Arithmetic of Derivatives: *The Basic Formulas and Rules of Differentiation*

Example. $f(x) = 5x + 4, a = -1$ $f'(-1) - ?$

According with the derivative definition:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \Rightarrow f'(-1) = \lim_{x \to -1} \frac{f(x) - f(-1)}{x - (-1)}$$

$$= \lim_{x \to -1} \frac{(5x + 4) - (5(-1) + 4)}{x - (-1)} = \lim_{x \to -1} \frac{5x + 5}{x + 1} = \left\|\frac{0}{0}\right\|$$

$$= \lim_{x \to -1} \frac{5\cancel{(x + 1)}}{\cancel{x + 1}} = 5$$

$f'(-1) = 5$

Moreover, we need to remember that the derivative of the function at every point is a function itself:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x}$$

Example 1. $\frac{d}{dx}(3x^2 + 5x + 7) - ?$

$$\frac{d}{dx}(3x^2 + 5x + 7) \quad = 3\frac{d}{dx}(x^2) + 5\frac{d}{dx}(x) + \frac{d}{dx}(7)$$
$$= 3 \cdot 2x + 5 \cdot 1 + 0$$
$$= 6x + 5.$$

Hence, if a task exists: $f(x) = (3x^2 + 5x + 7)$, $f'(2) - ?$

then

$$f'(x) = 6x + 5 \Rightarrow f'(2) = 6 \cdot 2 + 5 = 17$$

Example 2. Find the derivative of the function $y = \frac{2}{3x} + 3x^4$, $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{2}{3x} + 3x^4\right) - ?$ or $y' = \left(\frac{2}{3x} + 3x^4\right)' - ?$

$$y'(x) = (\frac{2}{3x} + 3x^4)' = (\frac{2}{3x})' + (3x^4)' = \frac{2}{3}(\frac{1}{x})' + 3(x^4)'$$

$$= \frac{2}{3} \cdot (-\frac{1}{x^2}) + 3 \cdot 4x^3 = -\frac{2}{3x^2} + 12x^3 = 12x^3 - \frac{2}{3x^2}.$$

Example 3. $\frac{\mathrm{d}}{\mathrm{d}x}\{(3x+9)(x^2+4x^3)\}$ - ? or $y' = ((3x+9)(x^2+4x^3))'$-?

$$\frac{\mathrm{d}}{\mathrm{d}x}\{(3x+9)(x^2+4x^3)\}$$

$$= \frac{\mathrm{d}}{\mathrm{d}x}(3x+9)\,(x^2+4x^3) + (3x+9)\frac{\mathrm{d}}{\mathrm{d}x}(x^2+4x^3)$$

$$= \left(\frac{\mathrm{d}}{\mathrm{d}x}(3x) + \frac{\mathrm{d}}{\mathrm{d}x}(9)\right)\,(x^2+4x^3) + (3x+9)\left(\frac{\mathrm{d}}{\mathrm{d}x}(x^2) + \frac{\mathrm{d}}{\mathrm{d}x}(4x^3)\right)$$

$$= (3+0)\,(x^2+4x^3) + (3x+9)(2x+12x^2)$$

$$= (3x^2+12x^3) + (18x+6x^2+114x^2+36x^3)$$

$$= 18x + 117x^2 + 48x^3$$

Example 4. $y = x^2 \sin x$, $y' = (x^2 \sin x)' - ?$

$$y'(x) = (x^2 \sin x)' = (x^2)' \sin x + x^2 (\sin x)' = 2x \sin x + x^2 \cos x$$

$$= x(2\sin x + x\cos x).$$

Example 5. $\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{4x^3-7x}{4x^2+1}\right\}$? or $y' = \left(\frac{4x^3-7x}{4x^2+1}\right)' - ?$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{4x^3-7x}{4x^2+1}\right\} = \frac{\frac{\mathrm{d}}{\mathrm{d}x}(4x^3-7x)(4x^2+1) - (4x^3-7x)\frac{\mathrm{d}}{\mathrm{d}x}(4x^2+1)}{(4x^2+1)^2}$$

$$= \frac{(12x^2-7)(4x^2+1) - (4x^3-7x)(8x)}{(4x^2+1)^2}$$

$$= \frac{(48x^4-16x^2-7) - (32x^4-56x^2)}{(4x^2+1)^2} = \frac{16x^4+40x^2-7}{(4x^2+1)^2}$$

Example 6. Find the derivative of the function $y = \frac{2x+1}{2x-1}$ at $x = 1$

$$y' = \left(\frac{2x+1}{2x-1}\right)' = \frac{2\cdot(2x-1)-(2x+1)\cdot 2}{(2x-1)^2} = \frac{\cancel{4x}-2-\cancel{4x}-2}{(2x-1)^2}$$

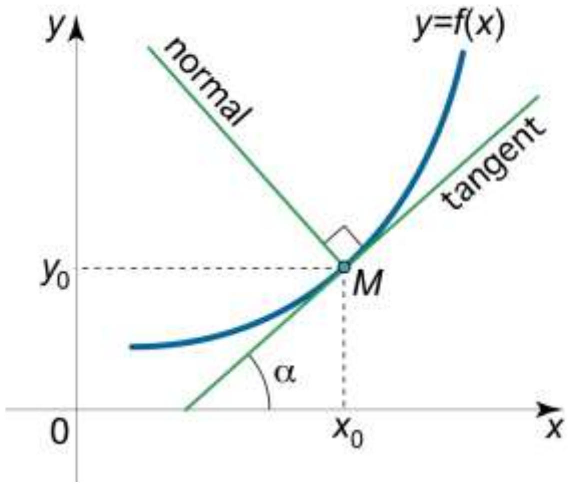
$$= -\frac{4}{(2x-1)^2}.$$

At $x = 1$

$$y'(1) = -\frac{4}{(2\cdot 1-1)^2} = -\frac{4}{1^2} = -4.$$

Using the Derivative

Example 7. A tangent line and a line normal to the curve $y = \sqrt{x}$ at $x = 4$;



By the geometrical meaning of the derivative, the tangent line to the curve $y = f(x)$ at $x = a$ is given by

$$y \quad = f(a) + f'(a)(x-a)$$

provided $f'(a)$ exists.

So, the derivative of $\sqrt{x}$ at $x = a$ is

$$f'(a) \quad = \frac{1}{2\sqrt{a}}$$

If $a = 4$ then

$$f'(a) = f'(4) = \frac{1}{2\sqrt{a}}\Big|_{a=4} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

and

$$f(a) = f(4) = \sqrt{x}|_{x=4} = \sqrt{4} = 2$$

Hence, the equation of the tangent line is

$$y = 2 + \frac{1}{4}(x - 4) \quad \text{or} \quad y = \frac{x}{4} + 1$$

Then, the slope of the line normal to the tangent line is calculated as

$$k_{normal} = -\frac{1}{k_{tangent}}$$

That is the equation of the normal

$$y - y_0 = -\frac{1}{f'(x_0)}(x - x_0), \text{ where } k_{normal} = -\frac{1}{f'(x_0)}$$

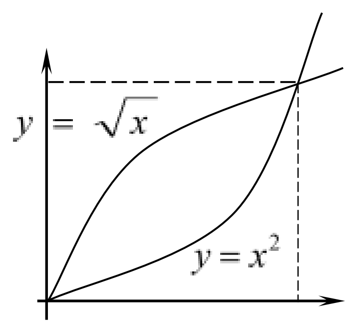
Hence,

$$k_{normal} = -\frac{1}{1/4} = -4$$

Finally, the equation of the normal is

$$y = 2 - 4(x - 4) \quad \text{or} \quad y = -4x + 18$$

Example 8. Find the angle of the intersection of the curves $y = x^2$ and $y = \sqrt{x}$ at the point $M(1,1)$.



Obvious the angle between curves is equal to the angle between their tangent lines. From the equations of the given lines we find the derivatives

$$y' = 2x, \; y' = \frac{1}{2\sqrt{x}},$$

Calculate the slopes of the lines tangent to the given curves at the point M (1,1)

$$k_1 = 2, \quad k_2 = \frac{1}{2}.$$

To find a sought angle α we can use the following formula $\tan\alpha = \dfrac{k_2 - k_1}{1 + k_1 k_2}$

(tangent of an angle between two straight lines with slopes k_1 and k_2). Then

$$\tan\alpha = \frac{2-\frac{1}{2}}{1+\frac{1}{2}\cdot 2} = \frac{3}{4}.$$

Example 9. Find the equation of the line tangent to the curve $y = 2x^2 + 4x - 1$ if it is known that the tangent line is parallel to the straight line $x + 2y - 3 = 0$

In the given case, a point (x_0, y_0) at which the line tangent to the given curve is not known. But it is known the slope of the line parallel to the tangent line, i.e. $k = -\frac{1}{2}$.

Therefore we can determine the derivative of the given curve and equite it to the value $-\frac{1}{2}$. So, $y' = 4x + 4$, then

$$4x_0 + 4 = -\frac{1}{2},$$

whence

$$x_0 + 1 = -\frac{1}{8},$$

It follows on that

$$x_0 = -\frac{9}{8},\ y_0 = \frac{81}{32} - \frac{9}{2} - 1 = -\frac{95}{32}.$$

So, the equation of the tangent line is:

$$y + \frac{95}{32} = -\frac{1}{2}\left(x + \frac{9}{8}\right),$$

or

$$16x + 32y + 113 = 0.$$

The Chain Rule

If the composition of functions $f(x)$ and $g(x)$ occurs

$$y(x) = f(g(x))$$

(represents a "two-layer" composite function or a function of a function)

Also, if $f(x)$ and $g(x)$ are differentiable functions, then the composite function $y(x)$ is also differentiable in x and its derivative is given by

$$\frac{dy}{dx} = \frac{d}{dx} f(g(x)) g'(x) \text{ or } \frac{dy}{dx} = \frac{df}{du}\frac{du}{dx}$$

where $u = g(x)$ is an inner function (intermediate argument), and $f(u)$ is an outer function

This rule is easily generalized for composite functions consisting of three and more "layers".

$$y' = [f(g(h(x)))]' = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x).$$

Example 10. $y = \ln x^2$, $y'-?$

$$y'(x) = (\ln x^2)' = \frac{1}{x^2} \cdot (x^2)' = \frac{1}{x^2} \cdot 2x = \frac{2x}{x^2} = \frac{2}{x} (x \neq 0).$$

Example 11. $y = \ln^2 x$, $y'-?$

$$y'(x) = (\ln^2 x)' = 2\ln x \cdot (\ln x)' = 2\ln x \cdot \frac{1}{x} = \frac{2\ln x}{x} (x > 0).$$

Example 12. $y = \cos(3x + 2)$, $y'-?$

$$y'(x) = [\cos(3x + 2)]' = -\sin(3x + 2) \cdot (3x + 2)' = -3\sin(3x + 2).$$

Example 13. $y = \sin^3 x$, $y'-?$

$$y'(x) = (\sin^3 x)' = 3\sin^2 x \cdot (\sin x)' = 3\sin^2 x \cos x.$$

Example 14. $y = 3^{\cos x}$, $y'-?$

$$y'(x) = (3^{\cos x})' = 3^{\cos x} \cdot \ln 3 \cdot (\cos x)' = -3^{\cos x} \ln 3 \sin x.$$

Example 15. $f(x) = \cos(\tan(3x))$.

$$[f(g(h(x)))]' = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$$

1) $f'(x) = -\sin(\tan(3x)) \times \cdots$
2) $f'(x) = -\sin(\tan(3x)) \cdot \sec^2(3x) \times \cdots$
3) $f'(x) = -\sin(\tan(3x)) \cdot \sec^2(3x) \cdot 3$

Example 16. $f(x) = (1 + \sin^9(2x + 3))^2$.

1) $f'(x) \; = 2(1+\sin^9(2x+3)) \times \cdots$

2) $f'(x) = 2(1+\sin^9(2x+3)) \cdot (0+9[\sin(2x+3)]^8) \times \cdots$

$$= 2(1+\sin^9(2x+3)) \cdot 9[\sin(2x+3)]^8 \times \cdots$$

$$= 2(1+\sin^9(2x+3)) \cdot 9\sin^8(2x+3) \times \cdots$$

3) $f'(x) == 2(1+\sin^9(2x+3)) \cdot 9\sin^8(2x+3) \cdot \cos(2x+3) \times \cdots$

4) $f'(x) = 2(1+\sin^9(2x+3)) \cdot 9\sin^8(2x+3) \cdot \cos(2x+3) \cdot 2$

Implicit Differentiation

The function can be defined in implicit form, that is by the equation

$$F(x,y) = 0$$

We do not need to convert an implicitly defined function into an explicit form to find the derivative $y'(x)$. In this case we proceed as follows:

Differentiate both sides of the equation $F(x,y) = 0$with respect to x, assuming that y is a differentiable function of x and using the chain rule. The derivative of zero (in the right side) will also be equal to zero.

Example 17. $x^2 + y^2 - 2x - 4y = 4$, $y'(x)$–?

We take the derivative of each term treating y as a function of x

$$(x^2)' + (y^2)' - (2x)' - (4y)' = 4', \Rightarrow 2x + 2yy' - 2 - 4y' = 0.$$

Solve this equation for y'

$$2yy' - 4y' = 2 - 2x, \Rightarrow yy' - 2y' = 1 - x, \Rightarrow y'(y-2) = 1 - x, \Rightarrow$$

$$y' = \frac{1-x}{y-2}.$$

Example 18. Let the function $y = y(x)$ be given by equation $xy + e^x + \sin y = 0$. In this case passing to explicit form of function is impossible. Then

$$y + xy' + e^x + \cos y \cdot y' = 0,$$

whence

$$y' = -\frac{y + e^x}{x + \cos y}.$$

Example 19. $x^3 + \tan y = 2$.

To find the derivative let us differentiate this equation as identity, regarding y as a function of x, i.e. $y = y(x)$. Then we obtain

$$3x^2 + \frac{1}{\cos^2 y} y' = 0,$$

whence

$$y' = -3x^2 \cos^2 y.$$

Obviously we can easy pass from implicit representation of the function to explicit form in this case. Indeed, we obtain that

$$\tan y = 2 - x^3,$$

whence

$$y = \arctan(2 - x^3) + n\pi, \; n = 0, \; \pm 1, \; \pm 2, \; ...,$$

then

$$y' = \frac{1}{1 + (2 - x^3)^2}(-3x^2) = -\frac{3x^2}{1 + (2 - x^3)^2},$$

which coincides with previous result, because $\tan y = 2 - x^3$ and consequently

$$\frac{1}{1 + (2 - x^3)^2} = \frac{1}{1 + \tan^2 y} = \cos^2 y.$$

Example 20. Calculate the derivative at the point (0,0) of the function given by the equation $x = y - 2\sin y$.

We differentiate both sides of the equation with respect to x and solve for y':

$$x' = y' - (2\sin y)', \Rightarrow 1 = y' - 2\cos y \cdot y', \Rightarrow y' = \frac{1}{1 - 2\cos y}.$$

Substitute the coordinates (0,0):

$$y'(0,0) = \frac{1}{1 - 2\cos 0} = \frac{1}{1 - 2 \cdot 1} = -1.$$

Example 21. Find the equation of the tangent line to the curve $x^4 + y^4 = 2$ at the point (1,1)

Differentiate both sides of the equation with respect to x:

$$\frac{d}{dx}(x^4 + y^4) = \frac{d}{dx}(2), \Rightarrow 4x^3 + 4y^3y' = 0, \Rightarrow x^3 + y^3y' = 0.$$

Then

$$y' = -\frac{x^3}{y^3}$$

At the point (1,1) we have $y'(1,1) = -1$

Hence, the equation of the tangent line is given by

$$\frac{x-1}{y-1} = -1 \text{ or } x + y = 2.$$

Logarithmic Differentiation

This approach allows calculating derivatives of power, rational and some irrational functions in an efficient manner. The steps are the following:

1) Take natural logarithms of both sides: $\ln y = \ln f(x)$.
2) Next, we differentiate this expression as an implicit function, i.e. using the chain rule and keeping in mind that y is a function of *x*:

$$(\ln y)' = (\ln f(x))', \Rightarrow \frac{1}{y} y'(x) = (\ln f(x))'.$$

3) So, $y' = y(\ln f(x))' = f(x)(\ln f(x))'$.

Example 22. Consider the function $y = (\sin x)^{\cos x}$. Then

$$\ln y = \cos x \cdot \ln \sin x,$$

consequently

$$\frac{1}{y} \cdot y' = -\sin x \cdot \ln \sin x + \cos x \cdot \frac{1}{\sin x} \cdot \cos x,$$

whence

$$y' = \left(-\sin x \cdot \ln \sin x + \frac{\cos^2 x}{\sin x} \right) y = (\text{ctg} x \cdot \cos x - \sin x \cdot \ln \sin x) \cdot (\sin x)^{\cos x}$$

Example 23. Find derivative of the function $y = x^x$. Applying formula (4.25) we can obtain at once

$$y' = x^x \cdot \ln x + x \cdot x^{x-1} = x^x(\ln x + 1).$$

We can also apply the method of logarithmic differentiation. Then,

$$\ln y = x \ln x \qquad \Rightarrow \qquad \frac{1}{y} y' = \ln x + 1,$$

$$y' = x^x \cdot (\ln x + 1).$$

Example 24. Find the derivative of the function

$$y = \frac{\sqrt[3]{1+x^2} \cdot 2^{\sin x} (\tan x - 1)^5}{x^3 \ln^2 x}.$$

Direct differentiation of this function is possible, but it is connected with difficulties, because of a large number of the multipliers. Therefore, it is easier, first, to take the logarithm of a given function, and then differentiate it. Indeed

$$\ln y = \frac{1}{3}\ln(1+x^2) + \sin x \cdot \ln 2 + 5\ln(\tan x - 1) - 3\ln x - 2\ln(\ln x).$$

Then

$$\frac{1}{y} y' = \frac{1}{3}\frac{2x}{1+x^2} + \cos x \cdot \ln 2 + \frac{5}{\tan x - 1} \cdot \frac{1}{\cos^2 x} - \frac{3}{x} - \frac{2}{\ln x} \cdot \frac{1}{x},$$

whence

$$y' = \left(\frac{2x}{3(1+x^2)} + \cos x \cdot \ln 2 + \frac{5}{(\tan x - 1)\cos^2 x} - \frac{3}{x} - \frac{2}{x \ln x}\right) y,$$

i. e.

$$y' = \left(\frac{2x}{3(1+x^2)} + \cos x \cdot \ln 2 + \frac{5}{(\tan x - 1)\cos^2 x} - \frac{3}{x} - \frac{2}{x \ln x}\right) *$$

$$* \frac{\sqrt[3]{1+x^2} \cdot 2^{\sin x} (\tan x - 1)^5}{x^3 \ln^2 x}.$$

Derivatives of Parametric Functions

The relationship between the variables x and y as a function $y(x)$ can be defined in parametric form using two equations:

$$\begin{cases} x & = x(t) \\ y & = y(t)' \end{cases}$$

Then its derivative is given by

$$y'_x = y'_t \cdot t'_x = y'_t \cdot \frac{1}{x'_t} = \frac{y'_t}{x'_t}.$$

Example 25. Find y'_x−? if $x = e^{2t}, y = e^{3t}$

Hence, the derivative is given by

$$\frac{dy}{dx} = y'_x = \frac{y'_t}{x'_t} = \frac{3e^{3t}}{2e^{2t}} = \frac{3}{2}e^{3t-2t} = \frac{3}{2}e^{t}.$$

Example 26. Find y'_x−? if $x = \sin^2 t, y = \cos^2 t$.

Differentiate with respect to the parameter *t*

$$x'_t = (\sin^2 t)' = 2\sin t \cdot \cos t = \sin 2t,$$

$$y'_t = (\cos^2 t)' = 2\cos t \cdot (-\sin t) = -2\sin t\cos t = -\sin 2t.$$

Then

$$\frac{dy}{dx} = y'_x = \frac{y'_t}{x'_t} = \frac{-\cancel{\sin 2t}}{\cancel{\sin 2t}} = -1, \text{where } t \neq \frac{\pi n}{2}, n \in \mathbb{Z}.$$

Example 27. Find the derivative y'_x−? for the function $x = \sin 2t, y = -\cos t$ at the point $t = \frac{\pi}{6}$.

Compute the derivatives with respect to *t*

$$x'_t = (\sin 2t)' = 2\cos 2t, y'_t = (-\cos t)' = \sin t.$$

So,

$$\frac{dy}{dx} = \frac{y'_t}{x'_t} = \frac{2\cos 2t}{\sin t}.$$

Compute the derivative at $t = \frac{\pi}{6}$:

$$\frac{dy}{dx}\left(t = \frac{\pi}{6}\right) = \frac{2\cos(2 \cdot \frac{\pi}{6})}{\sin\frac{\pi}{6}} = \frac{2\cos\frac{\pi}{3}}{\sin\frac{\pi}{6}} = \frac{2 \cdot \cancel{\frac{1}{2}}}{\cancel{\frac{1}{2}}} = 2.$$