Lecture #22: Higher-Order Derivatives

22.1 Higher-Order Derivatives of an Explicit Function

Let the function $y = f(x)$ have a finite derivative $f'(x)$ in a certain interval (a, b), i.e. the derivative $f'(x)$ is also a function in this interval. If this function is differentiable, we can find *the second derivative* of the original function $y = f(x)$, which is denoted by various notations as

$$f'' = (f')' = (\frac{dy}{dx})' = \frac{d}{dx}(\frac{dy}{dx}) = \frac{d^2y}{dx^2}.$$

So, *the second derivative* (or *the second order derivative*) of the function $f(x)$ may be denoted as

$$\frac{d^2f}{dx^2} \quad \text{or} \quad \frac{d^2y}{dx^2} \quad \text{(Leibniz's notation)}$$

$$f''(x) \quad \text{or} \quad y''(x) \quad \text{(Lagrange's notation)}$$

For example, if $y = x^5$, then

$$y' = 5x^4,\ y'' = \left(5x^4\right)' = 20x^3.$$

Similarly, if f'' exists and is differentiable, we can calculate *the third derivative* of the function $f(x)$:

$$f''' = \frac{d^3y}{dx^3} = y'''.$$

The result of taking the derivative n times is called *the n-th derivative* of $f(x)$ with respect to x and is denoted as

$$\frac{d^nf}{dx^n} = \frac{d^ny}{dx^n} \qquad \text{(in Leibnitz's notation),}$$

$$f^{(n)}(x) = y^{(n)}(x) \qquad \text{(in Lagrange's notation).}$$

Thus, the notion of the n-th order derivative is introduced inductively by sequential calculation of n derivatives starting from the first order derivative. Transition to the next higher-order derivative is performed using the recurrence formula

$$y^{(n)} = (y^{(n-1)})'$$

Notice: The order of the derivative is taken in parentheses so as to avoid confusion with the exponent of a power.

Also, derivatives of the fourth, fifth and higher orders are also denoted by Roman numerals: y^{IV}, y^{V}, y^{VI}, … . Herein, the order of the derivative may be written without brackets.

For instance, if $y = x^5$, then

$y' = 5x^4$,

$y'' = 20x^3$,

$y''' = 60x^2$,

$y^{IV} = y^{(4)} = 120x$,

$y^{V} = y^{(5)} = 120$,

$y^{(6)} = y^{(7)} = ... = 0$.

In some cases, we can derive a general formula for the derivative of an arbitrary *n*th order without computing intermediate derivatives. Some examples are considered below.

Basic functions:

1. Let's consider the a function $y = e^{kx}$ ($k = const$). The expression of its derivative of any order n is calculated as follows

$$y' = ke^{kx},\ y'' = k^2e^{kx},\ \ldots,\ y^{(n)} = k^n e^{kx}$$

So, the general formula for the derivative of an arbitrary nth order without computing intermediate derivatives is

$$y^{(n)} = k^n e^{kx}$$

2. Consider the function $y = \sin x$. Then

$$y' = \cos x = \sin\left(x + \frac{\pi}{2}\right);$$

$$y'' = -\sin x = \sin\left(x + 2\frac{\pi}{2}\right);$$

$$y''' = -\cos x = \sin\left(x + 3\frac{\pi}{2}\right);$$

$$y^{IV} = \sin x = \sin\left(x + 4\frac{\pi}{2}\right);$$

....

$$y^{(n)} = \sin\left(x + n\frac{\pi}{2}\right).$$

In similar manner we can also get the formulas for the derivatives of any order of the other elementary functions.

3. $y = x^k$,

$$(x^k)^{(n)} = \begin{cases} \frac{k!}{n!}x^{k-n}, n \le k \\ 0, \qquad n > k \end{cases}$$

4. $y = \cos x$,

$$(\cos x)^{(n)} = \cos\left(x + n\frac{\pi}{2}\right)$$

5. $y = \ln x$,

$$(\ln x)^{(n)} = \frac{(-1)^{n-1}(n-1)!}{x^n}$$

The derivatives of constant multiplication and sum

The following linear relationships can be used for finding higher-order derivatives:

$$(u+v)^{(n)} = u^{(n)} + v^{(n)},$$

$$(Cu)^{(n)} = Cu^{(n)}, C = \text{const}$$

Leibniz Formula

The Leibniz formula expresses the derivative on nth order of the product of two functions. Suppose that the functions $u\,(x)$ and $v\,(x)$ have the derivatives up to nth order. Consider the derivative of the product of these functions.

The first derivative is described by the well known formula:

$$(uv)' = u'v + uv'.$$

Differentiating this expression again yields the second derivative:

$$(uv)'' = [(uv)']' = (u'v + uv')' = (u'v)' + (uv')'$$
$$= u''v + u'v' + u'v' + uv'' = u''v + 2u'v' + uv''.$$

Likewise, we can find the third derivative of the product uv:

$$(uv)''' = [(uv)'']' = (u''v + 2u'v' + uv'')'$$
$$= (u''v)' + (2u'v')' + (uv'')'$$
$$= u'''v + u''v' + 2u''v' + 2u'v'' + u'v'' + uv'''$$
$$= u'''v + 3u''v' + 3u'v'' + uv'''.$$

It is easy to see that these formulas are similar to the binomial expansion raised to the appropriate exponent. Assuming that the terms with zero exponent u^0 and v^0 correspond to the functions u and v themselves, we can write the general formula for the derivative of th order of the product of functions uv as follows:

$$(uv)^{(n)} = \sum_{i=0}^{n} C_i^n u^{(n-i)} v^{(i)},$$

where C_i^n denotes the number of *i*-combinations of *n* elements, $C_i^n = \frac{n!}{i!(n-i)!}$

This formula is called *the Leibniz formula*.

Example 1. Find the 4-th derivative of the function $y = e^x \sin x$.

Let $u = \sin x$, $v = e^x$. Using the Leibniz formula, we can write

$$y^{(4)} = (e^x \sin x)^{(4)} = \sum_{i=0}^{4} C_i^4 u^{(4-i)} v^{(i)} = \sum_{i=0}^{4} C_i^4 (\sin x)^{(4-i)} (e^x)^{(i)}$$
$$= C_0^4 (\sin x)^{(4)} e^x + C_1^4 (\sin x)''' (e^x)' + C_2^4 (\sin x)'' (e^x)''$$
$$+ C_3^4 (\sin x)' (e^x)''' + C_4^4 (\sin x)(e^x)^{(4)}$$
$$= 1 \cdot \sin x \cdot e^x + \cancel{4 \cdot (-\cos x) \cdot e^x} + 6 \cdot (-\sin x) \cdot e^x + \cancel{4 \cdot \cos x \cdot e^x}$$
$$+ 1 \cdot \sin x \cdot e^x = -4 e^x \sin x.$$

Example 2. Find the 3-rd derivative of the function $y = x\sin x$. Let $u = x$, $v = \sin x$. By the Leibniz formula, we can write:

$$y''' = \sum_{i=0}^{3} C_i^3 u^{(3-i)} v^{(i)} = \sum_{i=0}^{3} C_i^3 (\sin x)^{(3-i)} x^{(i)}.$$

It is clear that

$$x' = 1, x'' = x''' \equiv 0.$$

Then the series expansion has only two terms:

$$y''' = C_0^3 (\sin x)''' x + C_1^3 (\sin x)'' x'.$$

Calculating the derivatives, we obtain

$$y''' = 1 \cdot (-\cos x) \cdot x + 3 \cdot (-\sin x) \cdot 1 = -x\cos x - 3\sin x.$$

Example 3. Find the third derivative of the function: $y = e^{2x}\ln x$. Let $u = e^{2x}$, $v = \ln x$. By the Leibniz formula, we can write:

$$u' = (e^{2x})' = 2e^{2x}, u'' = (2e^{2x})' = 4e^{2x}, u''' = (4e^{2x})' = 8e^{2x},$$

$$v' = (\ln x)' = \frac{1}{x}, v'' = (\frac{1}{x})' = -\frac{1}{x^2}, v''' = (-\frac{1}{x^2})' = -(x^{-2})' = 2x^{-3} = \frac{2}{x^3}.$$

The third-order derivative of the original function is given by the Leibniz rule:

$$y''' = (e^{2x}\ln x)''' = \sum_{i=0}^{3} C_i^3 u^{(3-i)} v^{(i)} = \sum_{i=0}^{3} C_i^3 (e^{2x})^{(3-i)} (\ln x)^{(i)}$$

$$= C_0^3 \cdot 8e^{2x}\ln x + C_1^3 \cdot 4e^{2x} \cdot \frac{1}{x} + C_2^3 \cdot 2e^{2x} \cdot (-\frac{1}{x^2}) + C_3^3 e^{2x} \cdot \frac{2}{x^3}$$

$$= 1 \cdot 8e^{2x}\ln x + 3 \cdot \frac{4e^{2x}}{x} - 3 \cdot \frac{2e^{2x}}{x^2} + 1 \cdot \frac{2e^{2x}}{x^3}$$

$$= 8e^{2x}\ln x + \frac{12e^{2x}}{x} - \frac{6e^{2x}}{x^2} + \frac{2e^{2x}}{x^3}$$

$$= 2e^{2x} \cdot (4\ln x + \frac{6}{x} - \frac{3}{x^2} + \frac{1}{x^3}).$$

22.2 Concept of Higher-Order Differentials

We consider a function $y = f(x)$, which is differentiable in the interval (a, b). The first-order differential of the function at the point $x \in (a, b)$ is defined by the formula

$$dy = f'(x)dx.$$

It can be seen that the differential dy depends on two quantities - the variable x (through the derivative $y' = f'(x)$) and the differential of the independent variable dx.

Let us fix the increment dx, i.e. we assume that dx is constant. Then the differential dy becomes a function only of the variable x for which we can also define the differential by taking the same differential dx as the increment Δx. As a result, we obtain *the second differential* or *differential of the second order*, which is denoted as d^2y or $d^2f(x)$. Thus, by definition:

$$d^2y = d(dy) = d[f'(x)dx] = df'(x)dx = f''(x)dxdx = f''(x)(dx)^2.$$

It is commonly denoted $(dx)^2 = dx^2$. Therefore, we get:

$$d^2y = f''(x)dx^2.$$

In the same way, we can establish that *the third differential* or *differential of the third order* has the form

$$d^3y = f'''(x)dx^3.$$

In the general case, the differential of an arbitrary order *n* is given by

$$d^ny = f^{(n)}(x)dx^n,$$

which can be rigorously proved by mathematical induction. This formula leads in particular to the following expression for the *n*th order derivative:

$$f^{(n)}(x) = \frac{d^ny}{dx^n}.$$

Note that for the linear function $y = ax + b,$ the second and subsequent higher-order differentials are zero. Indeed,

$$d^2(ax + b) = (ax + b)''dx^2 = 0 \cdot dx^2 = 0, \dots, d^n(ax + b) = 0.$$

In this case, it is obvious that

$$d^nx = 0 \quad \text{for} \quad n > 1.$$

Properties of Higher-Order Differentials

Let the functions u and v have the n-th order derivatives and α and β are arbitrary constants. Then the following properties are valid:

$$d^n(\alpha u + \beta v) = \alpha d^n u + \beta d^n v;$$

$$d^n(uv) = \sum_{i=0}^{n} C_n^i d^{n-i} u d^i v.$$

The last equality follows directly from the Leibniz formula.

Higher Order Differential of a Composite Function

Consider now the composition of two functions such that $y = f(u)$ and $u = g(x)$. In this case, y is a composite function of the independent variable x:

$$y = f(g(x)).$$

The first differential of y can be written as

$$dy = [f(g(x))]' dx = f'(g(x))g'(x)dx.$$

Compute the second differential d^2y (assuming dx is constant by definition). Using the product rule, we obtain:

$$\begin{aligned} d^2y &= [f'(g(x))g'(x)]' dx^2 \\ &= [f''(g(x))(g'(x))^2 + f'(g(x))g''(x)]dx^2 \\ &= f''(g(x))(g'(x)dx)^2 + f'(g(x))g''(x)dx^2. \end{aligned}$$

Take into account that

$$g'(x)dx = du \quad \text{and} \quad g''(x)dx^2 = d^2u.$$

Consequently,

$$d^2y = f''(u)du^2 + f'(u)d^2u$$

or in short form:

$$d^2y = y''du^2 + y'd^2u.$$

In the same way, we can obtain the expression for the third order differential of a composite function:

$$d^3y = f'''(u)du^3 + 3f''(u)dud^2u + f'(u)d^3u.$$

It follows from the above that the higher order differentials d^2y, d^3y, …,

$d^n y$ are generally not invariant.

Example 1. Find the differential $d^4 y$ of the function $y = x^5$.
The 4-th order differential is given by

$$d^4 y = f^{(4)}(x)dx^4 = (x^5)^{(4)}dx^4.$$

We find the fourth derivative of this function by successive differentiation:

$$(x^5)' = 5x^4,$$

$$(x^5)'' = (5x^4)' = 20x^3,$$

$$(x^5)''' = (20x^3)' = 60x^2,$$

$$(x^5)^{(4)} = (60x^2)' = 120x.$$

Hence,

$$d^4 y = 120x dx^4.$$

Example 2. Find the differential $d^5 y$ of the function $y = \sin 2x$.
It is known that the *n*th-order derivative of the sine function has the form

$$(\sin x)^{(n)} = \sin(x + \frac{\pi n}{2}).$$

One can show that the *n*th-order derivative of the function $y = \sin 2x$. is given by

$$y^{(n)} = (\sin 2x)^{(n)} = 2^n \sin(2x + \frac{\pi n}{2}).$$

Hence, the 5-th-order derivative is written as

$$y^{(5)} = (\sin 2x)^{(5)} = 2^5 \sin(2x + \frac{5\pi}{2}) = 32\sin(2x + 2\pi + \frac{\pi}{2})$$

$$= 32\sin(2x + \frac{\pi}{2}) = 32\cos 2x.$$

Hence,

$$d^5 y = 32\cos 2x dx^5.$$

Example 3. Find the second differential of the function $y = x^2 \cos 2x$.
Determine the second derivative of this function:

$$y' = (x^2\cos 2x)' = (x^2)'\cos 2x + x^2(\cos 2x)'$$
$$= 2x\cos 2x + x^2 \cdot (-2\sin 2x) = 2x\cos 2x - 2x^2\sin 2x,$$
$$y'' = (2x\cos 2x - 2x^2\sin 2x)' = 2(x\cos 2x - x^2\sin 2x)'$$
$$= 2[x'\cos 2x + x(\cos 2x)'(x^2)'\sin 2x - x^2(\sin 2x)']$$
$$= 2[\cos 2x - 2x\sin 2x - 2x\sin 2x - 2x^2\cos 2x]$$
$$= (2 - 2x^2)\cos 2x - 4x\sin 2x.$$

Then the second-order differential is written in the form:

$$d^2y = y''dx^2 = [(2 - 2x^2)\cos 2x - 4x\sin 2x]dx^2.$$

22.3 Higher-Order Derivatives of a Parametric Function

Consider a function $y = f(x)$ given parametrically by the equations

$$\begin{cases} x &= x(t) \\ y &= y(t) \end{cases}.$$

The first derivative of this function is given by

$$y' = y'_x = \frac{y'_t}{x'_t}.$$

Differentiating once more with respect to x we find the second derivative:

$$y'' = y''_{xx} = \frac{(y'_x)'_t}{x'_t}.$$

Herein

$$(y'_x)'_t = \left(\frac{y'_t}{x'_t}\right)'_t = \frac{(y'_t)'_t \cdot x'_t - y'_t \cdot (x'_t)'_t}{(x'_t)^2} = \frac{y''_{tt} \cdot x'_t - y'_t \cdot x''_{tt}}{(x'_t)^2}$$

Finally,

$$y'' = y''_{xx} = \frac{y''_{tt} \cdot x'_t - y'_t \cdot x''_{tt}}{(x'_t)^2}$$

Similarly, we define the derivatives of the third and higher order:

$$y''' = y'''_{xxx} = \frac{(y''_{xx})'_t}{x'_t}, \dots,.$$

$$y^{(n)} = y^{(n)}_{\underbrace{xx\dots x}_{n}} = \frac{\left(y^{(n-1)}_{\underbrace{xx\dots x}_{n-1}}\right)'_t}{x'_t},$$

where $(y''_{xx})'_t = \left(\frac{y''_{tt}\cdot x'_t - y'_t\cdot x''_{tt}}{(x'_t)^2}\right)'_t$, … ,

Example 1. The function $y = f(x)$ is given in parametric form by the equations

$$x = t + \cos t, y = 1 + \sin t,$$

where $t \in (0,2\pi)$. Find y''_{xx}.

Taking the first derivative of the parametric function, we have

$$y'_x = \frac{y'_t}{x'_t} = \frac{(1+\sin t)'_t}{(t+\cos t)'_t} = \frac{\cos t}{1-\sin t}.$$

Now we differentiate both sides of the expression for y'_x with respect to x This yields:

$$y''_{xx} = (y'_x)'_x = (y'_x)'_t \cdot t'_x = (\frac{\cos t}{1-\sin t})'_t \cdot t'_x = (\frac{\cos t}{1-\sin t})'_t \cdot \frac{1}{x'_t}$$

$$= \frac{(-\sin t)(1-\sin t) - \cos t(-\cos t)}{(1-\sin t)^2} \cdot \frac{1}{1-\sin t}$$

$$= \frac{-\sin t + \sin^2 t + \cos^2 t}{(1-\sin t)^3} = \frac{1-\sin t}{(1-\sin t)^3} = \frac{1}{(1-\sin t)^2}.$$

Example 2. Find the 3-rd derivative of the function given by the parametric equations

$$x = 1 + t^2, y = t - t^3$$

at $t = 1$

Take the first derivative:

$$y' = y'_x = \frac{y'_t}{x'_t} = \frac{(t-t^3)'}{(1+t^2)'} = \frac{1-3t^2}{2t} = \frac{1}{2t} - \frac{3t}{2}.$$

Continue differentiating:

$$y'' = y''_{xx} = \frac{(y'_x)'_t}{x'_t} = \frac{(\frac{1}{2t} - \frac{3t}{2})'}{(1+t^2)'} = \frac{\frac{1}{2}\cdot(-t^{-2}) - \frac{3}{2}}{2t} = -\frac{\frac{1}{2t^2} + \frac{3}{2}}{2t}$$

$$= -\frac{1}{4t^3} - \frac{3}{4t}.$$

Similarly we calculate the third derivative y'''_{xxx}:

$$y''' = y'''_{xxx} = \frac{(y''_{xx})'_t}{x'_t} = \frac{(-\frac{1}{4t^3} - \frac{3}{4t})'}{(1+t^2)'} = \frac{-\frac{1}{4}\cdot(-3t^{-4}) - \frac{3}{4}\cdot(-t^{-2})}{2t}$$

$$= \frac{\frac{3}{4t^4} + \frac{3}{4t^2}}{2t} = \frac{3}{8t^5} + \frac{3}{8t^3}.$$

At the point where $t = 1$ the third derivative is equal to

$$y'''_{xxx}(t=1) = \frac{3}{8\cdot 1^5} + \frac{3}{8\cdot 1^3} = \frac{3}{8} + \frac{3}{8} = \frac{3}{4}.$$

22.4 Higher-Order Derivatives of an Implicit Function

The *n*-th order derivative of an implicit function can be found by sequential (*n* times) differentiation of the equation $F(x,y) = 0$. At each step, after appropriate substitutions and transformations, we can obtain an explicit expression for the derivative, which depends only on the variables x and y, i.e. the derivatives have the form

$$y' = f_1(x,y), y'' = f_2(x,y), \dots, y^{(n)} = f_n(x,y).$$

Notice. We need to remember that it is assuming that *y* is a differentiable function of *x* and the chain rule for its differentiation should be used.

Example 1. Find the second derivative of the function given by the equation $x + y = e^{x-y}$.

Differentiating both sides in *x* we obtain:

$$(x+y)' = (e^{x-y})', \Rightarrow$$

$$1 + y' = e^{x-y}\cdot(x-y)', \Rightarrow$$

$$1 + y' = e^{x-y}(1-y') = e^{x-y} - e^{x-y}y', \Rightarrow$$

$$y' + e^{x-y}y' = e^{x-y} - 1, \Rightarrow$$

$$y' = \frac{e^{x-y} - 1}{e^{x-y} + 1}$$

Continuing the differentiation, we find the second derivative:

$$y'' = \left(\frac{e^{x-y} - 1}{e^{x-y} + 1}\right)' = \frac{2e^{x-y}(1 - y')}{(e^{x-y} + 1)^2}.$$

Substitute the expression for the first derivative:

$$y'' = \frac{2e^{x-y}(1 - y')}{(e^{x-y} + 1)^2} = \frac{2e^{x-y}\left(1 - \frac{e^{x-y} - 1}{e^{x-y} + 1}\right)}{(e^{x-y} + 1)^2} =$$

$$\frac{2e^{x-y} \cdot \frac{\cancel{e^{x-y}} + 1 - \cancel{e^{x-y}} + 1}{e^{x-y} + 1}}{(e^{x-y} + 1)^2} = \frac{4e^{x-y}}{(e^{x-y} + 1)^3}.$$

We now use the original equation, according to which

$$e^{x-y} = x + y.$$

As a result, we obtain the following expression for the derivative y''

$$y'' = \frac{4e^{x-y}}{(e^{x-y} + 1)^3} = \frac{4(x + y)}{(x + y + 1)^3}.$$

Example 2. Find the third derivative of the function given by the equation $x^2 - y^2 = 9$.

We differentiate both sides of the equation with respect to *x* keeping in mind that *y* is a function of *x*

$$(x^2 - y^2)' = 9', \Rightarrow 2x - 2yy' = 0, \Rightarrow x - yy' = 0, \Rightarrow y' = y'_x = \frac{x}{y}.$$

Continue differentiating to obtain y''_{xx}:

$$x - yy' = 0, \Rightarrow (x - yy')' = 0, \Rightarrow$$

$$1 - y'y' - yy'' = 0, \Rightarrow$$

$$yy'' = 1 - (y')^2, \Rightarrow$$

$$y'' = y''_{xx} = \frac{1 - (y')^2}{y} = \frac{1 - \left(\frac{x}{y}\right)^2}{y} = \frac{1 - \frac{x^2}{y^2}}{y} = \frac{y^2 - x^2}{y^3} = -\frac{x^2 - y^2}{y^3}$$

$$= -\frac{9}{y^3}.$$

Similarly we find the third derivative:

$$yy'' = 1 - (y')^2, \Rightarrow (yy'')' = (1 - (y')^2)', \Rightarrow$$

$$y'y'' + yy''' = -2y'y'', \Rightarrow$$

$$yy''' = -3y'y'', \Rightarrow$$

$$y''' = y'''_{xxx} = -\frac{3y'y''}{y} = -\frac{3 \cdot \frac{x}{y} \cdot (-\frac{9}{y^3})}{y} = \frac{\frac{27x}{y^4}}{y} = \frac{27x}{y^5}.$$