Lecture #24: Taylor & Maclaurin polynomials

A Taylor polynomial illustrates the first steps in the process of approximating complicated functions with polynomials. Using this process we can approximate trigonometric, exponential, logarithmic, and other non-polynomial functions as closely as we like (for certain values of x) with polynomials. This is extraordinarily useful in that it allows us to calculate values of these functions to whatever precision we like using only the operations of addition, subtraction, multiplication, and division, which are operations that can be easily programmed in a computer.

Idea of a Taylor polynomial

Polynomials are simpler than most other functions. This leads to the idea of approximating a complicated function by a polynomial. Taylor realized that this is possible provided there is an "easy" point at which you know how to compute the function and its derivatives. Given a function $f(x)$ and a value a, we will define for each degree n a polynomial $P_n(x)$ which is the "best *n*-th degree polynomial approximation to $f(x)$ near $x = a$."

It pays to start very simply. A *zero-degree polynomial* is a constant. What is the best constant approximation to $f(x)$ near $x = a$? Clearly, the constant $f(a)$. What is the best *linear approximation*? We already know this, and have given it the notation $L(x)$. It is the tangent line to the graph of $f(x)$ at $x = a$ and its equation is

$$\mathrm{L}(x) = f(a) + f'(a)(x - a)$$

So now we know now that zero- and first-degree polynomials are:

$$P_0(x) = f(a)$$

$$P_1(x) = f(a) + f'(a)(x - a)$$

Just one more idea is needed to bust this wide open, that is to figure out $P_n(x)$ for all *n*: the polynomial $P_n(x)$ matches all the derivatives of $f(x)$ at a up to the *n*th derivative. As *n* grows, notice how P_n should become a better approximation and stays close to $f(x)$ for longer. Find the form of $P_n(x)$?

Let the general form of a *n*-degree polynomial used to approximate a function $f(x)$ be

$$f(x) \approx P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n$$

Find the derivatives of the polynomial at a point $x = a$:

$$f'(x) = a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + \dots + na_n(x - x_0)^{n-1}$$

$$f''(x) = 2a_2 + 3 \cdot 2a_3(x - x_0) + \dots + n(n-1)a_n(x - x_0)^{n-2}$$

$$f'''(x) = 3 \cdot 2a_3 + \dots + n(n-1)(n-2)a_n(x - x_0)^{n-3}$$

……….

$$f^{(n)}(x) = n(n-1)(n-2) \cdot \dots \cdot 3 \cdot 2a_n$$

Plugging $x = a$ in the equalities we get

$f(a) = a_0 \Rightarrow a_0 = f(a)$

$f'(a) = a_1 \Rightarrow a_1 = \frac{f'(a)}{1!}$

$f''(a) = 2a_2 \Rightarrow a_2 = \frac{f''(a)}{2!}$

$f'''(a) = 3 \cdot 2a_3 \Rightarrow a_3 = \frac{f'''(a)}{3!}$

……….

$f^{(n)}(a) = n(n-1)(n-2) \cdot \dots \cdot 3 \cdot 2a_n \Rightarrow a_n = \frac{f^{(n)}(a)}{n!}$

Therefore, we have

$f(x) \approx P_n(x) = f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x - a)^n$

Definition: Let a be any real number and let $f(x)$ be a function that can be differentiated at least n times at the point $x = a$. The *Taylor polynomial* for $f(x)$ of order n about the point a is the polynomial $P_n(x)$ defined by

$$P_n(x) = f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x - a)^n =$$

$$= \sum_{k=1}^{n} \frac{f^{(n)}(a)}{n!}(x - a)^n \qquad (1)$$

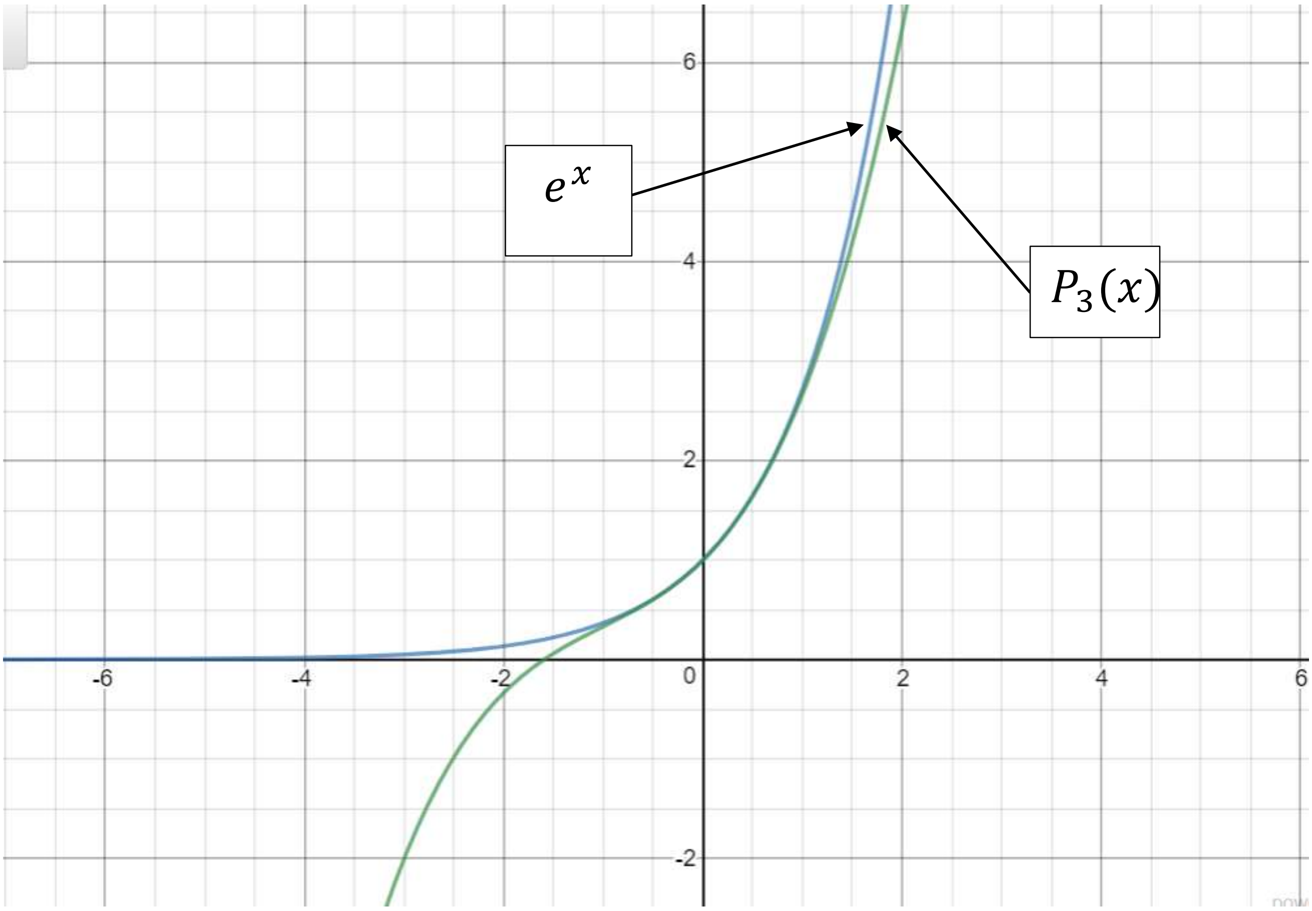
A special case of the Taylor polynomial is the *Maclaurin polynomial*, where $a = 0$. That is, the *Maclaurin polynomial* of degree n of $f\ (x)$ is

$$P_n(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ ... \ + \frac{f^{(n)}(0)}{n!}x^n = \sum_{k=1}^{n} \frac{f^{(n)}(0)}{n!}x^n \quad (2)$$

EXAMPLE: $f(x) = e^x$, $n = 3$ and $a = 0$. We list the function and its derivatives out to the third one.

k	$f^{(k)}(x)$	$f^{(k)}(a)$	$\frac{f^{(k)}(a)}{k!}(x-a)^k$
0	e^x	1	1
1	e^x	1	x
2	e^x	1	$\frac{x^2}{2}$
3	e^x	1	$\frac{x^3}{6}$

Summing the last column we find that $P_3(x) = 1 + x + x^2/2 + x^3/6$.



Taylor polynomials are used to approximate functions $f(x)$ in mainly two situations:

1) When $f(x)$ is known, but perhaps "hard" to compute directly. For instance, we can define $y = \cos x$ as either the ratio of sides of a right triangle ("adjacent over hypotenuse") or with the unit circle. However, neither of these provides a convenient way of computing $\cos 2$. A Taylor polynomial of sufficiently high degree can provide a reasonable method of computing such values using only operations usually hard-wired into a computer ($+$, $-$, $\times$ and $\div$).
2) When $f(x)$ is not known, but information about its derivatives is known. This occurs more often than one might think, especially in the study of differential equations.

In both situations, a critical piece of information to have is "How good is my approximation?" If we use a Taylor polynomial to compute $\cos 2$, how do we know how accurate the approximation is? he following theorem gives similar bounds for Taylor (and hence Maclaurin) polynomials.

Taylor's Theorem. Let $f(x)$ be a function whose $(n+1)$th derivative exists on an interval I and let $x = a$ be in I. Then, for each x in I, there exists θ between x and a such that

$$f(x) \approx P_n(x) = f(a) + f'(a)(x-a) + \frac{f'(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x) \quad (3)$$

where

$$R_n(x) = \frac{f^{(n+1)}(\theta)}{(n+1)!}(x-a)^{(n+1)}, x \leq \theta \leq a \quad (4)$$

In doing so, an estimation is valid

$$|R_n(x)| \leq \frac{max|f^{(n+1)}(\theta)|}{(n+1)!}|(x-a)^{(n+1)}|$$

The *first part* of Taylor's Theorem states that $f(x) = P_n(x) + R_n(x)$ or $f(x) - P_n(x) = R_n(x)$, where $P_n(x)$ is the nth order Taylor polynomial and $R_n(x)$ is the *remainder*, or *error* given in the *Lagrange form* in the Taylor approximation.

The *second part* gives bounds on how big that error can be. If the $(n+1)$th derivative is large, the error may be large; if x is far from a, the error may also be large. However, the $(n+1)!$ term in the denominator tends to ensure that the error gets smaller as n increases.

Note. This is at first a little mysterious and difficult to use, which is why we'll be doing some practice. The exact value of θ will depend on *a*, *x*, *n* and $f(x)$ and will not be known. However, it will always be between *x* and *a*. This means we can often get bounds. We might know, for example, that $f^{n+1}(x)$ is always positive on $[x, a]$ and is greatest at *a*, which would lead to

$$P_n(x) \leq x \leq P_n(x) + \frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{(n+1)}$$

Example. Use the Theorem to find error bounds when approximating $\ln 1.5$ and $\ln 2$ with $P_6(x)$, the Taylor polynomial of degree 6 of $f(x) = \ln x$ at $x = 1$

We start with the approximation of $\ln 1.5$ with $P_6(x)$. We can compute $P_6(x)$ using our work above:

$$P_6(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 - \frac{1}{6}(x-1)^6.$$

Since $P_6(x)$ approximates $\ln x$ well near $x = 1$, we approximate $ln\, 1.5 \approx P_6(x)$:

$$P_6(1.5) = (1.5-1) - \frac{1}{2}(1.5-1)^2 + \frac{1}{3}(1.5-1)^3 - \frac{1}{4}(1.5-1)^4 + + \frac{1}{5}(1.5-1)^5 - \frac{1}{6}(1.5-1)^6 = \frac{259}{640} \approx 0.404688.$$

The Theorem references an open interval *I* that contains both *x* and *a*. The smaller the interval we use the better; it will give us a more accurate (and smaller!) approximation of the error. We let $I = (0.9,1.6$, as this interval contains both $a = 1$ and $x = 1.5$.

The theorem references $max|f^{(n+1)}(\theta)|$. In our situation, this is asking

"How big can the 7th derivative of $y = \ln x$ be on the interval $(0.9,1.6)$?" The seventh derivative is $y^{(7)}(x) = -\frac{6!}{x^7}$. The largest value it attains on I is about 1506. Thus we can bound the error as:

$$|R_6(1.5)| \leq \frac{max|f^{(7)}(\theta)|}{7!}|(1.5-1)^7| \leq \frac{1506}{5040} \cdot \frac{1}{2^7} \approx 0.0023$$

We computed $P_6(1.5)$=0.404688; using a calculator, we find ln1.5≈0.405465, so the actual error is about 0.000778 , which is less than our bound of 0.0023 . This affirms Taylor's Theorem; the theorem states that our approximation would be within about 2 thousandths of the actual value, whereas the approximation was actually closer.

EXAMPLE: Let $f(x) = e^{-x}$, $a = \ln 10$ and $n = 1$. How well does $P_2(x) = \frac{1}{10} - \frac{1}{10}(x - \ln 10)$ approximate e^{-x} for $x = \ln 10 + 0.2 \approx 2.502$? The remainder $R = e^x - P_n(x)$ will equal $f''(u)/2!$ times $(0.2)^2$ for some u between $\ln 10$ and $\ln 10 + 2$. Because $f''(u) = e^{-u}$, we know that $0 < f''(u) < f''(a) = 1/10$. Therefore, with $x = \ln 10 + 0.2$,

$$\frac{1}{10} - \frac{0.2}{10} < e^{-x} < \frac{1}{10} - \frac{0.2}{10} + \frac{1}{20}(0.2)^2 .$$

Numerically, $0.08 < e^{-(\ln 10 + 0.2)} < 0.082$. The actual value is $0.081873\ldots$.

Here is another example.

EXAMPLE: Let $f(x) = \cos(x)$, $a = 0$ and $n = 4$. Then $P_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$. This is also P_5 because $f^{(5)}(0) = 0$. How close is this to the correct value of $\cos x$ at $x = \pi/4$? Because the sixth derivative of cos is $-$cos, Taylor's theorem says

$$\cos(\pi/4) - P_4(\pi/4) = c(\pi/4)^6$$

where $c = -\cos u/6!$ for some $u \in [0, \pi/4]$. The maximum value of $-\cos$ on $[0, \pi/4]$ is $-\sqrt{1/2}$ and the minimum value is -1, therefore

$$-\frac{1}{720}\left(\frac{\pi}{4}\right)^6 \leq \cos(\pi/4) - P_4(\pi/4) \leq -\frac{1}{720\sqrt{2}}\left(\frac{\pi}{4}\right)^6 .$$

For bounds one can compute mentally, we can use the fact that $\pi/4$ is a little less than 1 to get

$$-\frac{1}{720} \leq \cos(\pi/4) - P_4(\pi/4) \leq 0$$

to see that $P_4(\pi/4)$ overestimates $\cos(\pi/4)$ but not by more than 1/720 which is a little over 0.001.

Example. Approximating an unknown function: A function $y = f(x)$ is unknown save for the following two facts.

1) $y(0) = f(0) = 1$ 2) $y' = y^2$

Find the degree 3 Maclaurin polynomial $P_3(x)$ of $y = f(x)$.

One might initially think that not enough information is given to find $P_3(x)$. However, note how the second fact above actually lets us know what $y'(0)$ is:

$$y' = y^2 \Rightarrow y'(0) = y^2(0).$$

Since $y(0) = 1$, we conclude that $y'(0) = 1$.

Now we find information about y''. Starting with $y' = y^2$, take derivatives of both sides, with respect to x. That means we must use implicit differentiation.

$$y' = y^2$$

$$\frac{d}{dx}(y') = \frac{d}{dx}(y^2) \Rightarrow y'' = 2y \cdot y'$$

Now evaluate both sides at $x = 0$:

$$y''(0) = 2y(0) \cdot y'(0) \Rightarrow y''(0) = -2$$

We repeat this once more to find $y'''(0)$. We again use implicit differentiation; this time the Product Rule is also required.

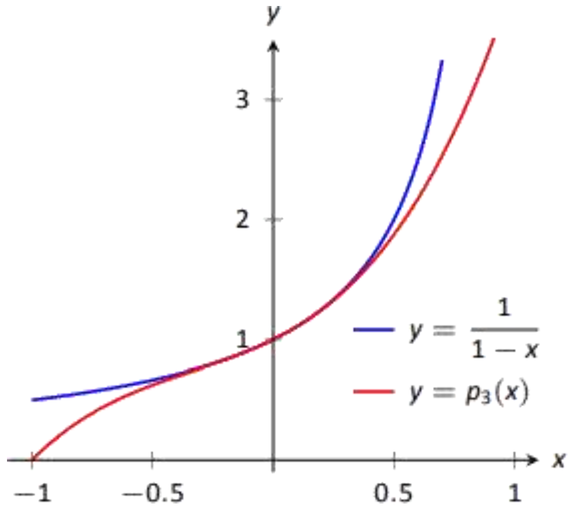
$$\frac{d}{dx}(y'') = \frac{d}{dx}(2yy') \Rightarrow y''' = 2y' \cdot y' + 2y \cdot y''$$

Now evaluate both sides at $x = 0$:

$$y'''(0) = 2y'(0)^2 + 2y(0)y''(0) \Rightarrow y'''(0) = 2 + 4 = 6$$

In summary, we have:

$$y(0) = 1, y'(0) = 1, y''(0) = 2, y'''(0) = 6.$$



We can now form $P_3(x)$

$$P_3(x) = 1 + x + \frac{2}{2!}x^2 + \frac{6}{3!}x^3 = 1 + x + x^2 + x^3$$

It turns out that the differential equation we started with, $y' = y^2$, where $y(0) = 1$, can be solved without too much difficulty: $y = \frac{1}{1-x}$. Figure shows this function plotted with $P_3(x)$. Note how similar they are near $x = 0$.

Also, in a neighborhood of the point x_0, a function can be expanded in a Taylor series with *Peano's form of remainder*:

$$R_n(x) = \frac{f^{(n+1)}(x_0) + \alpha}{(n+1)!}(x - x_0)^{n+1},$$

that is proportional to $(x - x_0)^{n+1}$.