Lecture #23: The Mean-Value Theorem of Differential Calculus

23.1 The Mean-Value Theorem (MVT)

The Mean Value Theorem is one of the most important theorems in calculus.

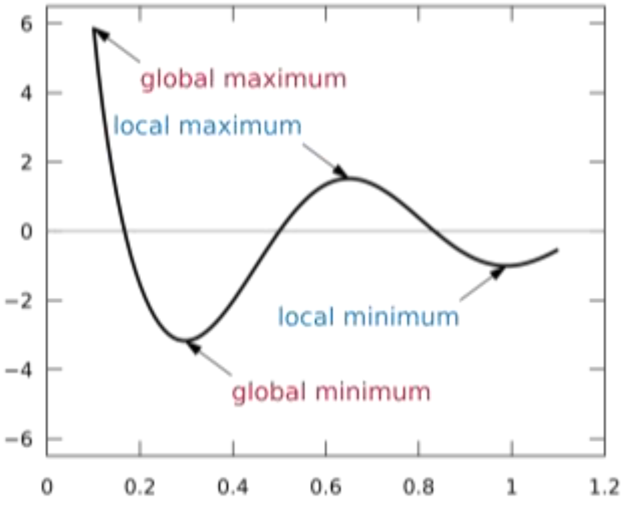
23.1.1 Fermat's theorem.

Let's start with Fermat's theorem (also known as *interior extremum theorem*). In essence, it is a method to find *local extremum* (i.e. *local maximum* or *minimum*) of differentiable functions on open sets by showing that every local extremum of the function is a stationary point (the function's derivative is zero at that point).

Definition: A function $f(x)$ has a *local maximum* at x_0 if there exists an open interval (a, b) containing x_0 such that (a, b) is contained in the domain of $f(x)$ and $f(x_0) \geq f(x)$ for all $x \in (a, b)$.

A function $f(x)$ has a *local minimum* at x_0 if there exists an open interval (a, b) containing x_0 such that (a, b) is contained in the domain of $f(x)$ and $f(x_0) \leq f(x)$ for all $x \in (a, b)$.

A function $f(x)$ has a *local extremum* at x_0 if $f(x)$ has a local maximum at x_0 or $f(x)$ has a local minimum at x_0.



We need to distinguish the *global* or *absolute extrema* (i.e. *global maximum* or *minimum*) of $f(x)$ which are the greatest (or the smallest) values the function $f(x)$ in the given domain of $f(x)$ and *local extrema* (i.e. *local* *maximum* or *minimum*) of $f(x)$ which are the local greatest (or the smallest) values the function $f(x)$ at a given point of the domain of $f(x)$.

Theorem. Let a function $f(x)$ be defined on a closed interval $[a, b]$ and it takes the local greatest (or the smallest) value at an interior point x_0 of the open interval (a, b). Then if the function is differentiable at this point its derivative vanishes at this point, i.e.

$$f'(x_0) = 0.$$

■ For the sake of definiteness let us assume that the function $f(x)$ takes the greatest value at the point x_0. Then the ratio

$$\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

has a numerator which is always negative value (of course, if $(x_0 + \Delta x) \in [a, b]$). Therefore, if $\Delta x > 0$, then

$$\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} < 0.$$

Whence the limit of the negative variable is not a positive value, that is

$$\lim_{\Delta x \to +0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \le 0. \tag{1}$$

Hence

$$f'(x_0) \le 0. \tag{2}$$

If $\Delta x < 0$, then

$$\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} > 0$$

Hence

$$\lim_{\Delta x \to -0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \ge 0, \tag{3}$$

that is

$$f'(x_0) \ge 0^{*)}. \tag{4}$$

We came to the situation that the relations (2) and (4) hold true

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*) The passing from (1) to (2) and from (3) to (4) may be carried out because the function $f(x)$ is differentiable at the point x_0.

simultaneously if and only if $f'(x_0)=0$. □

The proved theorem geometrically means that the line tangent to graph of function $f(x)$ is parallel to axis Ox at the point x_0 (Fig. 1).

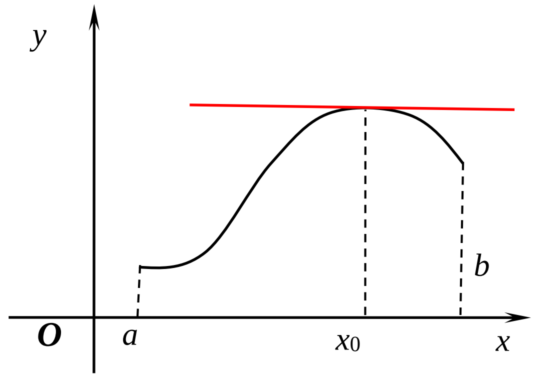


Fig. 1

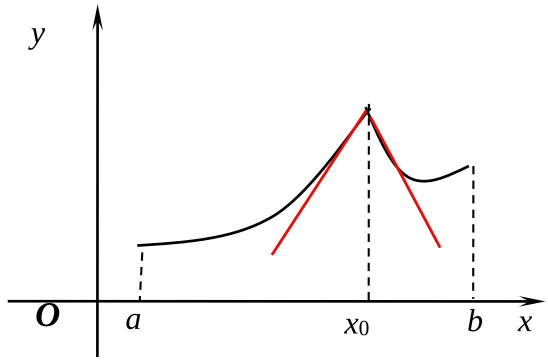


Fig. 2

Notice: If function $f(x)$ is not differentiable at the point x_0, then conditions of Fermat's theorem are not fulfilled and its assertion is not true (Fig. 2).

Definition: Let x_0 be an interior point in the domain of $f(x)$. We say that x_0 is a *critical point* of $f(x)$ if x_0 is a stationary point, i.e. $f'(x_0) = 0$ or $f'(x_0)$ is undefined.

23.1.2 Rolle's theorem

Before formulation of the MVT we start with a special case of the Mean Value Theorem, called Rolle's theorem.

Theorem. Let a function $f(x)$ be:

1) continuous on the closed interval $[a,b]$;
2) differentiable at all interior points of the open interval (a,b);
3) such that it takes the same value at the end points $f(a)=f(b)$.

Then inside of the interval there exists at least one point at which the derivative of the function vanishes. That is $f'(\xi)=0$.

■ Since the function is continuous on the closed interval $[a,b]$ it has a maximum M and a minimum m values on that interval I accordance with Weierstrass' Theorem.

If $M = m$ the function $f(x)$ is constant, that means that for all values of x of the segment $[a,b]$, it has a constant value $f(x) = m$ (Fig. 3). As a consequence of this $f'(x) \equiv 0$ at any point $\forall x \in [a,b]$ and the theorem is proved.

Suppose that $M \neq m$. Since $f(a) = f(b)$, then the function takes on that value at least one of these values at the interior point $\xi \in (a,b)$ (Fig.°4). Then due to theorem by Fermat $f'(\xi) = 0$. □

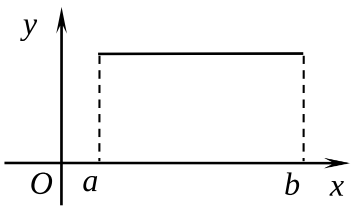


Fig. 3

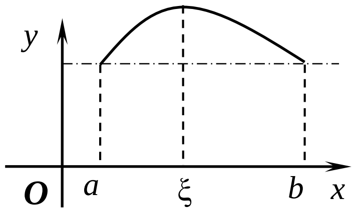


Fig. 4

Note 1. If function $f(x)$ is continuous in the interval (a,b) (open interval), but it is not continuous on the closed interval, the segment $[a,b]$, see Fig. 5, then one of conditions for the theorem is not fulfilled and the assertion of this theorem might be not fulfilled also. Thus, the condition to be continuous on the closed interval is *necessary requirement* of Rolle's theorem.

Note 2. On the contrary the requirement of the differentiability of the function on closed interval is *optional*, because the points A and B may be those at which $x = a$ and $x = b$ the function is not differentiable, (Fig. 6).

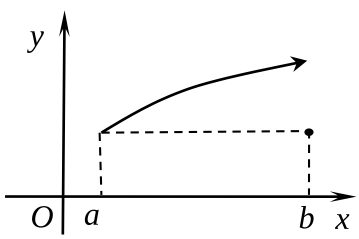


Fig. 5

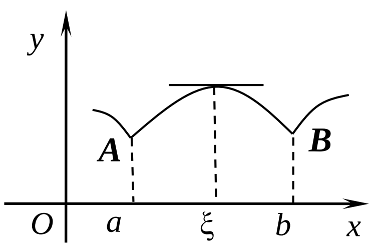


Fig. 6

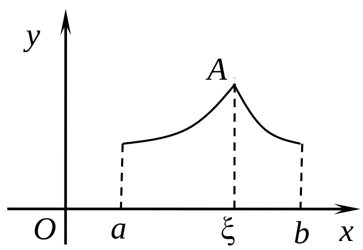


Fig. 7

Note 3. But at all interior points of the closed interval $[a,b]$ function $f(x)$ has to been differentiable otherwise the point A may be singular point (for example see graph 7) and point ξ at which $f'(\xi) = 0$ might not exist.

Note 4. It is obvious that points like to ξ may be several ones (Fig.8).
Note 5. Let $f(a)=f(b)=0$ (in this case the numbers *a* and *b* are called the *roots of the functions* $f(x)$).
Then this particular case follows from Rolle's theorem (Fig. 9).

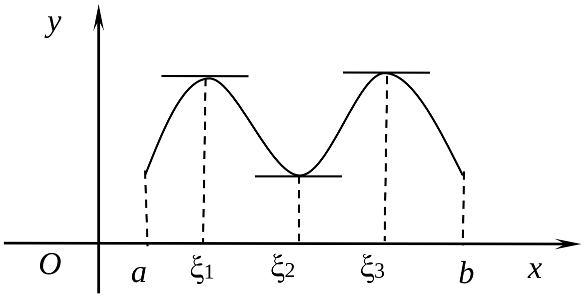


Fig. 8

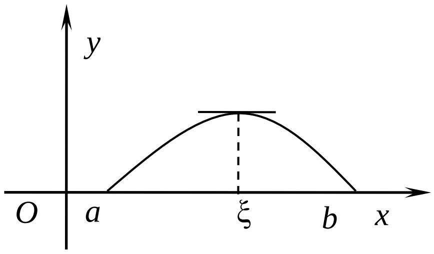


Fig. 9

Theorem*. Between two roots of the differentiable function there is at least one root of its derivative.

Example. Let function be given $f(x)=x(x-1)(x-2)(x-3)$. It is polynomial of the fourth power, which has the following roots $x_1=0,\ x_2=1,\ x_3=2,\ x_4=3$. Then derivative $f'(x)$, that is polynomial of the third power, has three roots. By virtue of theorem 2^* all roots are real and lie (only one) on intervals $(0,1)$, $(1,2)$ and $(2,3)$.

23.1.3 Mean-Value Theorem (or Lagrange's theorem)

Theorem. Let a function $f(x)$ be:

1) continuous on the closed interval $[a,b]$;
2) differentiable at all interior points of the open interval (a,b).

Then there will be, at least one point ξ $(a<\xi<b)$ within $[a,b]$, such that the following equality

$$\frac{f(b)-f(a)}{b-a}=f'(\xi)$$

is valid.

■ To prove this theorem let us introduce into consideration the auxiliary function

$$F(x)=f(x)-\lambda x.$$

Here λ is some a real number. Let us choose λ so that equality $F(a)=F(b)$

holds true.

It means that

$$f(a)-\lambda a = f(b)-\lambda b.$$

Whence

$$\lambda = \frac{f(b)-f(a)}{b-a}.$$

So, the function $F(x)=f(x)-\frac{f(b)-f(a)}{b-a}x$ satisfies conditions of Rolle's theorem. But it means that there is at least one point $\xi \in (a,b)$, such that the following equality $F'(\xi)=0$ is true. Or

$$f'(\xi)-\frac{f(b)-f(a)}{b-a}=0,$$

hence

$$f'(\xi)=\frac{f(b)-f(a)}{b-a}. \qquad (5)$$

The theorem is thus proved. □

In geometric sense this theorem means that there is at least one point M on curve $\overset{\cup}{AB}$ at which the line tangent to the graph of function $f(x)$ is parallel to chord AB (Fig. 10).

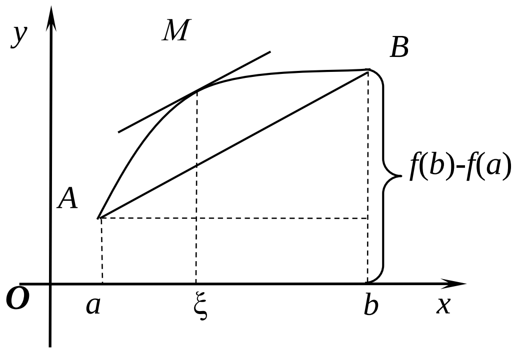


Fig. 10

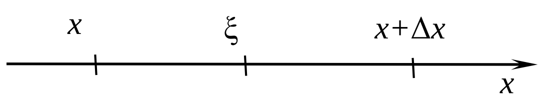


Fig. 11

Note 1. If $f(b)=f(a)$ then from (5) it follows that $f'(\xi)=0$. It means that mean-value (or Lagrange's) theorem is reduced to Rolle's theorem.

Note 2. Let function $f(x)$ be differentiable one. Let us add to x increment Δx and apply the mean-value (or Lagrange's) theorem on the closed interval $[x, x+\Delta x]$ (Fig. 11). Then we obtain equality

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}=f'(\xi),$$

or

$$\frac{\Delta y}{\Delta x}=f'(\xi),$$

whence

$$\Delta y=f'(\xi)\Delta x. \tag{6}$$

Actually it is another formulation of the mean-value (or Lagrange's) theorem. Formula (6) is called the *formula of finite increments*.

From the mean-value (or Lagrange's) theorem it follows the other fact, which will be used further. Let us assume that the following identity

$$f'(x)\equiv 0$$

is fulfilled for a function $f(x)$. Take arbitrary values x_1, x_2 and apply theorem by Lagrange on the closed interval $[x_1, x_2]$. Then we get

$$\frac{f(x_1)-f(x_2)}{x_1-x_2}=f'(\xi).$$

But $f'(\xi)=0$. It means that $f(x_1)=f(x_2)$. Since the values x_1 and x_2 are arbitrary then $f(x)\equiv const$ for all *x*. if $f(x)\equiv const$, then $f'(x)\equiv 0$.

23.1. Extended Mean Value Theorem (or Cauchy's theorem)

Theorem. Let functions $f(x)$ and $\varphi(x)$ be:

1) continuous on the closed interval $[a,b]$;
2) differentiable within the open interval (a,b);
3) $\varphi'(x)$ does not vanish anywhere inside of the interval (a,b).

Then there will be at least one point $\xi\in(a,b)$, such that the relation

$$\frac{f(b)-f(a)}{\varphi(b)-\varphi(a)}=\frac{f'(\xi)}{\varphi'(\xi)}$$

holds true.

■ To prove this theorem let us introduce into consideration the auxiliary function $F(x)=f(x)-\lambda\varphi(x)$. Choose the number λ in such manner that the equality $F(a) = F(b)$, that is

$$f(a) - \lambda\varphi(a) = f(b) - \lambda\varphi(b)$$

is fulfilled.

Whence one can get

$$\lambda = \frac{f(b) - f(a)}{\varphi(b) - \varphi(a)}.$$

At such value of λ the function $F(x)$ satisfies conditions of the Rolle's theorem. Therefore there exists at least one point $\xi \in (a,b)$, such that $F'(\xi) = 0$, that is

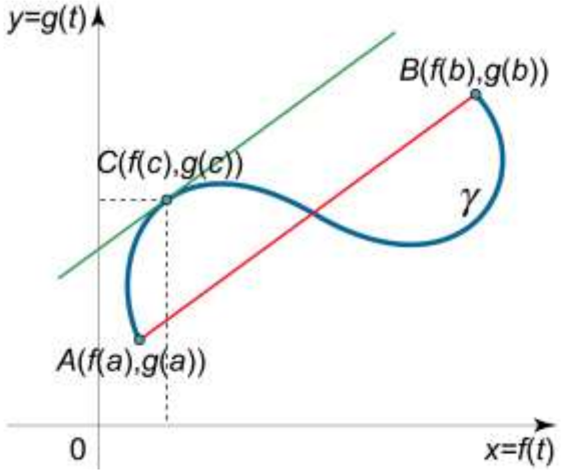
$$f'(\xi) - \frac{f(b) - f(a)}{\varphi(b) - \varphi(a)}\varphi'(\xi) = 0.$$

Whence it follows that

$$\frac{f(x_1) - f(x_2)}{\varphi(x_1) - \varphi(x_2)} = \frac{f'(\xi)}{\varphi'(\xi)}. \ \square$$

***Note*:** It is obvious that Lagrange's theorem is particular case of Cauchy's theorem if function $\varphi(x) = x$.

***Note*:** Cauchy's mean value theorem has the following *geometric meaning*. Suppose that a curve is described by the parametric equations $x = f(t), y = g(t)$, where the parameter *t* ranges in the interval $[a, b]$. When changing the parameter *t* the point of the curve in Figure runs from $A(f(a), g(a))$ to $B(f(b), g(b))$. According to the theorem, there is a point $(f(c), g(c))$ on the curve where the tangent is parallel to the chord joining the ends A and B of the curve.



23.2. L'Hospital's Rule for Evaluating Indeterminate Forms $\frac{0}{0}$

Theorem. Let functions $f(x)$ and $\varphi(x)$ vanish at point $x = a$ $\left(f(a) = \varphi(a) = 0\right)$ that is an indeterminate form of the type $\frac{0}{0}$ occurs and the functions satisfy the conditions of the Cauchy theorem in some neighbourhood of this point, and the derivative of the function $\varphi(x)$ does not vanish everywhere in this neighborhood, $\varphi'(x) \neq 0$. Then if $\lim\limits_{x\to a}\frac{f'(x)}{\varphi'(x)}$ exists (finite or infinite), then $\lim\limits_{x\to a}\frac{f(x)}{\varphi(x)}$ exists too and both the limits are equal:

$$\lim_{x\to a}\frac{f(x)}{\varphi(x)} = \lim_{x\to a}\frac{f'(x)}{\varphi'(x)}$$

■ Let $\lim\limits_{x\to a}\frac{f'(x)}{\varphi'(x)} = A$, where A is some number (or $A = \infty$). We must prove that

$$\lim_{x\to a}\frac{f(x)}{\varphi(x)} = A.$$

Let us take some arbitrary x, which is near by a and apply the Cauchy theorem on the segment $[a, x]$. We obtain

$$\frac{f(x) - f(a)}{\varphi(x) - \varphi(a)} = \frac{f'(\xi)}{\varphi'(\xi)}.$$

In accordance with the condition $f(a) = \varphi(a) = 0$, we can write

$$\frac{f(x)}{\varphi(x)} = \frac{f'(\xi)}{\varphi'(\xi)}$$

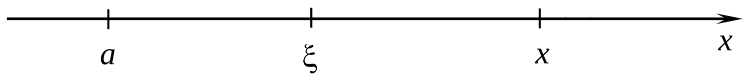


Fig. 12

Let $x \to a$. Then $\xi \to a$ (Fig. 12), so $\frac{f'(\xi)}{\varphi'(\xi)} \to A$. From this it follows that

$$(x \to a) \Rightarrow \left(\frac{f(x)}{\varphi(x)} \to A \right),$$

that was to be proved. □

We considered the case when a is a finite number. If $x \to \infty$ then the proof given above is inapplicable.

Nevertheless, we can prove that in this case

$$\lim_{x \to \infty} \frac{f(x)}{\varphi(x)} = \lim_{x \to \infty} \frac{f'(x)}{\varphi'(x)}.$$

■ Let us assign that $x = \frac{1}{t}$, where t is a new variable. Then from $(x \to \infty) \Rightarrow (t \to 0)$ substitute the expression for x in the given ratio

$$\lim_{x \to \infty} \frac{f(x)}{\varphi(x)} = \lim_{t \to 0} \frac{f\left(\frac{1}{t}\right)}{\varphi\left(\frac{1}{t}\right)} = \lim_{t \to 0} \frac{f'\left(\frac{1}{t}\right)\left(-\frac{1}{t^2}\right)}{\varphi'\left(\frac{1}{t}\right)\left(-\frac{1}{t^2}\right)} = \lim_{t \to 0} \frac{f'\left(\frac{1}{t}\right)}{\varphi'\left(\frac{1}{t}\right)} = \lim_{x \to \infty} \frac{f'(x)}{\varphi'(x)},$$

that has to be proved. □

*Example 1***.** Let us calculate limit $A = \lim\limits_{x \to 0} \frac{a^x - b^x}{c^x - d^x}$. Direct substitution gives $A = \frac{0}{0}$, it is obvious that condition of the theorem is fulfilled so we can use L'Hospital's rule:

$$A = \lim_{x \to 0} \frac{a^x \ln a - b^x \ln b}{c^x \ln c - d^x \ln d} = \frac{\ln a - \ln b}{\ln c - \ln d} = \frac{\ln \frac{a}{b}}{\ln \frac{c}{d}}.$$

Example 2. $\lim\limits_{x \to \pi} (\pi - x) \tan \frac{x}{2} = \|0 \cdot \infty\| = \lim\limits_{x \to \pi} \frac{\pi - x}{\operatorname{ctan} \frac{x}{2}} =$

$$= \left\| \frac{0}{0} \right\| = \lim_{x \to \pi} \frac{-1}{-\frac{1}{2\sin^2 \frac{x}{2}}} = 2 \lim_{x \to \pi} \sin^2 \frac{x}{2} = 2.$$

Example 3. Calculate $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{x^3}$.

Using L'Hospital's rule we obtain

$$A=\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}=\left\|\frac{0}{0}\right\|=\lim_{x\to 0}\frac{\dfrac{1}{\cos^2 x}-\cos x}{3x^2}=\lim_{x\to 0}\frac{1-\cos^3 x}{3x^2\cos^2 x}=$$

$$=\left\|\frac{0}{0}\right\|=\frac{1}{3}\lim\frac{(1-\cos x)(1+\cos x+\cos^2 x)}{x^2\cos^2 x}=\lim_{x\to 0}\frac{1-\cos x}{x^2}=\left\|\frac{0}{0}\right\|.$$

As indetermination $\dfrac{0}{0}$ is obtained, then we use L'Hospital's rule again.

$$A=\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{\sin x}{2x}=\frac{1}{2}.$$

Example 4. $\lim\limits_{x\to 0}\dfrac{x-\sin x}{x^3}=\left\|\dfrac{0}{0}\right\|=\lim\limits_{x\to 0}\dfrac{1-\cos x}{3x^2}=\lim\limits_{x\to 0}\dfrac{\frac{x^2}{2}}{3x^2}=\dfrac{1}{6}$.

23.3. L'Hospital's Rule for Evaluating Indeterminate Forms $\dfrac{\infty}{\infty}$

Theorem. Let functions $f(x)$ and $\varphi(x)$ be differentiable in some neighborhood of the point $x=a$, except may be this point. Derivative of the function $\varphi(x)$ does not vanish everywhere in this neighborhood, that is, $\varphi'(x)\neq 0$ and functions $f(x)\to\infty$ and $\varphi(x)\to\infty$ as $x\to a$. Then if $\lim\limits_{x\to a}\dfrac{f'(x)}{\varphi'(x)}$ exists then $\lim\limits_{x\to a}\dfrac{f(x)}{\varphi(x)}$ exists and both the limits are equal:

$$\lim_{x\to a}\frac{f(x)}{\varphi(x)}=\lim_{x\to a}\frac{f'(x)}{\varphi'(x)}$$

■ Let us take some x_1 which lies near by a and some arbitrary x_2 which lies between x_1 and a (Fig. 13).

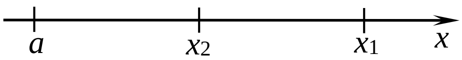


Fig. 13

Then by Lagrange's theorem we obtain

$$\frac{f(x_1)-f(x_2)}{\varphi(x_1)-\varphi(x_2)}=\frac{f'(\xi)}{\varphi'(\xi)},$$

where $x_2 < \xi < x_1$. Rewrite the last relation in the following form

$$\frac{f(x_2)}{\varphi(x_2)} \cdot \frac{\dfrac{f(x_1)}{f(x_2)} - 1}{\dfrac{\varphi(x_1)}{\varphi(x_2)} - 1} = \frac{f'(\xi)}{\varphi'(\xi)},$$

i.e.

$$\frac{f(x_2)}{\varphi(x_2)} = \frac{1 - \dfrac{\varphi(x_1)}{\varphi(x_2)}}{1 - \dfrac{f(x_1)}{f(x_2)}} \cdot \frac{f'(\xi)}{\varphi'(\xi)}.$$

Let us fix the point x_1 and take x_2 however close by a. Taking into account that $\varphi(x) \to \infty$ and $f(x) \to \infty$ as $x \to a$ we make values $\dfrac{\varphi(x_1)}{\varphi(x_2)}$ and $\dfrac{f(x_1)}{f(x_2)}$ however small, i.e. we make first factor on the right of the last equality however close by 1. If we take x_1 however close by a, we make ξ however close by a that is

$$\lim_{x \to a+0} \frac{f(x)}{\varphi(x)} = \lim_{x \to a+0} \frac{f'(x)}{\varphi'(x)},$$

that has to be proved.

Taking points x_1 and x_2 such as $x_1 < x_2 < a$ we can by the same way prove that

$$\lim_{x \to a-0} \frac{f(x)}{\varphi(x)} = A. \ \square$$

Note. As in the case of indeterminate form $\dfrac{0}{0}$, it is easy to show that L'Hospital's rule is true when $x \to \infty$.

*Example 1***.** Let us consider $\lim\limits_{x \to +\infty} \dfrac{a^x}{x^n}$, where $a > 1$, n is some natural number.

Direct substitution gives indeterminate form $\dfrac{\infty}{\infty}$. Applying n times L'Hospital's rule we obtain

$$\lim_{x \to +\infty} \frac{a^x}{x^n} = \lim_{x \to +\infty} \frac{a^x \ln a}{n x^{n-1}} = \lim_{x \to +\infty} \frac{a^x (\ln a)^2}{n(n-1)x^{n-2}} = \ldots = \lim_{x \to +\infty} \frac{a^x (\ln a)^n}{n!} = +\infty.$$

Example 2.

$$\lim_{x\to+0} x\ln x = \|0\cdot\infty\| = \lim_{x\to+0}\frac{\ln x}{\frac{1}{x}} = \lim_{x\to+0}\frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to+0}(-x) = 0.$$

Example 3. Using L'Hospital's rule we obtain

$$\lim_{x\to\infty}\frac{x-\sin x}{x+\sin x} = \left\|\frac{\infty}{\infty}\right\| = \lim_{x\to\infty}\frac{1-\cos x}{1+\cos x} = \lim_{x\to\infty}\tan^2\frac{x}{2}.$$

The last limit does not exist through previous limit exists and is equal to 1. But it does not contradict to L'Hospital's rule, which states that if $\lim\limits_{x\to a}\frac{f'(x)}{\varphi'(x)}$ exists then $\lim\limits_{x\to a}\frac{f(x)}{\varphi(x)}$ exists and both the limits are equal.

23.4. Evaluating Power-Exponential Indeterminate Forms

Let us calculate $\lim\limits_{x\to a}[f(x)]^{\varphi(x)}$ if one of three cases occur:

1) $\lim\limits_{x\to a} f(x) = 1,\ \lim\limits_{x\to a}\varphi(x) = \infty$ (indeterminate form 1^∞);
2) $\lim\limits_{x\to a} f(x) = 0, \lim\limits_{x\to a}\varphi(x) = 0$ (indeterminate form 0^0);
3) $\lim\limits_{x\to a} f(x) = \infty, \lim\limits_{x\to a}\varphi(x) = 0$ (indeterminate form ∞^0).

In all cases we should do the same. Let us assign $[f(x)]^{\varphi(x)} = y$. Then $\ln(y) = \varphi(x)\cdot\ln f(x)$, so

$$\lim_{x\to a}\ln y = \lim_{x\to a}\varphi(x)\ln f(x).$$

It is easy to see that in all the three cases we have on the right indeterminate form $0\cdot\infty$, which can be reduced to forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Evaluating this indeterminate form by L'Hospital's rule we obtain

$$\lim_{x\to a}\varphi(x)\ln f(x) = A,$$

where A is some number, so $\lim\limits_{x\to a}\ln y = A$, i.e. $\ln\lim\limits_{x\to a} y = A$. So

$$\lim_{x\to a} y = e^A.$$

Example. Let us calculate $\lim\limits_{x\to 0}\left(e^{-x}+3x\right)^{\frac{1}{x}}$. Direct substitution gives indeterminate form 1^{∞}. Let us assign $y=\left(e^{-x}+3x\right)^{\frac{1}{x}}$. Then

$$\ln y=\frac{1}{x}\ln\left(e^{-x}+3x\right),$$

so

$$\lim_{x\to 0}\ln y=\lim_{x\to 0}\frac{\ln\left(e^{-x}+3x\right)}{x}=\left\|\frac{0}{0}\right\|=\lim_{x\to 0}\frac{\dfrac{-e^{-x}+3}{e^{-x}+3x}}{1}=\frac{-1+3}{1}=2,$$

$$\lim_{x\to 0}y=e^{2}.$$