Lecture #20: DIFFERENTIAL CALCULUS OF FUNCTIONS OF ONE VARIABLE

19.4 The Derivative of Composite Functions

We have built up most of the tools that we need to express derivatives of complicated functions in terms of derivatives of simpler known functions by using the derivatives of sums, products and quotients.

The final tool we add is called the chain rule. It tells us how to take the derivative of a composition of two functions. That is if we know $f(x)$ and $g(x)$ and their derivatives, then the chain rule tells us the derivative of $f(g(x))$.

Theorem: *The chain rule*

If a function $u=\varphi(x)$ is differentiable at the point x_0, and a function $y=f(u)$ is differentiable at the point u_0, where $u_0=\varphi(x_0)$, then the composite function $F(x)=f[\varphi(x)]$ will be differentiable at the point x_0 and its derivative is calculated by the following formula:

$$\boxed{F'(x_0)=f'(u_0)\varphi'(x_0)}.$$

■ Let us determine the increment in the function $u=\varphi(x)$ and $y=f(u)$

$$\Delta u=\varphi(x_0+\Delta x)-\varphi(x_0),\ \Delta y=f(u_0+\Delta u)-f(u_0).$$

Since the considered functions are differentiable ones then the following relation $\Delta y=f'(u_0)\Delta u+\alpha\Delta u$, where $\alpha\to 0$ as $\Delta u\to 0$ is valid. Hence,

$$F'(x_0)=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\left(f'(u_0)\frac{\Delta u}{\Delta x}+\alpha\frac{\Delta u}{\Delta x}\right)=f'(u_0)\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}+\alpha\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}.$$

Since the function $u=\varphi(x)$ is differentiable at the point x_0, then it is continuous at this point, therefore $(\Delta x\to 0)\Leftrightarrow(\Delta u\to 0)$. Consequently $\lim_{\Delta x\to 0}\alpha=\lim_{\Delta u\to 0}\alpha=0$. So we obtain

$$F'(x_0)=f'(u_0)\cdot\varphi'(x_0)+0\cdot u'(x_0)=f'(u_0)\cdot\varphi'(x_0).\square$$

Here, as was the case earlier, we have been very careful to give the point at which the derivative is evaluated a special name (i.e. x_0). But of course this evaluation point can really be any point (where the derivative is defined). So it is very common to just call the evaluation point " x " rather than give it a special name like " x_0 ", i.e.

Let f and φ be differentiable functions then

$$\frac{d}{dx}f(\varphi(x)) \quad = f'(\varphi(x)) \cdot \varphi'(x) \text{ or } \frac{dy}{dx} \quad = \frac{dy}{du} \cdot \frac{du}{dx} \text{ (or } y'_x = y'_u u'_x \text{)}$$

This latter particular form is easy to remember because it looks like we can just "cancel" the du between the two terms.

$$\frac{dy}{dx} \quad = \frac{\mathrm{d}y}{\cancel{du}} \cdot \frac{\cancel{du}}{\mathrm{d}x}$$

Notice that when we form the composition $f(\varphi(x))$ there is an "outside" function (namely $f(x)$) and an "inside" function (namely $\varphi(x)$). The chain rule tells us that when we differentiate a composition that we have to differentiate the outside and then multiply by the derivative of the inside.

$$\frac{d}{dx}f(\varphi(x)) \quad = \quad \underbrace{f'(\varphi(x))}_{\text{differentiate outside}} \quad \cdot \quad \underbrace{\varphi'(x)}_{\text{differentiate inside}}$$

Example. $\frac{d}{dx}(\sin(x))^5$?

Let define $f(u) = u^5$ and $u = \varphi(x) = \sin x$. Then set $F(x) = f\big(\varphi(x)\big) = (\sin(x))^5$.

To find the derivative of $F(x)$ we can simply apply the chain rule — the pieces of the composition have been laid out for us. Here they are

$$\begin{aligned} f(u) \quad &= u^5 \quad & f'(u) \quad &= 5u^4 \\ \varphi(x) \quad &= \sin(x) \quad & \varphi'(x) \quad &= \cos x \end{aligned}$$

We now just put them together as the chain rule tells us

$$\frac{dF}{dx} = f'(\varphi(x)) \cdot \varphi'(x)$$

$$= 5(\varphi(x))^4 \cdot \cos(x) \text{ since } f'(u) = 5u^4$$

$$= 5(\sin(x))^4 \cdot \cos(x)$$

Example. Let the function $y = 3^{\tan x}$ be given. It means that $y = 3^u$, where

$u = \tan x$, thus $y' = 3^u \ln 3 \dfrac{1}{\cos^2 x} = 3^{\tan x} \cdot \ln 3 \cdot \dfrac{1}{\cos^2 x}$.

Example. Consider the function $y = \left(x^3 + 2\right)^{100}$. It means that $y = u^{100}$, where $u = x^3 + 2$. So we obtain $y' = 100 \cdot u^{99} \cdot 3x^2 = 300 \cdot \left(x^3 + 2\right)^{99} x^2$.

Notice that it is quite easy to extend this rule to any basic formula we derived. Thereby,

1. $\left(u(x)^a\right)' = au(x)^{a-1} \cdot u'(x)$
2. $\left(e^{u(x)}\right)' = e^{u(x)} \cdot u'(x)$
3. $\left(a^{u(x)}\right)' = a^{u(x)} \ln a \cdot u'(x)$
4. $\left(\ln u(x)\right)' = \dfrac{1}{u(x)} \cdot u'(x)$
5. $\left(\log_a u(x)\right)' = \dfrac{1}{u(x) \cdot \ln a} \cdot u'(x)$
6. $\left(\sin u(x)\right)' = \cos u(x) \cdot u'(x)$
7. $\left(\cos u(x)\right)' = -\sin u(x) \cdot u'(x)$
8. $\left(\tan u(x)\right)' = \dfrac{1}{\cos^2 u(x)} \cdot u'(x) = \sec^2 u(x) \cdot u'(x)$
9. $\left(\cot u(x)\right)' = -\dfrac{1}{\sin^2 u(x)} \cdot u'(x) = -\csc^2 u(x) \cdot u'(x)$
10. $\left(\mathrm{sh}\, u(x)\right)' = \mathrm{ch}\, u(x) \cdot u'(x)$
11. $\left(\mathrm{ch}\, u(x)\right)' = \mathrm{sh}\, u(x) \cdot u'(x)$
12. $\left(\mathrm{th}\, u(x)\right)' = \dfrac{1}{\mathrm{ch}^2 u(x)} \cdot u'(x)$
13. $\left(\mathrm{cth}\, u(x)\right)' = -\dfrac{1}{\mathrm{sh}^2 u(x)} \cdot u'(x)$
14. $\left(\arcsin u(x)\right)' = \dfrac{1}{\sqrt{1 - u^2(x)}} \cdot u'(x)$
15. $\left(\arccos u(x)\right)' = -\dfrac{1}{\sqrt{1 - u^2(x)}} \cdot u'(x)$

16. $(\arctan u(x))' = \frac{1}{1+u^2(x)} \cdot u'(x)$

17. $(\operatorname{arccot} u(x))' = -\frac{1}{1+u^2(x)} \cdot u'(x)$

Derivative of a double-composition

Find the derivative of $\frac{d}{dx} f(g(h(x)))$.

This is very similar to the previous example. Let us set $F(x) = f(g(h(x)))$ with $u = g(h(x))$. Then the chain rule tells us

$$\frac{dF}{dx} = \frac{df}{du} \cdot \frac{du}{dx} = f'\left(g\left(h(x)\right)\right) \cdot \frac{d}{dx} g\left(h(x)\right) =$$

We now just apply the chain rule again

$$= f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x).$$

Indeed, it is not too hard to generalize further (in the manner of previous example) to find the derivative of the composition of 4 or more functions (though things start to become tedious to write down):

$$\frac{d}{dx} f_1(f_2(f_3(f_4(x)))) = f_1'(f_2(f_3(f_4(x)))) \cdot \frac{d}{dx} f_2(f_3(f_4(x)))$$

$$= f_1'(f_2(f_3(f_4(x)))) \cdot f_2'(f_3(f_4(x))) \cdot \frac{d}{dx} f_3(f_4(x))$$

$$= f_1'(f_2(f_3(f_4(x)))) \cdot f_2'(f_3(f_4(x))) \cdot f_3'(f_4(x)) \cdot f_4'(x)$$

Example. Let the function $y = e^{\sqrt{\arcsin x}}$ be given.

Here $y = e^u$, where $u = \sqrt{v}$, $v = \arcsin x$.

$$y' = e^u \frac{1}{2\sqrt{v}} \cdot \frac{1}{\sqrt{1-x^2}} = \frac{e^{\sqrt{\arcsin x}}}{2\sqrt{\arcsin x}\sqrt{1-x^2}}.$$

Example. Consider the function $y = \operatorname{ch}\left[\ln^2(1-x)\right]$. Its derivative is defined as

$$y' = \operatorname{sh}\left[\ln^2(1-x)\right] \cdot 2 \cdot \ln(1-x) \cdot \frac{1}{1-x}(-1) = = -\frac{2}{1-x} \cdot \ln(1-x) \cdot \operatorname{sh}\left[\ln^2(1-x)\right].$$

19.5 The Derivative of Implicit Functions

Implicit differentiation is a simple trick that is used to compute derivatives of functions either

- when you don't know an explicit formula for the function, but you know an equation that the function obeys or
- even when you have an explicit, but complicated, formula for the function, and the function obeys a simple equation.

The trick is just to differentiate both sides of the equation and then solve for the derivative we are seeking.

Let the values of two variables x and y be defined by the equation $f(x,y)=0$.

If the function $y=\varphi(x)$ defined on the interval (a,b) is such that the equation $f(x,y)=0$ becomes an identity after substituting the expression $y=\varphi(x)$ in place of y. Then the function $y=\varphi(x)$ is assumed to be *an implicit function* defined by equation $f(x,y)=0$.

Let us suppose that the equation $f(x,y)=0$ determines an implicit function. We will give the rule for finding the derivative of an implicit function without transforming it into its explicit form, i.e. without representing it in the explicit form $y=\varphi(x)$.

In order to find the derivative y'_x it is required to differentiate the equation $f(x,y)=0$ as identity, taking into account that y is the function of x, i.e. $y=\varphi(x)$. In this case the variable y should be considered as “inside” function of the composite function.

Example. Let us consider the equation $x^3+\tan y=2$. To find the derivative let us differentiate this equation as identity, regarding y as a function of x, i.e. $y=y(x)$. Then we obtain

$$3x^2+\frac{1}{\cos^2 y}y'=0,$$

whence

$$y' = -3x^2 \cos^2 y.$$

Obviously in the given case we can easy pass from implicit representation of the function to explicit form. Indeed we obtain that

$$\tan y = 2 - x^3,$$

whence

$$y = \arctan\left(2 - x^3\right) + n\pi, \; n = 0, \; \pm 1, \; \pm 2, \; ...,$$

then

$$y' = \frac{1}{1 + \left(2 - x^3\right)^2}\left(-3x^2\right) = -\frac{3x^2}{1 + \left(2 - x^3\right)^2},$$

which coincides with previous result, because $\tan y = 2 - x^3$ and consequently

$$\frac{1}{1 + \left(2 - x^3\right)^2} = \frac{1}{1 + \tan^2 y} = \cos^2 y.$$

Example. Let the function $y = y(x)$ be given by equation $xy + e^x + \sin y = 0$. In this case passing to explicit form of function is impossible. Then

$$y + xy' + e^x + \cos y \cdot y' = 0,$$

whence

$$y' = -\frac{y + e^x}{x + \cos y}$$

Example. Finding a tangent line using implicit differentiation.

Find the equation of the tangent line to $y = y^3 + xy + x^3$ at $x = 1$

- First notice that when $x = 1$ the equation $y = y^3 + xy + x^3$ of the curve simplifies to $y = y^3 + y + 1$ or $y^3 = -1$, which we can solve: $y = -1$. So we know that the curve passes through $(1, -1)$ when $x = 1$.
- Now, to find the slope of the tangent line at $(1, -1)$, pretend that our curve is $y = f(x)$ so that $f(x)$ obeys

$$f(x) \quad = f(x)^3 + xf(x) + x^3$$

for all x. Differentiating both sides gives

$$f'(x) = 3f(x)^2 f'(x) + f(x) + xf'(x) + 3x^2$$

or

$$y' \quad = 3y^2 y' + xy' + y + 3x^2$$

- At this point we could isolate for $f'(x)$ and write it in terms of $f(x)$ and x, but since we only want answers when $x = 1$, let us substitute in $x = 1$ and $f(1) = -1$ (since the curve passes through $(1, -1)$) and clean things up before doing anything else.
- Substituting in $x = 1, f(1) = -1$ gives

$$f'(1) \quad = 3f'(1) - 1 + f'(1) + 3 \quad \text{and so } f'(1) = -\frac{2}{3}$$

- The equation of the tangent line is

$$y = y_0 + f'(x_0)(x - x_0) = -1 - \frac{2}{3}(x - 1) = -\frac{2}{3}x - \frac{1}{3}$$

We can further clean up the equation of the line to write it as

$$2x + 3y = -1.$$

19.6 Logarithmic Differentiation

The method of differentiating functions by first taking logarithms and then differentiating is called logarithmic differentiation. We use *logarithmic differentiation* in situations where it is easier to differentiate the logarithm of a function than to differentiate the function itself. This approach allows calculating derivatives of power, rational and some irrational functions in an efficient manner.

Consider this method in more detail. Let $y = f(x)$. Take natural logarithms of both sides:

$$\ln y = \ln f(x).$$

Next, we differentiate this expression using the chain rule and keeping in mind that y is a function of x.

$$(\ln y)' = (\ln f(x))', \Rightarrow \frac{1}{y}y'(x) = (\ln f(x))'.$$

It's seen that the derivative is

$$y' = y(\ln f(x))' = f(x)(\ln f(x))'.$$

The derivative of the logarithmic function is called the logarithmic derivative of the initial function $y = f(x)$. This differentiation method allows to effectively compute derivatives of power-exponential functions, that is functions of the form.

$$y = u(x)^{v(x)},$$

where $u(x)$ and $v(x)$ are differentiable functions of x.

In the examples below, find the derivative of the function $y = f(x)$. using logarithmic differentiation.

Example. $y = x^x, x > 0$, y'–?

First we take logarithms of the left and right side of the equation:

$$\ln y = \ln x^x, \Rightarrow \ln y = x\ln x.$$

Now we differentiate both sides meaning that y is a function of x

$$(\ln y)' = (\mathrm{x}\ln x)', \Rightarrow \frac{1}{y} \cdot y' = x' \ln x + x(\ln x)', \Rightarrow \frac{y'}{y} = 1 \cdot \ln x + x \cdot \frac{1}{x},$$

$$\Rightarrow \frac{y'}{y} = \ln x + 1, \Rightarrow y' = y(\ln x + 1),$$

$$\Rightarrow y' = x^x(\ln x + 1), \text{where} x > 0.$$

Example. $y = (x-1)^2(x-3)^5$, y'–?

First we take logarithms of both sides:

$$\ln y = \ln[(x-1)^2(x-3)^5], \Rightarrow \ln y = \ln(x-1)^2 + \ln(x-3)^5, \Rightarrow \ln y = 2\ln(x-1) + 5\ln(x-3).$$

Now it is easy to find the logarithmic derivative:

$$(\ln y)' = [2\ln(x-1) + 5\ln(x-3)]', \Rightarrow \frac{1}{y} \cdot y' = 2 \cdot \frac{1}{x-1} + 5 \cdot \frac{1}{x-3},$$

$$\Rightarrow y' = y\left(\frac{2}{x-1} + \frac{5}{x-3}\right), \Rightarrow$$

$$y' = (x-1)^2(x-3)^5 \cdot \left(\frac{2}{x-1} + \frac{5}{x-3}\right).$$

19.7 Derivatives of Parametric Functions

The relationship between the variables x and y can be defined in parametric form using two equations:

$$\begin{cases} x & = x(t) \\ y & = y(t) \end{cases}$$

where the variable t is called a parameter.

Find an expression for the derivative of a parametrically defined function. Suppose that the functions $x = x(t)$ and $y = y(t)$ are differentiable in the interval $\alpha < t < \beta$ and $x'(t) \neq 0$. Moreover, we assume that the function $x = x(t)$ has an inverse function $t = \varphi(x)$.

By the inverse function theorem we can write:

$$\frac{dt}{dx} = t'_x = \frac{1}{x'_t}.$$

The original function $y(x)$ can be considered as a composite function:

$$y(x) = y(t(x)).$$

Then its derivative is given by

$$y'_x = y'_t \cdot t'_x = y'_t \cdot \frac{1}{x'_t} = \frac{y'_t}{x'_t}.$$

This formula allows to find the derivative of a parametrically defined function without expressing the function $y(x)$ in explicit form.

In the examples below, find the derivative of the parametric function.

Example. $x = t^2, y = t^3$. y'_x—?

We find the derivatives of x and y with respect to t:

$$x'_t = (t^2)' = 2t, y'_t = (t^3)' = 3t^2.$$

Hence

$$\frac{dy}{dx} = y'_x = \frac{y'_t}{x'_t} = \frac{3t^2}{2t} = \frac{3t}{2} (t \neq 0).$$

Example. $x = e^{2t}, y = e^{3t}$. y'_x—?

$$x'_t = (e^{2t})' = 2e^{2t}, y'_t = (e^{3t})' = 3e^{3t}.$$

Hence, the derivative y'_x is given by

$$\frac{dy}{dx} = y'_x = \frac{y'_t}{x'_t} = \frac{3e^{3t}}{2e^{2t}} = \frac{3}{2}e^{3t-2t} = \frac{3}{2}e^t.$$