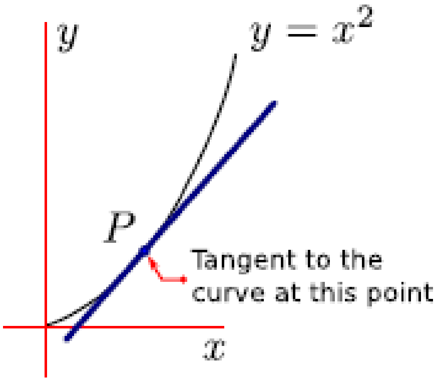
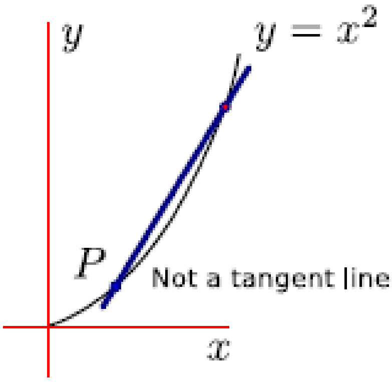
Lecture #19: DIFFERENTIAL CALCULUS OF FUNCTIONS OF ONE VARIABLE

19.1 The Definition of the Derivative

Let us start with the "*tangent line*" problem. Of course, we need to define "tangent", but we won't do this formally. Instead let us draw some pictures.





To find the equation of a line we either need

- the slope of the line and a point on the line (x_1, y_1), or
- two points on the line (x_1, y_1) and (x_2, y_2), from which we can compute the slope via the formula

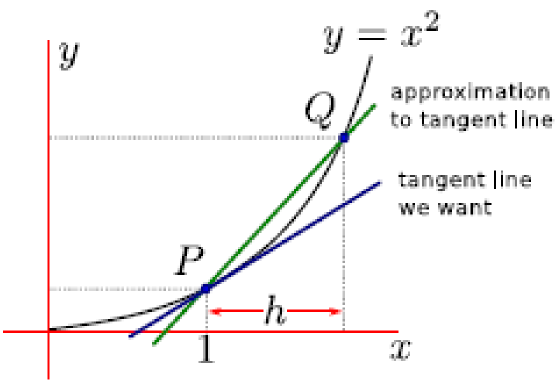
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

and then write down the equation for the tangent line at the point (x_1, y_1) via a formula such as

$$y = m \cdot (x - x_1) + y_1.$$

We cannot use this formula because we do not know what the slope of the tangent line should be. To work out the slope we need calculus — so we'll be able to use this method once we get to the definition of "*differentiation*".

Let's approximate the tangent line, by drawing a line that passes through $P(1,1)$ and some nearby point — call it Q whose x -coordinate is equal to



that of P plus a little bit — where the little bit is some small number h. And since this point lies on the curve $y = x^2$, and Q's x-coordinate is $1 + h$, Q's y-coordinate must be $(1 + h)^2$.

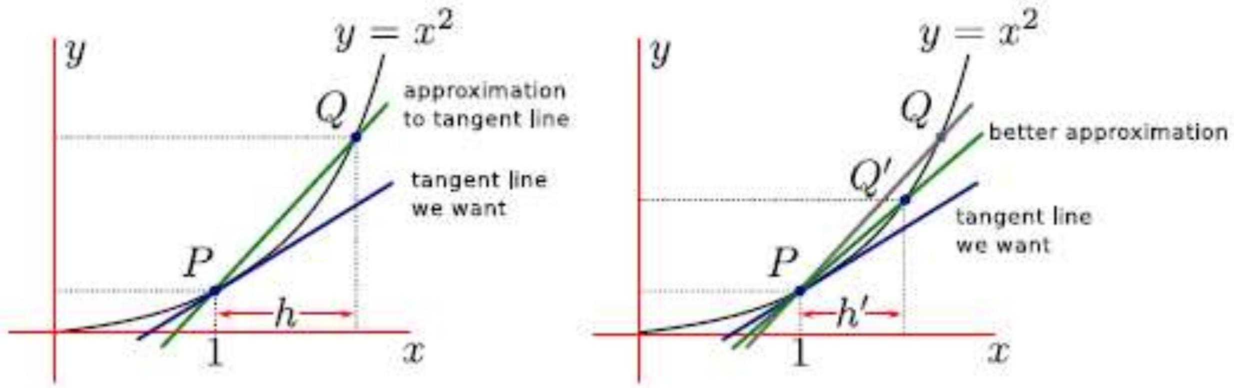
This line that passes through the curve in two places P and Q is called a "*secant line*". The slope of the line is then

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{(1+h)^2 - 1}{(1+h) - 1} = \frac{1 + 2h + h^2 - 1}{h} = \frac{2h + h^2}{h} = 2 + h$$

It is obvious that this isn't our tangent line because it passes through 2 nearby points on the curve — however it is a reasonable approximation of it.

Now we can make that approximation better and so obtain the tangent line by considering what happens when we move this point Q closer and closer to P, i.e. make the number h closer and closer to zero.



The original choice of Q is on the left, while on the right we have drawn what happens if we choose h' to be some number a little smaller than h, so that our point Q becomes a new point Q' that is a little closer to P. The new approximation is better than the first. So as we make h smaller and smaller, we bring Q closer and closer to P, and make our secant line a better and better approximation of the tangent line.

Thereby, our tangent line can be thought of as the end of this process — namely as we bring Q closer and closer to P, the slope of the secant line comes closer and closer to that of the tangent line we want. Given this, we can work out the equation for the tangent line more mathematically as

$$m = \boxed{\lim_{h \to 0} \frac{(1+h)^2 - 1}{h}} = \lim_{h \to 0} (2 + h) = 2.$$

That is our tangent line is

$$y = 2x - 1.$$

As we go further and learn more about limits and derivatives we will be able to get closer to real problems and their solutions. For instance, consider the following problem:

If an object is moving at *a constant velocity* in the positive direction, then that velocity is just the distance travelled divided by the time taken. That is

$$v = \frac{\text{distance moved}}{\text{time taken}}$$

When velocity is constant everything is easy.

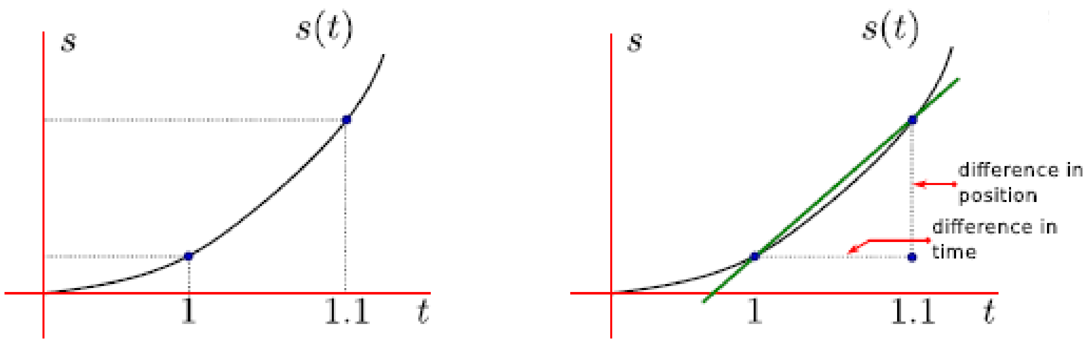
However, if the object is being moved with its definitely *not constant speed*. Instead of asking for the velocity, let us examine the "*average velocity*" of the object over a certain window of time. In this case the formula is very similar

$$\text{average velocity} = \frac{\text{distance moved}}{\text{time taken}}$$

We can rewrite it as — the distance moved is the difference in position, and the time taken is just the difference in time — and the latter is more mathematically precise, and is easy to translate into the following equation

$$\text{average velocity} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}.$$

If we sketch a graph of the function $s(t)$, we can see that this task is reduced to the "tangent line" problem. Indeed, the differences in position and time, and the line joining the two points on the graph of $s(t)$ is the same as the previous problem.



Remember that the slope of this line is

$$\text{slope} \quad = \frac{\text{change in } y}{\text{change in } x} = \frac{\text{difference in } s}{\text{difference in } t}$$

Squeezing the window between t_2 and t_1 down towards zero, the average velocity becomes the "*instantaneous velocity*" — just as the slope of the secant line becomes the slope of the tangent line.

Thereby, we define the instantaneous velocity at time $t = a$ to be the second limit as

$$v(a) \quad = \boxed{\lim_{h \to 0} \frac{s(a+h) - s(a)}{h}}$$

It is clear that in all similar problems the limit is used to analyze instantaneous rate of change or the "*derivative*, based on the limiting slope ideas of the previous two problems.

Let us now generalize what we did as to find "the slope of the curve $y = f(x)$ at (x_0, y_0) " for any continuous function $f(x)$.

As before, let (x_0, y_0) be any point on the curve $y = f(x)$. So we must have $y_0 = f(x_0)$. Now let (x_1, y_1) be any other point on the same curve. So, $y_1 = f(x_1)$ and $x_1 \neq x_0$. Think of (x_1, y_1) as being pretty close to (x_0, y_0) as a result the difference

$$\Delta x = x_1 - x_0$$

in x –coordinates is pretty small.

In terms of this Δx we have

$$x_1 = x_0 + \Delta x \quad \text{and} \quad y_1 = f(x_0 + \Delta x)$$

We can construct a secant line through (x_0, y_0) and (x_1, y_1) just as we have done above. It has a slope

$$\frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

If $f(x)$ is continuous, then as x_1 approaches x_0, i.e. as Δx approaches 0, we would expect the secant through (x_0, y_0) and (x_1, y_1) to approach the tangent line to the curve $y = f(x)$ at (x_0, y_0), just as previous cases. So, the slope of the secant through (x_0, y_0) and (x_1, y_1) should approach the slope of the tangent line to the curve $y = f(x)$ at (x_0, y_0), which is to be

$$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

This limit defines the *derivative of function* $f(x)$ at the point $x = x_0$. A more rigorous formulation of this definition is sound as:

Definition: *Derivative at a point.*

Let $a \in R$ and let $f(x)$ be defined on an open interval that contains x_0.

The derivative of $f(x)$ at $x = x_0$ is denoted $f'(x_0)$ and is defined by

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

if the limit exists.

When the above limit exists, the function $f(x)$ is said to be *differentiable at* $x = x_0$. When the limit does not exist, the function $f(x)$ is said to be not differentiable at $x = x_0$.

We can equivalently define the derivative $f'(x_0)$ by the limit

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{\Delta x \to 0} \frac{\Delta f(x_0)}{\Delta x}$$

To see that these two definitions are the same, we set $h \equiv \Delta x$ and $x = x_0 + \Delta x$ and then the limit as Δx goes to 0 is equivalent to the limit as x goes to x_0.

The important thing here is that we can move from the derivative being computed at a specific point to the derivative being a function itself — input any value of a and it returns the slope of the tangent line to the curve at the point $x = a, y = f(a)$. The variable a is a dummy variable. We can rename a to anything we want, like x, for example, which gives us the following definition.

Definition: *Derivative as a function*

Let $f(x)$ be a function.

The derivative of $f(x)$ with respect to x is

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x}$$

provided the limit exists.

If the derivative $f'(x)$ exists for all $x \in (a,b)$ we say that $f(x)$ is *differentiable on* (a,b).

Note: As the derivative was discovered independently by Newton and Leibniz in the late 17th century. Because their discoveries were independent, Newton and Leibniz did not have exactly the same notation. Stemming from this, and from the many different contexts in which derivatives are used, there are quite a few alternate notations for the derivative - "*the derivative of* $f(x)$ *with respect to* x ":

$$f'(x), \frac{\mathrm{d}f(x)}{\mathrm{d}x}, \dot{f}(x), Df(x), D_x f(x)$$

while the following notations are for "*the derivative of* $f(x)$ *at* $x = a$ "

$$f'(a), \frac{\mathrm{d}f(a)}{\mathrm{d}x}, \dot{f}(a), Df(a), D_x f(a)$$

We will generally use the first three, but you should recognize them all.

- The notation $f'(a)$ is due to Lagrange, while the notation $\frac{df(a)}{dx}$ is due to Leibniz. They are both very useful. Neither can be considered "better".
- Leibniz notation writes the derivative as a "fraction" — however it is definitely not a fraction and should not be thought of in that way. It is just shorthand, which is read as "*the derivative of* f *with respect to* x".
- You read $f'(x)$ as " f –prime of x ", and $\frac{df}{dx}$ as "*dee–* f *–dee–* x ", and $\frac{df(x)}{dx}$ as "*dee-by-dee–* x *of* f ".
- Similarly you read $\frac{df(a)}{dx}$ as "*dee–* f *–dee–* x *at* a ", and $\frac{df(x)}{dx}|_{x=a}$ as "*dee-by-dee-* x *of* f *of* x *at* x *equals* a ".
- The notation $\dot{f}$ is due to Newton. In physics, it is common to use $\dot{f}(t)$ to denote the derivative of f with respect to time *t*.

Following the *Theorem on Limits* a connection between continuity and differentiability of a function occurs.

Let a function $y = f(x)$ be differentiable at a point x_0. It means that there exists the following limit

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

Whence it follows that the value $\frac{\Delta y}{\Delta x}$ differs from its limit by infinitesimal value, that is,

$$\frac{\Delta y}{\Delta x} = f'(x_0) + \alpha,$$

where $\lim_{\Delta x \to 0} \alpha = 0$. So,

$$\Delta y = f'(x_0)\Delta x + \alpha \Delta x. \qquad (*)$$

Theorem. If a function is differentiable at a point, then it is continuous at this point.

■ Indeed by virtue of theorem condition the value $f'(x_0)$ is a finite number. But on the base of equality (*), we can write

$$\lim_{\Delta x \to 0} \Delta y = \lim_{\Delta x \to 0} [f'(x_0) + \alpha]\Delta x = 0,$$

which is what had to be proved. □

The contrary statement is not always true. For example, the function $y - \sqrt[3]{x}$ is continuous at the point $x = 0$, but it is not differentiable at this point.

Let us calculate the derivative of the function $y = \sqrt[3]{x}$ at the point $x = 0$. It is obvious that

$$\Delta y = \sqrt[3]{0 + \Delta x} - \sqrt[3]{0} = \sqrt[3]{\Delta x},$$

consequently

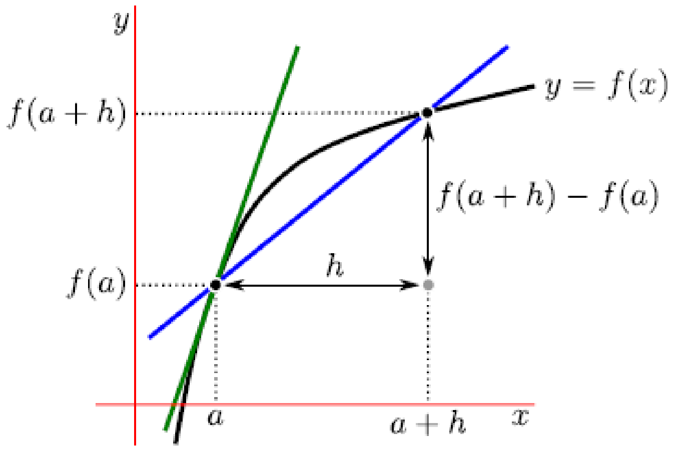
$$y' = \lim_{\Delta x \to 0} \frac{\sqrt[3]{\Delta x}}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\sqrt[3]{(\Delta x)^2}} = +\infty.$$

So the function $y = \sqrt[3]{\Delta x}$ is not differentiable at the point $x = 0$, $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ does not exist.

19.2 Interpretations of the Derivative

In the previous sections we defined the derivative as the slope of a tangent line, using a particular limit. This allows us to compute "*the slope of a curve*" and provides us with one interpretation of the derivative so-called its *geometrical meaning*.

Suppose that $y = f(x)$ is the equation of a curve in the xy –plane. That is, $f(x)$ is the y –coordinate of the point on the curve whose x –coordinate is x.



Then, as we have already seen that the slope of the tangent line to $y = f(x)$ at $x = a$, as we do this, when during the limiting procedure the secant through $(a, f(a))$ and $(a + h, f(a + h))$ approaches the tangent line to $y = f(x)$ at $x = a$, is

$$[\text{the slope of the tangent line to } y = f(x) \text{ at } x = a] \equiv k = \tan\alpha$$

$$= \lim_{h\to 0}\frac{f(a+h) - f(a)}{h} = f'(a)$$

Thereby, the geometrical meaning of the derivative is that the tangent line passing through the point $(a, f(a))$ has slope defined by $f'(a)$.

Hence, *the equation of the tangent line* to curve $y = f(x)$ at $x = a$ is

$$y - f(a) = f'(a)(x - a) \quad \text{or} \quad y = f(a) + f'(a)(x - a)$$

Also, *the equation of the line normal to* curve $y = f(x)$ at $x = a$ is

$$y - f(a) = -\frac{1}{f'(a)}(x - a) \quad \text{or} \quad y = f(a) - \frac{1}{f'(a)}(x - a)$$

However, the main importance of derivatives does not come from this application. Instead, (arguably) it comes from the interpretation of the derivative as *the instantaneous rate of change of a quantity*.

For instance, the instantaneous rate of change of a function $f(x)$ at a point $x = a$ is its *slope* $k = \tan \alpha$ defined by the derivative $f'(a)$ at this point.

The *mechanical (physical) meaning of the derivative* is the instantaneous rate of change of a mechanical (physical) value (for example, velocity at a given time).

19.3 Arithmetic of Derivatives

So far, we know derivatives only by applying *Definition* to the function at hand and then computing the required limits directly.

It is quite obvious that as the function being differentiated becomes even a little complicated, this procedure quickly becomes extremely unwieldy. It is many orders of magnitude more efficient if we will be equipped by

1. a list of derivatives of some simple functions and
2. a collection of rules for breaking down complicated derivative computations into sequences of simple derivative computations.

We started with limits of simple functions and then used “arithmetic of limits” to computed limits of complicated functions.

19.3.1 A list of derivatives of some simple functions

The Derivatives of Exponential Function $y = a^x$.

$$f'(x) = \lim_{\Delta x \to 0} \frac{a^{x+\Delta x} - a^x}{\Delta x} = a^x \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x} = a^x \ln a$$

$$(a^x)' = a^x \ln a$$

In particular case, $y = e^x$, we have $(e^x)' = e^x$

The Derivatives of Trigonometric Function $y = \sin x$, then

$$y' = \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin x}{\Delta x} = \lim_{\Delta x \to 0} \frac{2\sin\left(\frac{\Delta x}{2}\right)\cos\left(x + \frac{\Delta x}{2}\right)}{\Delta x} = \cos x,$$

so

$$\boxed{(\sin x)' = \cos x}$$

Analogously we can show, that

$$\boxed{(\cos x)' = -\sin x}$$

The Derivatives of Logarithmic Function $y = \ln x$. Then

$$y' = \lim_{\Delta x \to 0} \frac{\ln(x+\Delta x) - \ln(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\ln\left(\frac{x+\Delta x}{x}\right)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\ln\left(1 + \frac{\Delta x}{x}\right)}{\Delta x},$$

since $\ln\left(1 + \frac{\Delta x}{x}\right) \sim \frac{\Delta x}{x}$, as $\Delta x \to 0$, then we obtain

$$y' = \lim_{\Delta x \to 0} \frac{\frac{\Delta x}{x}}{\Delta x} = \frac{1}{x},$$

So

$$\boxed{(\ln x)' = \frac{1}{x}}$$

Note. If $y = \log_a x$, then $y' = \frac{1}{x \ln a}$ or

$$\boxed{(\log_a x)' = \frac{1}{x \ln a}}$$

The Derivatives of the Power Function

Let us consider the power function $y = x^a$, where a is any real number. Then

$$y' = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^a - x^a}{\Delta x} = x^a \lim_{\Delta x \to 0} \frac{\left(1 + \frac{\Delta x}{x}\right)^a - 1}{\Delta x} =$$

$$=\left\|\left(1+\frac{\Delta x}{x}\right)^{a}-1\sim\frac{a\cdot\Delta x}{x}\right\|=ax^{a}\lim_{\Delta x\to 0}\frac{\Delta x}{x\Delta x}=ax^{a-1}.$$

So

$$\boxed{y'=ax^{a-1}}.$$

The derivative of other simple function we will calculate later. First, a collection of differentiation rules has to be deduced.

19.3.2 A Collection of Differentiation Rules

We have to consider derivatives of sums, differences, products and quotients of functions. These theorems are not too difficult to prove from the definition of the derivative (which we know) and the arithmetic of limits (which we also know). Now we show how to construct these rules.

Theorem 1. The derivative of a constant is equal to zero.

■ Let $y=C$, where $C=const$. Then for any increament in argument Δx corresponding increament in the function will be equal to zero, that is $\Delta y=0$, so

$$y'=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=0,\ (C)'=0.\square$$

Theorem 2. The derivative of the sum of a finite number of differentiable functions is equal to the corresponding sum of the derivatives of these functions.

■ Let us prove this theorem for two functions. Consider the sum $y=u(x)+v(x)$. Then

$$y'=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta u+\Delta v}{\Delta x}=\lim_{\Delta x\to 0}\left(\frac{\Delta u}{\Delta x}+\frac{\Delta v}{\Delta x}\right)=\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}+\lim_{\Delta x\to 0}\frac{\Delta v}{\Delta x}.$$

It means

$$y'=u'+v'$$

or

$$\boxed{(u+v)'=u'+v'}.$$

Which is what had to be proved.□

Example. If $y = x^4 + \frac{1}{\sqrt[3]{x}} - 2$.

Then

$$y' = 4x^3 - \frac{1}{3}x^{-4/3} + 0 = 4x^3 - \frac{1}{3x\sqrt[3]{x}}.$$

Theorem 3. The derivative of a product of two differentiable functions is equal to the product of the derivative of the first function by the second one plus the product of the first function by the derivative of the second function, that is, if

$$y = u(x)v(x),$$

then

$$y' = u'\nu + \nu' u.$$

■ Let $y = u(x)v(x)$, then

$$\Delta y = u(x+\Delta x)\nu(x+\Delta x) - u(x)\nu(x).$$

Due to relation

$$u(x+\Delta x) - u(x) = \Delta u$$

we can write that

$$u(x+\Delta x) = u(x) + \Delta u$$

and analogously

$$v(x+\Delta x) = v(x) + \Delta v.$$

Therefore

$$\Delta y = (u(x)+\Delta u)(\nu(x)+\Delta\nu) - u(x)\nu(x) = u\,\Delta\nu + \nu\Delta u + \Delta u\,\Delta\nu.$$

Whence

$$y' = \lim_{\Delta x\to 0}\frac{u\Delta\nu + \nu\Delta u + \Delta u\Delta\nu}{\Delta x} = \lim_{\Delta x\to 0}\left(\frac{u\Delta\nu}{\Delta x} + \frac{\nu\Delta u}{\Delta x} + \frac{\Delta u\,\Delta\nu}{\Delta x}\right) =$$

$$= u\left(\lim_{\Delta x\to 0}\frac{\Delta v}{\Delta x}\right) + v\left(\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}\right) + \lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}\cdot\lim_{\Delta x\to 0}\Delta v.$$

Since the function is differentiable then it is continuous, therefore

$$\lim_{\Delta x\to 0}\Delta\nu = 0.$$

So

$$y' = u'\nu + \nu' u + u'\cdot 0$$

or

$$\boxed{(uv)' = u'v + v'u}.\square$$

***Corollary* 1.** A constant factor may be taken outside the derivative sign, i.e. if $y = Cf(x)$, then $y' = Cf'(x)$ or

$$\boxed{(Cf(x))' = Cf'(x)}.$$

***Corollary* 2:** The Linearity of Differentiation:

$$\frac{\mathrm{d}}{\mathrm{d}x}\{\alpha f(x) + \beta g(x)\} = \alpha f'(x) + \beta g'(x)$$

Example. Consider the function $y = (3x^2 + 4x)\left(\frac{5}{x} + \sqrt[3]{x}\right)$. Then

$$y' = (6x + 4)\left(\frac{5}{x} + \sqrt[3]{x}\right) + (3x^2 + 4x)\left(-\frac{5}{x^2} + \frac{1}{3\sqrt[3]{x^2}}\right).$$

Theorem 5. The derivative of a fraction is equal to a fraction whose denominator is the square of the denominator of the given fraction and the numerator is the difference between the product of the denominator by the derivative of the numerator, and the product of the numerator by the derivative of the denominator, i.e., if

$$y = \frac{u(x)}{v(x)},$$

then

$$y' = \frac{u'v - uv'}{v^2}.$$

■ Let the function $y = \dfrac{u(x)}{v(x)}$ be given. Then

$$\Delta y = \frac{u(x + \Delta x)}{v(x + \Delta x)} - \frac{u(x)}{v(x)} = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} =$$

$$= \frac{(u + \Delta u)v - (v + \Delta v)u}{(v + \Delta v)v} = \frac{v\Delta u - u\Delta v}{(v + \Delta v)v}.$$

Whence we obtain

$$\frac{\Delta y}{\Delta x} = \frac{v\dfrac{\Delta u}{\Delta x} - u\dfrac{\Delta v}{\Delta x}}{(v + \Delta v)v},$$

consequently

$$y' = \lim_{\Delta x \to 0} \frac{\nu \frac{\Delta u}{\Delta x} - u \frac{\Delta \nu}{\Delta x}}{\nu(\nu + \Delta \nu)} = \frac{\lim\limits_{\Delta x \to 0} \nu \frac{\Delta u}{\Delta x} - \lim\limits_{\Delta x \to 0} u \frac{\Delta \nu}{\Delta x}}{\lim\limits_{\Delta x \to 0} \nu(\nu + \Delta \nu)} = \frac{\nu \lim\limits_{\Delta x \to 0} \frac{\Delta u}{\Delta x} - u \lim\limits_{\Delta x \to 0} \frac{\Delta \nu}{\Delta x}}{\nu \lim\limits_{\Delta x \to 0} (\nu + \Delta \nu)}.$$

Since the function is differentiable then it is continuous, therefore

$$\lim_{\Delta x \to 0} \Delta \nu = 0.$$

So we obtain finally

$$y' = \frac{u'\nu - u\nu'}{\nu^2}$$

or

$$\boxed{\left(\frac{u}{\nu}\right)' = \frac{u'\nu - u\nu'}{\nu^2}}. \square$$

Example. Let the function $y = \dfrac{x^2}{x^3 + 4}$ be given. Then

$$y' = \frac{2x(x^3 + 4) - x^2 3x^2}{(x^3 + 4)^2} = \frac{8x - x^4}{(x^3 + 4)^2}.$$

The Derivatives of Trigonometric Function $y = \tan x = \frac{\sin x}{\cos x}$, then

$$y' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)' \cos x - \sin x (\cos x)'}{\cos^2 x} = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} =$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x},$$

that is

$$\boxed{(\tan x)' = \frac{1}{\cos^2 x} = \sec^2 x}$$

Analogously, if $y = \cot x$, then

$$y' = \left(\frac{\cos x}{\sin x}\right)' = \frac{-\sin x \sin x - \cos x \cos x}{\sin^2 x} = -\frac{\cos^2 x + \sin^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x},$$

that is

$$\boxed{(\cot x)' = -\frac{1}{\sin^2 x} = -\csc^2 x}$$

The Derivatives of the Hyperbolic Functions

Let us consider the function $y = \operatorname{sh} x = \frac{e^x - e^{-x}}{2}$, then

$$y' = \left(\frac{e^x - e^{-x}}{2}\right)' = \frac{\left(e^x\right)' - \left(e^{-x}\right)'}{2} = \frac{e^x + e^{-x}}{2} = \operatorname{ch} x.$$

Therefore

$$y' = (\operatorname{sh} x)' = \operatorname{ch} x$$

In the similar way we can show that if $y = \operatorname{ch} x = \frac{e^x + e^{-x}}{2}$, then

$$y' = \left(\frac{e^x + e^{-x}}{2}\right)' = \frac{\left(e^x\right)' + \left(e^{-x}\right)'}{2} = \frac{e^x - e^{-x}}{2} = \operatorname{sh} x.$$

Further,

$$(\operatorname{th} x)' = \left(\frac{\operatorname{sh} x}{\operatorname{ch} x}\right)' = \frac{\operatorname{ch} x \operatorname{ch} x - \operatorname{sh} x \operatorname{sh} x}{\operatorname{ch}^2 x} = \frac{\operatorname{ch}^2 x - \operatorname{sh}^2 x}{\operatorname{ch}^2 x} = \frac{1}{\operatorname{ch}^2 x},$$

$$(\operatorname{cth} x)' = \left(\frac{\operatorname{ch} x}{\operatorname{sh} x}\right)' = \frac{\operatorname{sh} x \operatorname{sh} x - \operatorname{ch} x \operatorname{ch} x}{\operatorname{sh}^2 x} = -\frac{\operatorname{ch}^2 x - \operatorname{sh}^2 x}{\operatorname{sh}^2 x} = -\frac{1}{\operatorname{sh}^2 x}.$$

19.3.3 Derivative of the Inverse Functions

Theorem. If a function $y = f(x)$ is continuous and strictly monotonic on some interval E and there exists the derivative $y_0' = f'(x_0) \neq 0$ at some point $x_0 \in E$**,** then the inverse function $x = \varphi(y)$ has the derivative $x_0' = \varphi'(y_0)$ at the corresponding point $y_0 = f(x_0)$, which is defined as

$$\varphi'(y_0) = \frac{1}{f'(x_0)}.$$

■ Let us determine the increment in the function

$$\Delta y = f(x_0 + \Delta x) - f(x_0).$$

We had shown that if function is differentiable at the point then its increment may be presented in the form $\Delta y = f'(x_0)\Delta x + \alpha \Delta x$, where α is infinitesimal value as $\Delta x \to 0$. Whence it follows that

$$\Delta x = \frac{\Delta y}{f'(x_0) + \alpha}.$$

Then

$$\varphi'(y_0) = \lim_{\Delta y \to 0} \frac{\Delta x}{\Delta y} = \lim_{\Delta y \to 0} \frac{1}{f'(x_0) + \alpha}.$$

From continuity of the function $y = f(x)$ it follows that

$$(\Delta x \to 0) \Leftrightarrow (\Delta y \to 0).$$

Therefore

$$\varphi'(y_0) = \lim_{\Delta x \to 0} \frac{1}{f'(x_0) + \alpha} = \frac{1}{f'(x_0)}. \square$$

Proved formula is written as

$$\boxed{x'_y = \frac{1}{y'_x}}$$

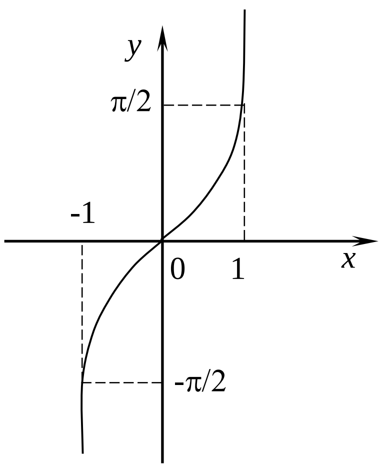
Derivatives of the Inverse Trigonometric Functions

Let us find the derivative of the function $y = \arcsin x$. Determine the inverse function to the given one $x = \sin y$. Then

$$y'_x = \frac{1}{x'_y} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}}.$$

In the given case $\cos y = +\sqrt{1 - \sin^2 y}$, because the function $y = \arcsin x$ has the range $\left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$ (Fig.). So

$$\boxed{(\arcsin x)' = \frac{1}{\sqrt{1 - x^2}}}$$



Analogously we can find the derivative of the function $y = \arccos x$. But more simply if take into account the following relation

$$\arcsin x + \arccos x = \frac{\pi}{2}$$

Hence

$$\arccos x = \frac{\pi}{2} - \arcsin x,$$

then

$$(\arccos x)' = \left(\frac{\pi}{2}\right)' - (\arcsin x)' = -\frac{1}{\sqrt{1-x^2}}.$$

So

$$\boxed{(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}}$$

Let us find the derivative of the function $y = \arctan x$. From this it follows that $x = \tan y$, consequently

$$y'_x = \frac{1}{x'_y} = \frac{1}{(\tan y)'} = \frac{1}{\dfrac{1}{\cos^2 y}} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}.$$

So

$$\boxed{(\arctan x)' = \frac{1}{1+x^2}}$$

By virtue of identity $\arctan x + \operatorname{arccot} x = \frac{\pi}{2}$, we get

$$\boxed{(\operatorname{arccot} x)' = -\frac{1}{1+x^2}}.$$

19.3.4 The Basic Formulas and Rules of Differentiation

Summarizing all the formulas we have deduced above, we collect them below as Table (a list) of Rules:

Rules of Differentiation

1. $(Const)' \equiv 0$
2. $(u+v)' = u' + v'$
3. $(u \cdot v)' = u'v + uv'$
4. $(Cf(x))' = Cf'(x)$
5. $\left(\dfrac{u}{v}\right)' = \dfrac{u'v - uv'}{v^2}$

Basic Formulas

1. $(x^a)' = ax^{a-1}$
2. $(e^x)' = e^x$
3. $(a^x)' = a^x \ln a$
4. $(\ln x)' = \frac{1}{x}$
5. $(\log_a x)' = \frac{1}{x \cdot \ln a}$
6. $(\sin x)' = \cos x$
7. $(\cos x)' = -\sin x$
8. $(tan\, x)' = \frac{1}{\cos^2 x} = \sec^2 x$
9. $(cot\, x)' = -\frac{1}{\sin^2 x} = -\csc^2 x$
10. $(\mathrm{sh}\, x)' = \mathrm{ch}\, x$
11. $(\mathrm{ch}\, x)' = \mathrm{sh}\, x$
12. $(\mathrm{th}\, x)' = \frac{1}{\mathrm{ch}^2 x}$
13. $(\mathrm{cth}\, x)' = -\frac{1}{\mathrm{sh}^2 x}$
14. $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$
15. $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$
16. $(\arctan x)' = \frac{1}{1+x^2}$
17. $(\mathrm{arccot}\, x)' = -\frac{1}{1+x^2}$

Example 1. $\frac{\mathrm{d}}{\mathrm{d}x}\{(3x+9)(x^2+4x^3)\}$?

$$\frac{\mathrm{d}}{\mathrm{d}x}\{(3x+9)(x^2+4x^3)\}$$

$$= \frac{\mathrm{d}}{\mathrm{d}x}(3x+9)\,(x^2+4x^3) + (3x+9)\frac{\mathrm{d}}{\mathrm{d}x}(x^2+4x^3)$$

$$= 18x + 117x^2 + 48x^3$$

Example 2. $\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{4x^3-7x}{4x^2+1}\right\}$?

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{4x^3-7x}{4x^2+1}\right\} = \frac{(12x^2-7)(4x^2+1)-(4x^3-7x)(8x)}{(4x^2+1)^2}$$

$$= \frac{(48x^4-16x^2-7)-(32x^4-56x^2)}{(4x^2+1)^2} = \frac{16x^4+40x^2-7}{(4x^2+1)^2}$$

Example 3. A tangent line to the curve $y = \sqrt{x}$ at $x = 4$;

By the geometrical meaning of the derivative, the tangent line to the curve $y = f(x)$ at $x = a$ is given by

$$y \quad = f(a) + f'(a)(x-a)$$

provided $f'(a)$ exists.

So, the derivative of $\sqrt{x}$ at $x = a$ is

$$f'(a) \quad = \frac{1}{2\sqrt{a}}$$

If $a = 4$ then

$$f'(a) = f'(4) = \frac{1}{2\sqrt{a}}\Big|_{a=4} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

and

$$f(a) = f(4) = \sqrt{x}\big|_{x=4} = \sqrt{4} = 2$$

Hence, the equation of the tangent line is

$$y = 2 + \frac{1}{4}(x-4) \quad \text{or} \quad y = \frac{x}{4} + 1$$