1. *Problem-Solving Strategy: Calculating a Limit when* $\frac{f(x)}{g(x)}$ *has the Indeterminate Form* $\left\|\frac{\infty}{\infty}\right\|$.

Example 1: Consider a limit of the quotient of two functions: $\lim_{x\to\infty}\frac{2x^3-4x^2+3}{4x^2+7x-5}$

1. First, we need to make sure that our function has the appropriate form and cannot be evaluated immediately using the limit laws.

Since $2x^3-4x^2+3$ and $4x^2+7x-5$ are third-degree and second-degree polynomials with positive leading coefficients, respectively, and $\lim_{x\to\infty}(2x^3-4x^2+3) = \infty$ and $\lim_{x\to\infty}(4x^2+7x-5) = \infty$. So, we have the form $\left\|\frac{\infty}{\infty}\right\|$, i.e. we cannot apply any theorems of the limit laws immediately.

2. To evaluate this limit, we will divide the numerator and denominator by the highest power of x in the numerator and denominator.

In doing so, we saw that

$$\frac{2x^3-4x^2+3}{x^3} = 2-\frac{4}{x}+\frac{3}{x^3}$$

and

$$\frac{4x^2+7x-5}{x^3} = \frac{4}{x}+\frac{7}{x^2}-\frac{5}{x^3}$$

Then,

$$\lim_{x\to\infty}\frac{2x^3-4x^2+3}{2x^3-4x^2+3} = \left\|\frac{\infty}{\infty}\right\| = \lim_{x\to\infty}\frac{\frac{2x^3-4x^2+3}{x^3}}{\frac{4x^2+7x-5}{x^3}}$$

$$= \lim_{x\to\infty}\frac{2-\frac{4}{x}+\frac{3}{x^3}}{\frac{4}{x}+\frac{7}{x^2}-\frac{5}{x^3}} = \left\{\begin{matrix}\lim_{x\to\infty}\frac{4}{x}\to 0,\ \lim_{x\to\infty}\frac{3}{x^3}\to 0\\ \lim_{x\to\infty}\frac{7}{x^2}\to 0,\ \lim_{x\to\infty}\frac{5}{x^3}\to 0\end{matrix}\right\} = \lim_{x\to\infty}\frac{2}{0}$$

$$= \infty$$

Example 2: $\lim_{x\to+\infty}\frac{\sqrt[3]{5x^4-3x^3+7+x}+\sqrt[3]{x^2-4x+3}}{\sqrt[5]{x^2+3x-2}+\sqrt[6]{x+5}}$

Both the numerator and denominator now approach ∞ as $x \to +\infty$, i.e. the form $\left\|\frac{\infty}{\infty}\right\|$ occurs.

Hence, we divide numerator and denominator by $x^{\frac{4}{3}} = \sqrt[3]{x^4}$ the highest power of the fraction. Then

$$\lim_{x\to\infty} \frac{\sqrt[3]{5x^4-3x^3+7+x}+\sqrt[3]{x^2-4x+3}}{\sqrt[5]{x^2+3x-2}+\sqrt[6]{x+5}} = \left\|\frac{\infty}{\infty}\right\|$$

$$= \lim_{x\to\infty} \frac{\frac{\sqrt[3]{5x^4-3x^3+7+x}}{\sqrt[3]{x^4}}+\frac{\sqrt[3]{x^2-4x+3}}{\sqrt[3]{x^4}}}{\frac{\sqrt[5]{x^2+3x-2}}{\sqrt[3]{x^4}}+\frac{\sqrt[6]{x+5}}{\sqrt[3]{x^4}}}$$

$$= \lim_{x\to\infty} \frac{\sqrt[3]{\frac{5x^4-3x^3+7+x}{x^4}}+\sqrt[3]{\frac{x^2-4x+3}{x^4}}}{\sqrt[15]{\frac{(x^2+3x-2)^3}{(x^4)^5}}+\sqrt[6]{\frac{x+5}{(x^4)^2}}}$$

$$= \lim_{x\to\infty} \frac{\sqrt[3]{5-\frac{3}{x}+\frac{7}{x^4}+\frac{1}{x^3}}+\sqrt[3]{\frac{1}{x^2}-\frac{4}{x^3}+\frac{3}{x^4}}}{\sqrt[15]{\frac{(x^2+3x-2)^3}{\left(x^{\frac{20}{3}}\right)^3}}+\sqrt[6]{\frac{x+5}{x^8}}}$$

$$= \lim_{x\to\infty} \frac{\overbrace{\sqrt[3]{5-\underbrace{\frac{3}{x}}_{\to 0}+\underbrace{\frac{7}{x^4}}_{\to 0}+\underbrace{\frac{1}{x^3}}_{\to 0}}}^{\to\sqrt[3]{5}}+\overbrace{\sqrt[3]{\frac{1}{x^2}-\frac{4}{x^3}+\frac{3}{x^4}}}^{\to 0}}{\underbrace{\sqrt[15]{\left(\frac{1}{x^{\frac{14}{3}}}+\frac{3}{x^{\frac{17}{3}}}-\frac{2}{x^{\frac{20}{3}}}\right)^3}}_{\to 0}+\underbrace{\sqrt[6]{\frac{1}{x^7}+\frac{5}{x^8}}}_{\to 0}} = \lim_{x\to\infty}\frac{\sqrt[3]{5}}{0} = \infty$$

Example 3. $\lim_{x\to\infty}\left(\sqrt{3x^2+4x-3}-\sqrt{3x^2-2x+7}\right)$

If $x \to \infty$, then

$$\lim_{x\to\infty}\sqrt{3x^2+4x-3} = \infty \text{ and } \lim_{x\to\infty}\sqrt{3x^2-2x+7} = \infty.$$

Thus, we deal here with an indeterminate form of type $\|\infty\text{-}\infty\|$

By multiplying this expression (both the numerator and the denominator) by the corresponding conjugate expression, we get

$$\lim_{x\to\infty}\left(\sqrt{3x^2+4x-3}-\sqrt{3x^2-2x+7}\right) = \|\infty-\infty\|$$

$$= \lim_{x\to\infty}\frac{\left(\sqrt{3x^2+4x-3}-\sqrt{3x^2-2x+7}\right)\left(\sqrt{3x^2+4x-3}+\sqrt{3x^2-2x+7}\right)}{\sqrt{3x^2+4x-3}+\sqrt{3x^2-2x+7}}$$

$$= \lim_{x\to\infty}\frac{3x^2+4x-3-(3x^2-2x+7)}{\sqrt{3x^2+4x-3}+\sqrt{3x^2-2x+7}}$$

$$= \lim_{x\to\infty}\frac{6x-10}{\sqrt{3x^2+4x-3}+\sqrt{3x^2-2x+7}} = \left\|\frac{\infty}{\infty}\right\|$$

Both the numerator and denominator now approach ∞ as $x \to \infty$ Hence, we divide numerator and denominator by x the highest power of the fraction. Then

$$= \lim_{x\to\infty}\frac{\dfrac{6x-10}{x}}{\dfrac{\sqrt{3x^2+4x-3}+\sqrt{3x^2-2x+7}}{x}}$$

$$= \lim_{x\to\infty}\frac{6-\dfrac{\overbrace{10}^{\to 0}}{x}}{\sqrt{3+\underbrace{\dfrac{4}{x}-\dfrac{3}{x^2}}_{\to 0}}+\sqrt{3-\underbrace{\dfrac{2}{x}+\dfrac{7}{x^2}}_{\to 0}}} = \frac{6}{2\sqrt{3}}$$

2. *Problem-Solving Strategy: Calculating a Limit When* $\frac{f(x)}{g(x)}$ *has the Indeterminate Form* $||\frac{0}{0}||$.

Example 4. $\lim\limits_{x\to -1}\frac{x^3-x^2-5x-3}{-3x^3-4x^2+x+2}$.

1. First, we need to make sure that our function has the appropriate form and cannot be evaluated immediately using the limit laws.

By substituting -1 for x returns the familiar indeterminate form of $\left\|\frac{0}{0}\right\|$.

2. We then need to find a function that is equal to $h(x)=\frac{f(x)}{g(x)}$ for all $x\neq -1$ over some interval containing -1. To do this, we may need to try one or more of the following steps:
   (a) If $f(x)$ and $g(x)$ are polynomials, we should factor each function and cancel out any common factors.
   (b) If the numerator or denominator contains a difference involving a square root, we should try multiplying the numerator and denominator by the conjugate of the expression involving the square root.
   (c) If $\frac{f(x)}{g(x)}$ is a complex fraction, we begin by simplifying it.

Since the numerator and denominator are each polynomials, we know that $(x-(-1))$ is factor of each. Using whatever method is most comfortable to you, factor out $(x+1)$ from each (using polynomial division). We find that

$$\frac{x^3-x^2-5x-3}{-3x^3-4x^2+x+2}=\frac{(x+1)(x^2-2x-3)}{(x+1)(-3x^2-x+2)}$$

It could be done by division each polynomials by the factor $(x+1)$ as follows:

$$
\begin{array}{l|l}
x^3 - x^2 - 5x - 3 & \underline{x+1} \\
-\underline{(x^3 + x^2)} & x^2 - 2x - 3 \\
-2x^2 - 5x & \\
-\underline{(-2x^2 - 2x)} & \\
-3x - 3 & \\
-\underline{(-3x - 3)} & \\
0 &
\end{array}
$$

Similarly,

$$
\begin{array}{l|l}
-3x^3 - 4x^2 + x + 2 & \underline{x+1} \\
-\underline{(-3x^3 - 3x^2)} & -3x^2 - x + 2 \\
-x^2 + x & \\
-\underline{(-x^2 - x)} & \\
2x + 2 & \\
-\underline{(2x + 2)} & \\
0 &
\end{array}
$$

Then, we can cancel the $(x+1)$ terms as long as $x \neq -1$. So,

$$
\lim_{x\to -1} \frac{x^3 - x^2 - 5x - 3}{-3x^3 - 4x^2 + x + 2} = \left\|\frac{0}{0}\right\| = \lim_{x\to -1} \frac{\cancel{(x+1)}(x^2 - 2x - 3)}{\cancel{(x+1)}(-3x^2 - x + 2)}
$$

3. Last, we apply the limit laws.

$$
= \lim_{x\to -1} \frac{(x^2 - 2x - 3)}{(-3x^2 - x + 2)} = ||\frac{0}{0}||
$$

In fact, if we substitute -1 into the fraction we get $||\frac{0}{0}||$ again, which is undefined. Factoring and canceling is a strategy to remove this indeterminacy once more:

$$
= \lim_{x\to -1} \frac{\cancel{(x+1)}(x-3)}{\cancel{(x+1)}(-3x+2)} = \lim_{x\to -1} \frac{(x-3)}{(-3x+2)}
$$

Applying the limit laws again we get

$$= \lim_{x \to -1} \frac{(x-3)}{(-3x+2)} = -\frac{4}{5}$$

Example 5. $\lim\limits_{x \to 5} \frac{\sqrt[3]{x^2+2}-3}{2+\sqrt[3]{x-13}}$.

By substituting 5 for x returns the indeterminate form of $\left\|\frac{0}{0}\right\|$.

We then need to simplify the expression to avoid the indeterminacy.

The numerator or denominator contains a difference involving a third-order root, we should try multiplying the numerator and denominator by the expressions leading to difference and sum of cubed terms such as

$$(a-b)(a^2+ab+b^2) = a^3 - b^3$$
$$(a+b)(a^2-ab+b^2) = a^3 + b^3$$

in the numerator or denominator, correspondingly.

Then,

$$\lim_{x \to 5} \frac{\sqrt[3]{x^2+2}-3}{2+\sqrt[3]{x-13}} = \left\|\frac{0}{0}\right\|$$

$$= \lim_{x \to 5} \frac{\left(\sqrt[3]{x^2+2}-3\right)\left(\left(\sqrt[3]{x^2+2}\right)^2 + 3\sqrt[3]{x^2+2} + 3^2\right)}{\left(2+\sqrt[3]{x-13}\right)\left(2^2 - 2\sqrt[3]{x-13} + \left(\sqrt[3]{x-13}\right)^2\right)}$$

$$= \lim_{x \to 5} \frac{\left(\sqrt[3]{x^2+2}\right)^3 - 3^3}{2^3 + \left(\sqrt[3]{x-13}\right)^3} = \lim_{x \to 5} \frac{x^2+2-27}{8+x-13} = \lim_{x \to 5} \frac{x^2-25}{x-5}$$

$$= \left\|\frac{0}{0}\right\| = \lim_{x \to 5} \frac{\cancel{(x-5)}(x+5)}{\cancel{x-5}} = 10$$

3. *Problem-Solving Strategy: Calculating a Limit by using the consequences of the first and second remarkable limits*

Example 6. $\lim_{x\to 0}\frac{\sqrt{5-2x}-\sqrt{5+2x}}{\sin\left(\frac{\pi}{6}-3x\right)\sin 4x}$;

By substituting 0 for x returns the indeterminate form of $\left\|\frac{0}{0}\right\|$.

To apply the rules of the remarkable limits, we rewrite the terms in the numerator and denominator as follows:

$\sqrt{5-2x}=\sqrt{5}\left(1-\frac{2}{5}x\right)^{\frac{1}{2}}$ and $\sqrt{5+2x}=\sqrt{5}\left(1+\frac{2}{5}x\right)^{\frac{1}{2}}$, then

$$\sqrt{5-2x}-\sqrt{5+2x}=\sqrt{5}\left(1-\frac{2}{5}x\right)^{\frac{1}{2}}-\sqrt{5}\left(1+\frac{2}{5}x\right)^{\frac{1}{2}}$$

$$=\sqrt{5}\left\{\left(1-\frac{2}{5}x\right)^{\frac{1}{2}}-\left(1+\frac{2}{5}x\right)^{\frac{1}{2}}\underbrace{+1-1}_{additional\ terms}\right\}$$

$$=\sqrt{5}\left\{\left[\left(1-\frac{2}{5}x\right)^{\frac{1}{2}}-1\right]-\left[\left(1+\frac{2}{5}x\right)^{\frac{1}{2}}-1\right]\right\}=$$

Since $-\frac{2}{5}x\to 0$ and $\frac{2}{5}x\to 0$ as $x\to 0$, we can apply the rule:

$$(1+x)^{\frac{1}{\alpha}}-1\sim\frac{x}{\alpha},\ x\to 0$$

Hence,

$$=\sqrt{5}\left\{\frac{1}{2}\left(-\frac{2}{5}x\right)-\frac{1}{2}\left(\frac{2}{5}x\right)\right\}=-\frac{x}{\sqrt{5}}$$

The trigonometric function can be expended as follows:

$$\sin\left(\frac{\pi}{6}-3x\right)=\sin\frac{\pi}{6}\cos 3x-\sin 3x\cos\frac{\pi}{6}=\frac{1}{2}\cos 3x-\frac{\sqrt{3}}{2}\sin 3x$$

Since $3x\to 0$ as $x\to 0$, then $\cos 3x\to 1$, but for the sinus function of $3x$ we can apply the rule:

$$\sin x\sim x,\ x\to 0$$

Then

$$=\frac{1}{2}1-\frac{\sqrt{3}}{2}3x$$

Also,

Since $4x \to 0$ as $x \to 0$, then $\sin 4x \sim 4x$

Summarizing all these expressions we get

$$\lim_{x\to 0}\frac{\sqrt{5-2x}-\sqrt{5+2x}}{\sin\left(\frac{\pi}{6}-3x\right)\,\sin 4x}=\left\|\frac{0}{0}\right\|$$

$$=\lim_{x\to 0}\frac{\sqrt{5}\left\{\left[\left(1-\frac{2}{5}x\right)^{\frac{1}{2}}-1\right]-\left[\left(1+\frac{2}{5}x\right)^{\frac{1}{2}}-1\right]\right\}}{\left(\frac{1}{2}\cos 3x-\frac{\sqrt{3}}{2}\sin 3x\right)\sin 4x}$$

$$=\lim_{x\to 0}\frac{-\frac{\cancel{x}}{\sqrt{5}}}{\left(\frac{1}{2}1-\frac{\sqrt{3}}{2}3x\right)4\cancel{x}}=-\lim_{x\to 0}\frac{1}{\sqrt{5}\left(\frac{1}{2}-\underbrace{\frac{3\sqrt{3}}{2}x}_{\to 0}\right)}=\frac{2}{\sqrt{5}}$$

Example 7. $\lim\limits_{x\to 0}\frac{e^{x}+e^{-x}-2}{\sin^{2}3x}$.

By substituting 0 for x returns the indeterminate form of $\left\|\frac{0}{0}\right\|$.

To apply the rules of the remarkable limits, we rewrite the terms in the numerator and denominator as follows:

$$e^{x}+e^{-x}-2=\frac{2(e^{x}+e^{-x})}{2}-2=2\underbrace{\frac{(e^{x}+e^{-x})}{2}}_{=\operatorname{ch}x}-2=2(\operatorname{ch}x-1)$$

$$=-2(1-\operatorname{ch}x)=$$

where the rule

$$(1-\operatorname{ch}x)\sim\frac{x^{2}}{2}\text{ as }x\to 0$$

leads to the final expression:

$$= -2\frac{x^2}{2} = -x^2$$

And

$$\sin^2 3x = 1 - \cos^2 3x = (1 - \cos 3x)(1 + \cos 3x)$$

In accordance with the rule:

$$1 - \cos x \sim \frac{x^2}{2} \text{ as } x \to 0$$

Since $3x \to 0$ as $x \to 0$, then we get

$$\frac{(3x)^2}{2}\left(1 + \underbrace{\cos 3x}_{\to 1}\right) = (3x)^2 = 9x^2$$

Therefore,

$$\lim_{x\to 0}\frac{e^x + e^{-x} - 2}{\sin^2 3x} = \left\|\frac{0}{0}\right\| = \lim_{x\to 0}\frac{-2(1 - ch\, x\,)}{(1 - \cos 3x)(1 + \cos 3x)} = \lim_{x\to 0}\frac{-x^2}{9x^2}$$

$$= -\frac{1}{9}$$

Example 8. $\lim\limits_{x\to\pi}\frac{2^{\cos^2\frac{x}{2}} - 1}{\ln(2 + \cos x)}$.

Substitution π for x gives the the indeterminate form of $\left\|\frac{0}{0}\right\|$. However, the variable x tends to π, not zero. Therefore, to apply the consequences of the first and/or second remarkable limits, we will change the variable x with a new one which goes to zero:

Since $x \to \pi$, then $x - \pi \to 0$. Let $t = x - \pi$ i.e. $t \to 0$

Substituting the new variable into the numerator and denominator of the functions in the limit we get

$$\lim_{x\to\pi}\frac{2^{\cos^2\frac{x}{2}} - 1}{\ln(2 + \cos x)} = \left\|\frac{0}{0}\right\| = \begin{Bmatrix} t = x - \pi \\ x = t + \pi \\ t \to 0 \end{Bmatrix} = \lim_{t\to 0}\frac{2^{\cos^2\left(\frac{t+\pi}{2}\right)} - 1}{\ln(2 + \cos(t + \pi))}$$

where

$$\cos^2\left(\frac{t + \pi}{2}\right) - 1 = \cos\left(\frac{t}{2} + \frac{\pi}{2}\right)\cos\left(\frac{t}{2} + \frac{\pi}{2}\right) = \sin^2\frac{t}{2}$$

$$2 + \cos(t + \pi) = 2 - \cos t = 1 + (1 - \cos t)$$

In accordance with the rules:

Since $\frac{t}{2} \to 0$ as $t \to 0$, then $\sin^2\frac{t}{2} = \sin\frac{t}{2}\sin\frac{t}{2} \sim \frac{t^2}{4}$ and $(1-\cos t) \sim \frac{t^2}{2}$

Hence,

$$= \lim_{t\to 0}\frac{2^{\frac{t^2}{4}}-1}{\ln(1+\frac{t^2}{2})}$$

As $t \to 0$, then $\frac{t^2}{4} \to 0$ and $\frac{t^2}{2} \to 0$. It allows us following the rules

$$a^x - 1 \sim x\ln a,\ x \to 0 \text{ and } \ln(1+x) \sim x,\ x \to 0$$

That is

$$2^{\frac{t^2}{4}} - 1 \sim \frac{t^2}{4}\ln 2 \text{ and } \ln(1+\frac{t^2}{2}) \sim \frac{t^2}{2}$$

Finally,

$$= \lim_{t\to 0}\frac{\frac{t^2}{4}\ln 2}{\frac{t^2}{2}} = \frac{\ln 2}{2}$$

Example 9. $\lim\limits_{x\to 3}\dfrac{5\arcsin\frac{x-3}{4}}{3^{x^2-8}-3}$.

Substitution 3 for x gives the indeterminate form of $\left\|\frac{0}{0}\right\|$. However, the variable x tends to 3, not zero. Therefore, to apply the consequences of the first and/or second remarkable limits, we will change the variable x with a new one which goes to zero:

Since $x \to 3$, then $x - 3 \to 0$. Let $t = x - 3$ i.e. $t \to 0$

Substituting the new variable into the numerator and denominator of the functions in the limit we get

$$\lim_{x\to 3}\frac{5\arcsin\frac{x-3}{4}}{3^{x^2-8}-3} = \left\|\frac{0}{0}\right\| = \begin{Bmatrix} t = x-3 \\ x = t+3 \\ t \to 0 \end{Bmatrix} = \lim_{t\to 0}\frac{5\arcsin\frac{t+3-3}{4}}{3^{(t+3)^2-8}-3}$$

$$= \lim_{t\to 0}\frac{5\arcsin\frac{t}{4}}{3^{t^2+6t+1}-3} = \lim_{t\to 0}\frac{5\arcsin\frac{t}{4}}{3\left(3^{t^2+6t}-1\right)}$$

where following the appropriate rules we can write that:

$$\text{since } \frac{t}{4} \to 0 \text{ as } t \to 0 \text{ then } \arcsin\frac{t}{4} \sim \frac{t}{4}$$

and

$$\text{since } (t^2 + 6t) \to 0 \text{ as } t \to 0 \text{ then } 3^{t^2+6t} - 1 \sim (t^2 + 6t)\ln 3$$

Hence,

$$= \lim_{t\to 0} \frac{5\frac{t}{4}}{3(t^2 + 6t)\ln 3} = \lim_{t\to 0} \frac{5\cancel{t}}{12\ln 3\,(t+6)\cancel{t}} = \lim_{t\to 0} \frac{5}{12\ln 3\,(\underbrace{t}_{\to 0} + 6)}$$

$$= \frac{5}{72\ln 3}$$

Example 10. $\lim\limits_{x\to\frac{\pi}{2}} \frac{\cos 3x - \cos x}{5(2x-\pi)^2}$.

$$\lim_{x\to\frac{\pi}{2}} \frac{\cos 3x - \cos x}{5(2x-\pi)^2} = \left\|\frac{0}{0}\right\| = \left\{\begin{matrix} t = x - \frac{\pi}{2} \\ x = t + \frac{\pi}{2} \\ t \to 0 \end{matrix}\right\}$$

$$= \lim_{t\to 0} \frac{\cos 3\left(t + \frac{\pi}{2}\right) - \cos\left(t + \frac{\pi}{2}\right)}{5\left(2\left(t + \frac{\pi}{2}\right) - \pi\right)^2} = \lim_{t\to 0} \frac{\sin 3t + \sin t}{20t^2}$$

$$= \lim_{t\to 0} \frac{2\sin 2t\,\overbrace{\cos t}^{\to 1}}{20t^2} =$$

Since $2t \to 0$ as $t \to 0$ then $\sin 2t \sim 2t$, i.e.

$$= \lim_{t\to 0} \frac{4t}{20t^2} = \lim_{t\to 0} \frac{\cancel{t}}{5t^{\cancel{2}}} = \lim_{t\to 0} \frac{1}{5t} = \infty$$

4. *Problem-Solving Strategy: Calculating a Limit of Power-Exponential Functions by using the second remarkable limit*

Example 11. $\lim\limits_{x\to\infty} \left(\frac{x^2 - x + 4}{2x^2 + x + 5}\right)^{\frac{x^3-5}{4x+3}}$

Substitution ∞ for x leads to the calculation of the appropriate limits of the functions in the basis and the power as follows:

$$\lim_{x\to\infty}\frac{x^2-x+4}{2x^2+x+5}=\left\|\frac{\infty}{\infty}\right\|=\lim_{x\to\infty}\frac{\frac{x^2-x+4}{x^2}}{\frac{2x^2+x+5}{x^2}}=\lim_{x\to\infty}\frac{1-\frac{1}{x}+\frac{4}{x^2}}{2+\frac{1}{x}+\frac{5}{x^2}}=\frac{1}{2}$$

Similarly,

$$\lim_{x\to\infty}\frac{x^3-5}{4x+3}=\left\|\frac{\infty}{\infty}\right\|=\lim_{x\to\infty}\frac{\frac{x^3-5}{x^3}}{\frac{4x+3}{x^3}}=\lim_{x\to\infty}\frac{1-\overbrace{\frac{5}{x^3}}^{\to 0}}{\underbrace{\frac{4}{x^2}+\frac{3}{x^3}}_{\to 0}}=\lim_{x\to\infty}\frac{1}{0}=\infty$$

Therefore, we get

$$\lim_{x\to\infty}\left(\frac{x^2-x+4}{2x^2+x+5}\right)^{\frac{x^3-5}{4x+3}}=\lim_{x\to\infty}\left(\frac{1}{2}\right)^{\infty}=0$$

Example 12. $\lim\limits_{x\to\infty}\left(\frac{2x^2-x+4}{x^2+x+5}\right)^{\frac{x^3-5}{4x+3}}$

Substitution ∞ for x leads to the calculation of the appropriate limits of the functions in the basis and the power as follows:

$$\lim_{x\to\infty}\frac{2x^2-x+4}{x^2+x+5}=\left\|\frac{\infty}{\infty}\right\|=\lim_{x\to\infty}\frac{\frac{2x^2-x+4}{x^2}}{\frac{x^2+x+5}{x^2}}=\lim_{x\to\infty}\frac{2-\frac{1}{x}+\frac{4}{x^2}}{1+\frac{1}{x}+\frac{5}{x^2}}=2$$

Similarly,

$$\lim_{x\to\infty}\frac{x^3-5}{4x+3}=\left\|\frac{\infty}{\infty}\right\|=\lim_{x\to\infty}\frac{\frac{x^3-5}{x^3}}{\frac{4x+3}{x^3}}=\lim_{x\to\infty}\frac{1-\overbrace{\frac{5}{x^3}}^{\to 0}}{\underbrace{\frac{4}{x^2}+\frac{3}{x^3}}_{\to 0}}=\lim_{x\to\infty}\frac{1}{0}=\infty$$

Therefore, we get

$$\lim_{x\to\infty}\left(\frac{2x^2-x+4}{x^2+x+5}\right)^{\frac{x^3-5}{4x+3}}=\lim_{x\to\infty}2^{\infty}=\infty$$

Example 13. $\lim\limits_{x\to\infty}\left(\frac{x^2-x+4}{x^2+x+5}\right)^{\frac{x^3-5}{4x+3}}$

Substitution ∞ for x leads to the calculation of the appropriate limits of the functions in the basis and the power as follows:

$$\lim_{x\to\infty}\frac{x^2-x+4}{x^2+x+5}=\left\|\frac{\infty}{\infty}\right\|=\lim_{x\to\infty}\frac{\frac{x^2-x+4}{x^2}}{\frac{x^2+x+5}{x^2}}=\lim_{x\to\infty}\frac{1-\frac{1}{x}+\frac{4}{x^2}}{1+\frac{1}{x}+\frac{5}{x^2}}=1$$

Similarly,

$$\lim_{x\to\infty}\frac{x^3-5}{4x+3}=\left\|\frac{\infty}{\infty}\right\|=\lim_{x\to\infty}\frac{\frac{x^3-5}{x^3}}{\frac{4x+3}{x^3}}=\lim_{x\to\infty}\frac{1-\overbrace{\frac{5}{x^3}}^{\to 0}}{\underbrace{\frac{4}{x^2}+\frac{3}{x^3}}_{\to 0}}=\lim_{x\to\infty}\frac{1}{0}=\infty$$

Therefore, we get

$$\lim_{x\to\infty}\left(\frac{x^2-x+4}{x^2+x+5}\right)^{\frac{x^3-5}{4x+3}}=\lim_{x\to\infty}(1)^{\infty}=\left\|1^{\infty}\right\|=$$

The second remarkable limit can be used to remove the Indeterminate Form $||1^{\infty}||$ as follows:

$$\lim_{x\to\infty}\left(1+\frac{x^2-x+4}{x^2+x+5}-1\right)^{\frac{x^3-5}{4x+3}}$$

$$=\lim_{x\to\infty}\left(1+\frac{(x^2-x+4)-(x^2+x+5\,)}{x^2+x+5}\right)^{\frac{x^3-5}{4x+3}}$$

$$=\underbrace{\boxed{\lim_{x\to\infty}\left(1+\frac{-2x-1}{x^2+x+5}\right)^{\frac{x^2+x+5}{-2x-1}}}}_{\to e}{}^{\cdot\frac{x^3-5}{4x+3}\cdot\frac{-2x-1}{x^2+x+5}}$$

$$=e^{\lim_{x\to\infty}\frac{x^3-5}{4x+3}\cdot\frac{-2x-1}{x^2+x+5}}=$$

Consider the limit of the function in the power

$$\lim_{x\to\infty}\frac{x^3-5}{4x+3}\cdot\frac{-2x-1}{x^2+x+5}=\lim_{x\to\infty}\frac{-2x^4-x^3+10x+5}{4x^3+7x^2+20x+15}=\left\|\frac{\infty}{\infty}\right\|$$

$$= \lim_{x\to\infty} \frac{\dfrac{-2x^4 - x^3 + 10x + 5}{x^4}}{\dfrac{4x^3 + 7x^2 + 20x + 15}{x^4}} = \lim_{x\to\infty} \frac{-2 - \overbrace{\frac{1}{x} + \frac{10}{x^3} + \frac{5}{x^4}}^{\to 0}}{\underbrace{\frac{4}{x} + \frac{7}{x^2} + \frac{20}{x^3} + \frac{15}{x^4}}_{\to 0}} = \lim_{x\to\infty} \frac{-2}{0} = -\infty$$

Then, finally we get

$$= e^{-\infty} = \frac{1}{e^{\infty}} = 0$$

Examples

(a).

$$\lim_{x\to\infty}\left(\frac{2x-1}{2x+3}\right)^x = \lim_{x\to\infty}\left[\frac{(2x+3)-4}{2x+3}\right]^x = \lim_{x\to\infty}\left(1-\frac{4}{2x+3}\right)^x =$$

$$= \lim_{x\to\infty}\left(1-\frac{2}{x}\right)^x = \frac{1}{e^2}.$$

(b)

$$\lim_{x\to 0}(1+\tan 3x)^{\cot x} = \lim_{x\to 0}(1+\tan 3x)^{\frac{1}{\tan x}} = \lim_{x\to 0}(1+3x)^{\frac{1}{x}} = e^3.$$

(c)

$$\lim_{x\to 0}(x\sin 2x + \cos x)^{\frac{1}{x^2}} = \lim_{x\to 0}\{1+[x\sin 2x - (1-\cos x)]\}^{\frac{1}{x^2}} =$$

$$= \lim_{x\to 0}\left(1+2x^2-\frac{x^2}{2}\right)^{\frac{1}{x^2}} = \lim_{x\to 0}\left(1+\frac{3}{2}x^2\right)^{\frac{1}{x^2}} = e^{\frac{3}{2}}.$$

25. *Problem-Solving Strategy: Find points of discontinuity, determine their type and draw a sketch of the function behavior in the neighborhood of the discontinuity points*

Example 14. $f(x) = \begin{cases} x^2, & x \le 1 \\ x+3, & x > 1 \end{cases}$

Is the function below continuous at its transition point? If not, identify the type of discontinuity occurring there.

Step 1. Identify the point(s) which is(are) suspected to have discontinuities

The suspected point is at $x = 1$ since this is where the function transitions from one formula to the next.

Step 2. Determine the left-sided limit at the transition point.

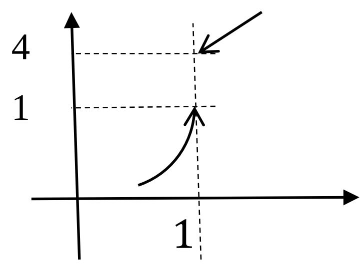
$$\lim_{x\to 1^{-0}} f(x) = \lim_{x\to 1^{-0}} x^2 = (1-0)^2 = \left(1^2 \underbrace{-2\cdot 0 + 0^2}_{\to 0}\right) = 1$$

Similarly, the right-sided limit at the transition point.

$$\lim_{x\to 1^{+0}} f(x) = \lim_{x\to 1^{+0}} (x+3) = (1+0+3) = 4$$

Step 3. We can conclude: Since the one-sided limits are different, the function has a jump discontinuity at $x = 1$

Step 4. Draw s sketch of the discontinuity at $x = 1$

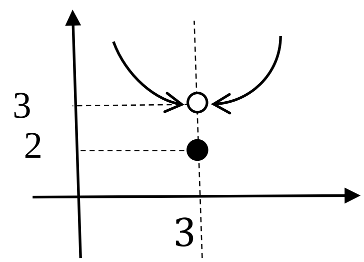


Example 15. $f(x) = \frac{x^2+2x-15}{x^2-2x-3}$

1. Without graphing, determine the type of discontinuity the function below has at $x = 3$ as
   Evaluating $f(3)$ we have
   $$f(3) = \frac{(3)^2 + 2(3) - 15}{(3)^2 - 2(3) - 3} = \frac{9+6-15}{9-6-3} = \frac{0}{0}$$
   That is, the function is undefined at $x = 3$, so there is a discontinuity at this point. To determine the type, we will need to evaluate the limit as x approaches 3.
   Since the function has a $\frac{0}{0}$ form at $x = 3$, we need to find and divide out the common factors in the numerator and denominator.
   $$\frac{x^2+2x-15}{x^2-2x-3} = \frac{(x+5)(x-3)}{(x-3)(x+1)} = \frac{x+5}{x+1}$$
2. Evaluate the limit of the simpler function as x approaches 3 (it is the same value for right – and left – sided limits).

$$\lim_{x\to 3}\frac{x+5}{x+1}=\frac{3+5}{3+1}=\frac{8}{4}=2$$

3. Since the limit exists, but the function value does not, we know the function has is a removable discontinuity at $x=3$.



Example 16. $f(x)=2^{\frac{x-1}{x^2-1}}$,

1. Find points which might give indeterminacy of the function (do not belong to the domain of function)

   One can see that the function is undetermined at $x=-1$

2. Calculate the one-sided limits at such points:
   The right-sided limit at $x=-1$ is

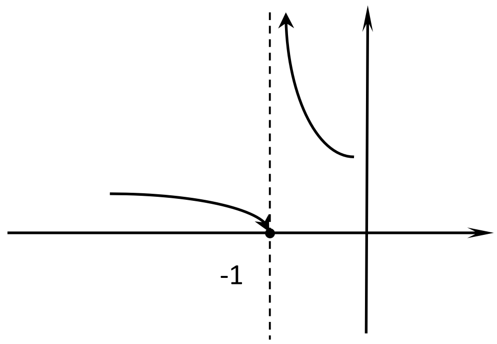
$$\lim_{x\to -1+0} 2^{\frac{x-1}{x^2-1}}$$
$$=\left\{\lim_{x\to -1+0}\frac{x-1}{x^2-1}=\lim_{x\to -1+0}\frac{\cancel{x-1}}{\cancel{(x-1)}(x+1)}=\frac{1}{-1+0+1}=\infty\right\}=2^{\infty}=\infty$$

   The left-sided limit at $x=-1$ is

$$\lim_{x\to -1-0} 2^{\frac{x-1}{x^2-1}}$$
$$=\left\{\lim_{x\to -1-0}\frac{x-1}{x^2-1}=\lim_{x\to -1+0}\frac{\cancel{x-1}}{\cancel{(x-1)}(x+1)}=\frac{1}{-1-0+1}=-\infty\right\}=2^{-\infty}=0$$

3. Define a kind of the discontinuity taking into account the obtained values of these limits.
   Following the definition, the function has a second kind discontinuity at the point $x=-1$

4. Draw a sketch of the function in the neighborhood of the discontinuity point $x = -1$



Example 15. Find the parameter at which a give function $f(x)=\begin{cases} x^2+3, -4<x\le 3, \\ \dfrac{A}{x-3}, 3<x<\infty. \end{cases}$ to be continuous (if it is possible)

Calculate the one-sided limits at the point, where the function changes its behavior.

The left-sided limit as $x \to 3$

$$\lim_{x\to 3-0}(x^2+3) = (3-0)^2 + 3 = 9 - 6\cdot 0 + 0^2 + 3 = 12$$

For the given function to be continuous, the one-sided limits as $x \to 3$ have to be equal, i.e. the right-sided limit of the function has to satisfy the condition:

$$\lim_{x\to 3+0}\frac{A}{x-3} = \frac{A}{3+0-3} = \frac{A}{0} \neq 12$$

Since the latter equality is impossible for any A; or there is no A for which this function is continuous.