will continuous at the point $x=0$.

Example 2. Investigate the function $f(x)=\frac{x^2-4}{x-2}$ for continuity.

Solution.

1. The function is defined for every x except $x_0=2$.

2. $\lim\limits_{x\to 2+0}\frac{x^2-4}{x-2}=\lim\limits_{x\to 2+0}(x+2)=4$, $\lim\limits_{x\to 2-0}\frac{x^2-4}{x-2}=\lim\limits_{x\to 2-0}(x+2)=4$.

3. One-sided limits coincide, i.e. there exists $\lim\limits_{x\to 2}\frac{x^2-4}{x-2}=4$, but the equality $\lim\limits_{x\to 2} f(x)=f(2)$ is false as the value $f(2)$ has no sense.

There is a removable discontinuity at the point $x_0=2$. Let us see how to make the point $x_0=2$ be the continuous point.

Let us consider $F(x)$ given as follows:

$$F(x)=\begin{cases}\dfrac{x^2-4}{x-2}, & \text{if} \quad x\neq 2\\ 4, & \text{if} \quad x=2\end{cases}.$$

The value $F(x)\equiv f(x)$ everywhere except $x_0=2$. At the point $x_0=2$ the function $f(x)$ is undetermined, and $F(x)$ is equal to 4, that coincides with the value $\lim\limits_{x\to 2} f(x)=4$, i.e. $F(x)$ is continuous at the point $x_0=2$.

We should mention that the main reason allowing us to obtain the continuous function $F(x)$ from the discontinuous function $f(x)$ is equality of its one-sided limits: $\lim\limits_{x\to 2-0} f(x)=\lim\limits_{x\to 2+0} f(x)$.

Example 3. Investigate the function $f(x)=\begin{cases}x^2+2, & x\le 0,\\ 2x, & x>0.\end{cases}$ for continuity.

Solution. The given function changes type of an analytical expression at the point $x=0$. So at this point it might have a break. The value of the function is equal to 2 at this point. Let us calculate one-sided limits:

$$\lim_{x\to -0}\left(x^2+2\right)=2, \quad \lim_{x\to +0}(2x)=0.$$

You can see that one-sided limits are not equal to each other. Therefore the function has a break of the I-st kind of the type "jump" (Fig. 1a)

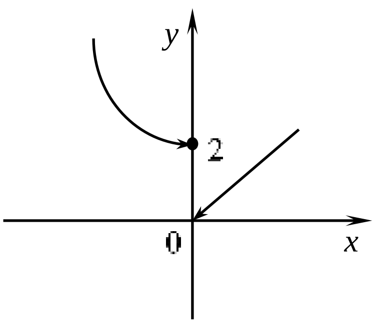


Fig. 1a

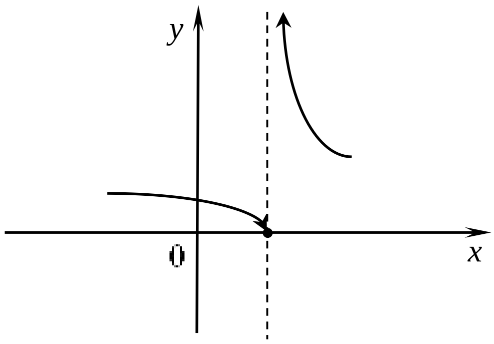


Fig. 1b

Example 4. Investigate the function $f(x)=3^{\frac{1}{x-1}}$ for continuity.

Solution.

1. An exponential function is continuous over range of definition, but the given function is undetermined at the point $x_0=1$.

2. Let us calculate $\lim\limits_{x\to 1-0} 3^{\frac{1}{x-1}} = 0$, $\lim\limits_{x\to 1+0} 3^{\frac{1}{x-1}} = \infty$.

3. The point $x_0 = 1$ is the point of discontinuity of the II-nd kind, as one of the one-sided limits tends to infinity (Fig. 1b).

Example 5. Research the function $f(x) = \dfrac{1}{x}$ for continuity.

Solution. The given function is not defined at the point $x = 0$. Let us calculate the one-sided limits

$$\lim_{x\to -0} f(x) = -\infty,\ \lim_{x\to +0} f(x) = +\infty.$$

Since these limits are infinite then the function has a break at the point $x = 0$ of the II-nd kind or infinite break (Fig. 2a).

Example 6. Investigate the function $y = \arctan\dfrac{1}{x}$ for continuity.

Solution. The function is not defined at the point $x = 0$. Let us calculate the one-sided limits

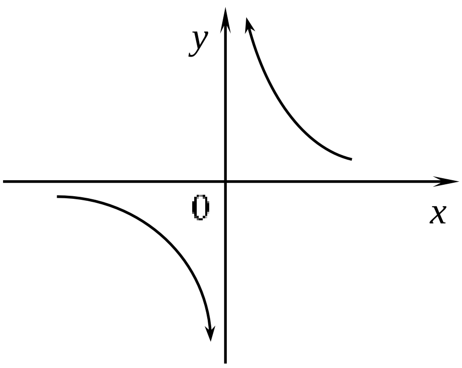


Fig. 2a

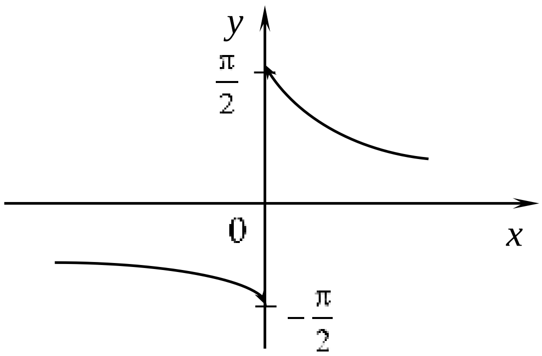


Fig. 2b

$$\lim_{\substack{x\to +0 \\ x\to -0}} \arctan\frac{1}{x} = \begin{cases} \dfrac{\pi}{2} \\ -\dfrac{\pi}{2} \end{cases}.$$

So the right and the left limits exist, but they do not equal to each other. This discontinuity point of the I-st kind of type “jump” (Fig. 2b).

Example 7. Let function $f(x)=\dfrac{x}{\ln(x-2)}$ be given. The function is not defined at two points $x_1=3$ and $x_2=2$. At that for point $x_2=2$ it is necessary to find the right limit only due to the function definition $(x-2>0,\ x>2)$. Let us calculate the one-sided limits.

$$\lim_{x\to 3+0} f(x)=+\infty,\ \lim_{x\to 3-0} f(x)=-\infty,\ \lim_{x\to 2+0} f(x)=0,$$

So, points $x_1=3$ and $x_2=2$ are discontinuity points of the II-nd kind (Fig. 3a).

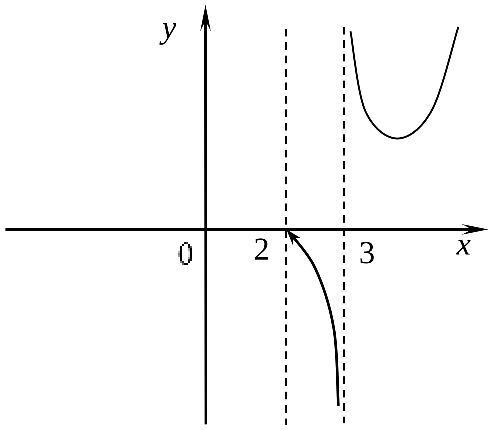


Fig. 3a

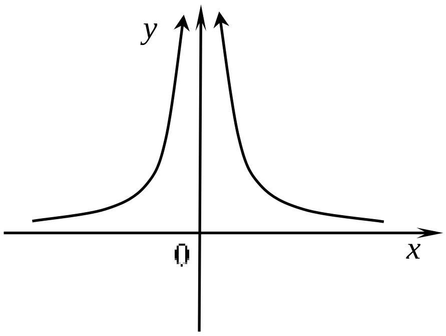


Fig. 3b

Example 8. Let us consider the function $f(x)=\dfrac{1}{x^2}$. Here

$$\lim_{x\to -0} f(x)=\lim_{x\to +0} f(x)=+\infty.$$

In the given case we deal with break of the II-nd kind as well (Fig. 3b).

17.8 Continuity of Functions given on Closed Intervals

We can extend the definition of continuity at a point to closed intervals by considering the appropriate one-sided limits at the endpoints of the interval.

Definition: A function $f(x)$ is called *continuous on closed interval* $[a, b]$, where a and b are real numbers if it is continuous at each point inside of this interval (a, b) and on the left end it is continuous from the right side and at the right end it is continuous from the left side:

$$\lim_{x \to a^+} f(x) = f(a)$$

and

$$\lim_{x \to b^-} f(x) = f(b)$$

Continuity of the Inverse Function

Let the function $y = f(x)$ have the domain defined by the segment $[a,b]$ and the range defined by the segment $[\alpha,\beta]$. Let each $y \in [\alpha,\beta]$ have the only one corresponding value from the domain $x \in [a,b]$, i.e. we have a one-to-one correspondence $f(x) = y$. Then, the segment $[\alpha,\beta]$ can be redefined as a domain of the function, denoted as $x = f^{-1}(y)$, putting in correspondence to each value $y \in [\alpha,\beta]$ the range of the function $x = f^{-1}(y)$ is the segment $x \in [a,b]$.

The function $x = f^{-1}(y)$ is called an *inverse function* to the given function $y = f(x)$.

Theorem. Let strictly defined monotonic continuous function $y = f(x)$ be given on the segment $[a,b]$ and $\alpha = f(a)$, $\beta = f(b)$ $(\alpha < \beta)$. Then this function has strictly monotonic and continuous *inverse* function $x = f^{-1}(y)$ or $x = \varphi(y)$ on the segment $[\alpha,\beta]$.

Properties of Continuous Functions

Let $f(x)$ and $g(x)$ be continuous functions on an interval I (close or open), let c be a real number and let n be a positive integer. The following functions are continuous on I:

1. Sums/Differences : $f(x) \pm g(x)$
2. Constant Multiples : $c \cdot f(x)$
3. Products : $f(x) \cdot g(x)$
4. Quotients : $\frac{f(x)}{g(x)}$ (as long as $g(x) \neq 0$ on I)
5. Powers : $f^n(x)$

6. Roots : $\sqrt[n]{f(x)}$ (if n is even then $f(x) \geq 0$ on I ; if n is odd, then true for all values of $f(x)$ on I .)
7. Compositions: Adjust the definitions of $f(x)$ and $g(x)$ to: Let $f(x)$ be continuous on I , where the range of $f(x)$ on I is J , and let $g(x)$ be continuous on J . Then $g(f(x))$, is continuous on I .

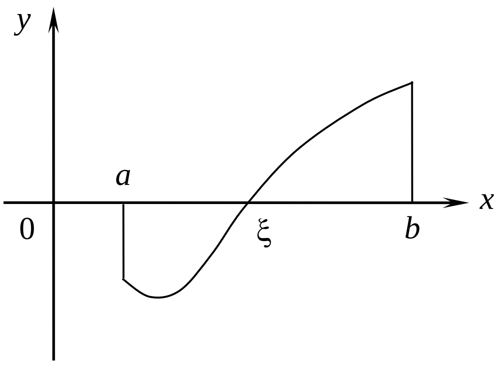
Note: The following basic elementary functions are continuous everywhere in their domains

- polynomials, rational functions
- roots and powers
- trig functions and their inverses
- exponential and the logarithm

Basic Theorems on Functions Continuous on Closed Interval

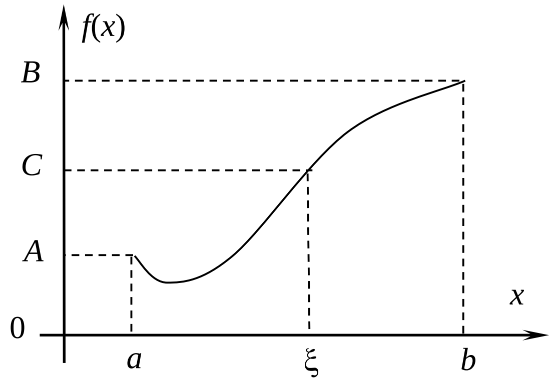
Theorem 1. (The first theorem by Boltsano-Cauchy).
If a function $f(x)$ is continuous on the closed interval $[a,b]$ and takes values of different signs at the end points of this interval, then there exists at least one point $x=\xi$ lying between the points *a* and *b* where the function vanishes, i.e. $f(\xi)=0$, $\xi \in (a,b)$.

Note. This theorem has a simple geometrical meaning: The graph of the continuous function $y=f(x)$ joining the points $M_1(a, f(a))$ and $M_2(b, f(b))$, where $f(a)<0,\ f(b)>0$, crosses the *x*-axis at least one point, as shown in Fig.



Theorem 2. (The second theorem by Boltsano-Cauchy called as Intermediate value theorem (IVT)). If a function $f(x)$ is continuous on the closed interval $[a,b]$ and takes unequal values $f(a)=A,\ f(b)=B$, at the end points of this interval, then no matter what a number C, which lies

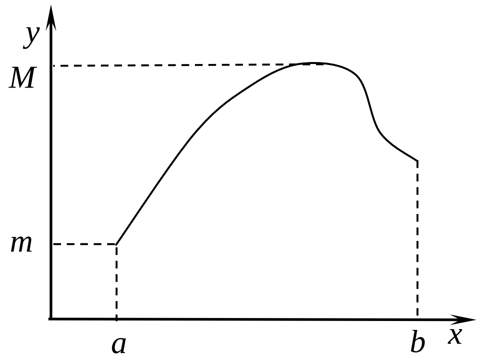
between the numbers A and B, there exists at least one a point $\xi \in (a,b)$, such that $f(\xi) = C$.



Note. The geometrical meaning is: a continuous function takes all intermediate values from the interval where this function is given.

Theorem 3. (The first theorem by Weierstrass). If a function is continuous on the closed interval, then it is bounded on this interval**.**

Note. From this theorem it follows that if the function $f(x)$ is continuous on the closed interval $[a,b]$, then there exist such finite real numbers *m* and *M*, that all values of the function lie between these numbers, that is, $m \le f(x) \le M$,$\forall x$, $x \in [a,b]$ as shown in Fig.



Herewith, the greatest of all possible numbers *m* is obviously $\inf\limits_{x \in [a,b]} f(x)$ and the smallest of all possible numbers M is $\sup\limits_{x \in [a,b]} f(x)$. We will remind that namely these numbers *m* and *M* are under consideration further.

Relating to this the following question arises: do exist a points x_0 and x_1 that belong to segment $[a,b]$ such that $f(x_0) = m$, $f(x_1) = M$? The answer gives the following theorem:

Theorem 4. (The second theorem by Wierstrass). If a function $f(x)$ is continuous on the closed interval $[a,b]$, then it attains on this interval the greatest value *M* and the smallest value *m* (at least one of them).

It means that there exist such points x_0 and x_1 on the interval $[a,b]$, that $f(x_0) = m$, $f(x_1) = M$.

Note. If a function $f(x)$ is bounded on the closed interval, but is not continuous, then the supremum and infimum might be not reached.

Uniform Continuity of a Function

Let a function $f(x)$ be continuous on an interval *I* and a point $x_0 \in I$ be an arbitrary point of this interval. Then for any $\varepsilon > 0$ there exists a number $\delta > 0$ such that $(|x - x_0| < \delta) \Rightarrow (|f(x) - f(x_0)| < \varepsilon)$.

Obviously δ depends on both ε and the point x_0. It means that for the same value ε and different points x_0 there might exist different values δ. That is different velocities of changing function $f(x)$ on different parts of the interval occur.

Then, the following question arises: does a value of δ corresponding to given ε exist such that for any two values of the function arguments x_0 and *x*, whose satisfy the inequality $|x - x_0| < \delta$, the inequality $|f(x) - f(x_0)| < \varepsilon$ holds true? That is if $x_0 \in I$ and $x \in I$ then do the inequalities $(|x - x_0| < \delta) \Rightarrow (|f(x) - f(x_0)| < \varepsilon)$ fulfill?

Another words the number δ is independent value on x_0, but it is dependent on ε only.

Definition. If the function $f(x)$ is continuous on an interval *I* and for any $\varepsilon > 0$ it is possible to point out a number $\delta(\varepsilon) > 0$, that the inequality $|f(x) - f(x_0)| < \varepsilon$ is fulfilled for any $x_0 \in I$ and $x \in I$ as the inequality $|x - x_0| < \delta$ holds true, then the function $f(x)$ is called *uniformly continuous on the interval I.*

Cantor's Theorem (without proof). If the function $f(x)$ is continuous on the closed interval $[a, b]$, then it is uniformly continuous on this interval.