# Lecture #21: The DIFFERENTIAL

## 21.1 The Definition of the Deferential

We have built up most of the tools that we need to express derivatives. In this section we revise a basic idea of the derivative.

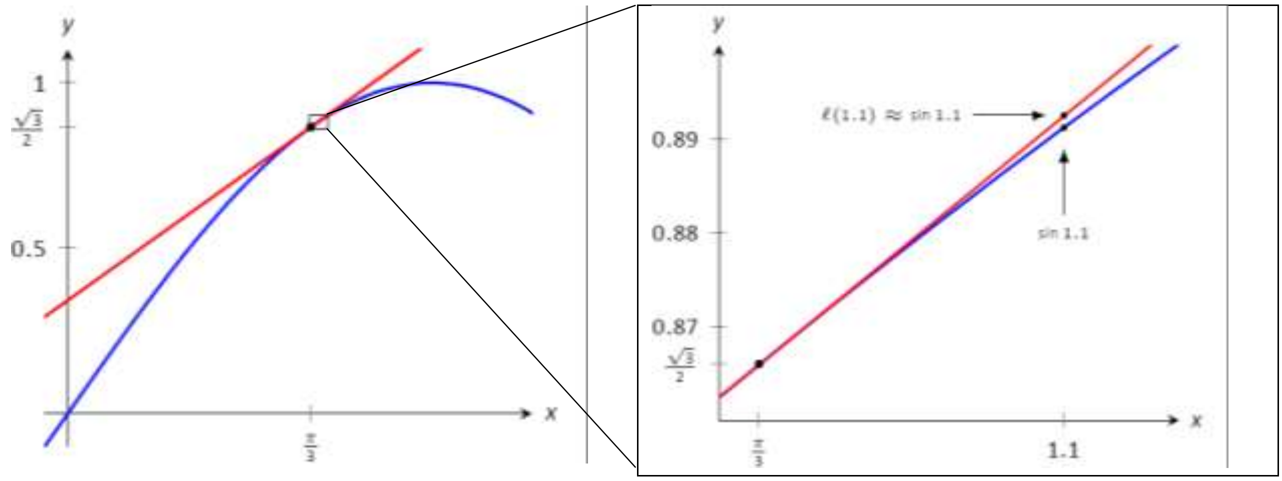
Recall that the derivative of a function $f(x)$ can be used to find the slopes of lines tangent to the graph of $f(x)$ at a point $x = a$. That is at $x = a$ , the tangent line to the graph of $f(x)$ is stated by the equation:

$$l(x) = f'(a)(x-a)+f(a)$$

In doing so, the tangent line can be used to find good approximations of $f(x)$ for values of $x$ near $a$.

For instance, we can approximate $\sin(1.1)$ using the tangent line to the graph of $f(x) = \sin x$ at $x = \frac{\pi}{3} \approx 1.05$. It is known that $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \approx 0.866$, and $f'(x) = \cos x$, then $f'(\frac{\pi}{3}) = cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ . Thus the tangent line to $f(x) = \sin x$ at $x = \frac{\pi}{3}$ is:

$$l(x) = f'\left(\frac{\pi}{3}\right)\left(x-\frac{\pi}{3}\right)+ f\left(\frac{\pi}{3}\right) \Rightarrow l(x) = \frac{1}{2}\left(x - \frac{\pi}{3}\right) + 0.866$$




(a) (b)

In Figure (a), we can see the plot of $f(x) = sinx$ graphed along with its tangent line at $x = \frac{\pi}{3} \approx 1.05$. The small rectangle shows the region that is zoomed in Figure (b). In this figure, we see how we are approximating $\sin(1.1)$ with the tangent line $l(x)$, evaluated at $x = 1.1$. Indeed, the two lines together show how close these values are. That is, using this tangent line to approximate $\sin(1.1)$ , we have:

$$\ell(1.1) = \frac{1}{2}(1.1 - \pi/3) + 0.866 = \frac{1}{2}(0.053) + 0.866 = 0.8925$$

We leave it for later to see how good of an approximation this is.

The linear approximation, or tangent line approximation is called the *linearization* of $f(x)$ at $x = a$

Now we generalize this concept:

Consider a function $f(x)$. At some point $x = a$, the tangent line is $\ell(x) = f'(a)(x - a) + f(a)$. Clearly, we can match $f(a) = \ell(a)$.

Let $\Delta x$ be a small number, representing a small change in $x$-value (an increment of $x$). We can assert that

$$f(a + \Delta x) \approx l(a + \Delta x)$$

since the tangent line to a function approximates the values of that function near $x = a$.

As the $x$-value changes from $a$ to $a + \Delta x$, the $y$-value of $f(x)$ changes from $f(a)$ to $f(a + \Delta x)$. We call this change of $y$-value as the increment of $\Delta y$. That is:

$$\Delta y = f(a + \Delta x) - f(a)$$

Replacing $f(a + \Delta x)$ with its tangent line approximation, we have

$$\Delta y \approx \ell(a + \Delta x) - f(a)$$
$$= f'(a)((a + \Delta x) - a) + f(a) - f(a) = f'(a)\Delta x$$

So,

$$\Delta y \approx f'(a)\Delta x$$

**Definition**: Let $y = f(x)$ be differentiable. The *differential* of $y(x)$, denoted by the symbol $dy$ , is

$$dy = df(x) = f'(x)\Delta x$$

Moreover, if we consider the function $y = x$, then its differential is $dy = dx = 1 \cdot \Delta x$, i.e. $dx = \Delta x$ the differential and the increment of $x$ are the same values.

Hence,

$$dy = df(x) = f'(x)dx$$

If follows from the formula:

$$f'(x) = \frac{dy}{dx}$$

**Note:** This is not the alternate notation for the derivative due to Leibniz, but again, it is one symbol and not a fraction.

Therefore, it is useful to understand that

$$\Delta y \approx dy$$

Thereby, we use differentials to approximate the value of a function. This technique can sometimes be used to easily compute something that looks rather hard. Indeed,

$$f(x + \Delta x) - f(x) \approx f'(x)\Delta x \Rightarrow f(x + \Delta x) \approx f'(x)\Delta x + f(x)$$

*Example.* Use the linear approximation of $f(x) = \sqrt{x}$ at $x = 9$ to estimate $\sqrt{9.1}$

Since we are looking for the linear approximation at $x = 9$, using Equation we know the linear approximation is given by

$$L(x) = f(9) + f'(9)(x - 9)$$

We need to find $f(9)$ and $f'(9)$

$$f(x) = \sqrt{x} \Rightarrow f(9) = \sqrt{9} = 3$$

$$f'(x) = \frac{1}{2\sqrt{x}} \Rightarrow f'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{6}$$

Therefore, the linear approximation is given by

$$L(x) = 3 + \frac{1}{6}(x - 9)$$

Using the linear approximation, we can estimate $\sqrt{9.1}$ by writing

$$\sqrt{9.1} = f(9.1) \approx L(9.1) = 3 + \frac{1}{6}(9.1 - 9) \approx 3.0167.$$

Any type of measurement is prone to a certain amount of error. Here we examine this type of error and study how differentials can be used to estimate the error.

Consider a function $y = f(x)$, which is continuous in the interval $[a, b]$, with an input that is a measured quantity. Suppose that at some point $x_0 \in [a, b]$ the independent variable is incremented by $\mathit{\Delta} x$. The increment of the function $\Delta y$ corresponding to the change of the independent variable $\Delta x$ is given by

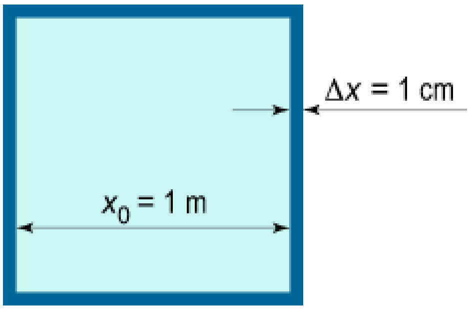
$$\Delta y = \Delta f(x_0) = f(x_0 + \Delta x) - f(x_0)$$

For any differentiable function, the increment $\Delta y$ can be represented as a sum of two terms:

$$\Delta y = A\Delta x + o(\Delta x) = f'(x_0)\Delta x + o(\Delta x)$$

where the first term (called the *principal part* of the increment) is linearly dependent on the increment $\Delta x$. The expression $A\Delta x$ is called the *differential of function* and is denoted by $dy$. The coefficient $A$ in the principal part of the increment of a function $\Delta y$ at a point $x_0$ is equal to the value of the derivative $f'(x_0)$ at this point. The second term has a higher order of smallness with respect to $\Delta x$. The latter is an error of the differential use to approximate the function increment.

Consider the idea of partition of the increment of the function $\Delta y$ into two parts in the following simple example. Given a square with side $x_0 = 1\text{m}$




Its area is obviously equal to

$$S_0 = x_0^2 = 1\text{m}^2.$$

If the side of the square is increased by $\Delta x = 1\text{cm}$, the exact value of the area of the square will be equal to

$$S = x^2 = (x_0 + \Delta x)^2 = 1{,}01^2 = 1{,}0201\text{m}^2,$$

that is the increment of the area $\Delta S$ is

$$\Delta S = S - S_0 = 1{,}0201 - 1 = 0{,}0201\text{m}^2 = 201\text{cm}^2.$$

We now represent this increment $\Delta S$ as follows:

$$\Delta S = S - S_0 = (x_0 + \Delta x)^2 - x_0^2 = \cancel{x_0^2} + 2x_0\Delta x + (\Delta x)^2 - \cancel{x_0^2}$$
$$= 2x_0\Delta x + (\Delta x)^2 = A\Delta x + o(\Delta x) = dy + o(\Delta x).$$

Thus, the increment $\Delta S$ consists of the principal part (the differential of the function), which is proportional to $\Delta x$ and is equal to

$$dy = A\Delta x = 2x_0\Delta x = 2 \cdot 1 \cdot 0{,}01 = 0{,}02\text{m}^2 = 200\text{cm}^2,$$

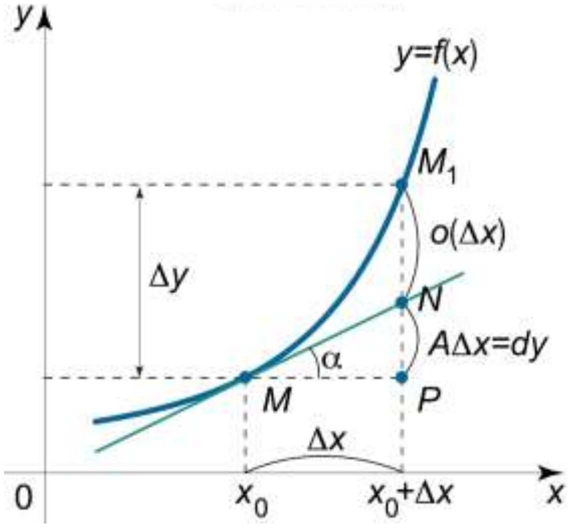
and the term of a higher order of smallness (the error), which in turn is equal to

$$o(\Delta x) = (\Delta x)^2 = 0,01^2 = 0{,}0001\text{m}^2 = 1\text{cm}^2.$$

In sum, both these terms comprise the full increment of the square area equal to $200 + 1 = 201\text{cm}^2$.

*Geometric Meaning of the Differential of a Function*

Figure schematically shows splitting of the increment $\Delta y$ into the principal part $dy$ (the differential of function) and the term of a higher order of smallness $o(\Delta x)$




The tangent $MN$ drawn to the curve of the function $y = f(x)$ at the point $M$ as it is known, has the slope angle $\alpha$, the tangent of which is equal to the derivative:

$$\tan\alpha = f'(x_0).$$

When the independent variable changes by $\Delta x$, the tangent line increments by $dy = A\Delta x$. This linear increment formed by the tangent is just the differential of the function. The remaining part of the full increment $\Delta y = MP$ (the segment $M_1N$) corresponds to the "nonlinear" additive of a higher order of smallness with respect to $\Delta x$.

*Properties of the Differential*

The differential has the following properties:

1. A constant can be taken out of the differential sign:
$$d(Cu) = Cdu,$$
where $C$ is a constant number.
2. The differential of the sum (difference) of two functions is equal to the sum (difference) of their differentials:
$$d(u \pm v) = du \pm dv,$$
where $u$ and $v$ be functions of the variable $x$.
3. The differential of a constant is zero:
$$d(C) = 0.$$
4. Differential of the product of two functions:
$$d(uv) = du \cdot v + u \cdot dv.$$
5. Differential of the quotient of two functions:
$$d\left(\frac{u}{v}\right) = \frac{du \cdot v - u \cdot dv}{v^2}.$$

*Form Invariance of the Differential*

Consider a composition of two functions $y = f(u)$ and $u = g(x)$. Its derivative can be found by the chain rule:
$$y'_x = y'_u \cdot u'_x,$$
where the subindex denotes the variable of differentiation.

The differential of the "outer" function $y = f(u)$ can be written as
$$dy = y'_u du.$$
The differential of the "inner" function $u = g(x)$ can be represented in a similar manner:
$$du = u'_x dx.$$
If we substitute $du$ in the last formula, we obtain
$$dy = y'_u du = y'_u u'_x dx.$$
Since $y'_x = y'_u \cdot u'_x$, then
$$dy = y'_x dx.$$
It can be seen that in the case of a composite function, we get an expression for the differential in the same form as for a "simple" function. This property is called the *form invariance of the differential*.

*Example* 1: Find the differential of the function $y = \sin x - x\cos x$.
Determine the derivative of the given function:

$$y' = (\sin x - x\cos x)' = \cos x - (x'\cos x + x(\cos x)')$$
$$= \cos x - (\cos x + x(-\sin x)) = \cancel{\cos x} - \cancel{\cos x} + x\sin x$$
$$= x\sin x.$$

The differential has the following form:

$$dy = y'dx = x\sin x dx.$$

*Example* 2: Find the differential of the function $y = 2x^2 + 3x + 1$ at the point $x = 1$ when $dx = 0{,}1$.

$$dy = f'(x)dx = (2x^2 + 3x + 1)'dx = (4x + 3)dx.$$

Substituting the given values, we calculate the differential:

$$dy = (4 \cdot 1 + 3) \cdot 0{,}1 = 0{,}7$$

*Example* 3: Use differential to approximate the change in $y = x^3 + x^2$ as $x$ changes from 1 to 0.95.
The differential $dy$ is defined by the formula

$$dy = y'dx = y'(1)dx.$$

Take the derivative

$$y' = (x^3 + x^2)' = 3x^2 + 2x,$$

So,

$$y'(1) = 3 \cdot 1^2 + 2 \cdot 1 = 5.$$

Calculate the differential $dx$:

$$dx = \Delta x = 0{,}95 - 1 = -0{,}05.$$

Hence,

$$dy = y'(1)dx = 5 \cdot (-0{,}05) = -0{,}25.$$

The approximate value of the function at $x = 0{,}95$ is

$$y(0{,}95) \approx y(1) + dy = (1^3 + 1^2) - 0{,}25 = 1{,}75.$$