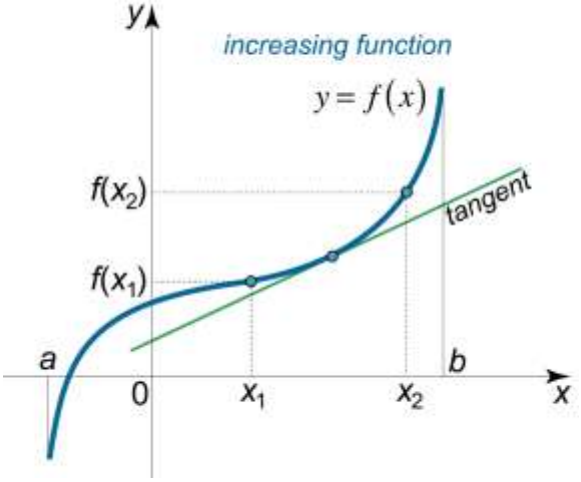
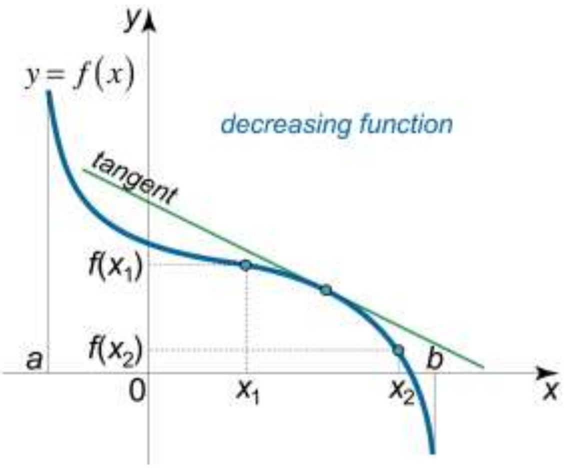
Lecture #25: Using the Derivative to Graph Functions

Given a particular function, we are often interested in creating its accurate graph by using an analytical method. The derivatives can be effectively used to build the graph for a function. The secret to creating a graph lies in properly carrying on appropriate steps and learning how to read it.

25.1 Increasing and Decreasing Function

Definition 1: Let $y = f(x)$ be a differentiable function on an interval (a, b). If for any two points x_1, $x_2 \in (a, b)$ such that $x_1 < x_2$, there holds the inequality $f(x_1) \leq f(x_2)$, the function is called *increasing* (or *non-decreasing*) in this interval.





If this inequality is strict, i.e. $f(x_1) < f(x_2)$, then the function $y = f(x)$ is said to be *strictly increasing* on the interval (a, b).

This concept can be formulated in a more compact form. A function $f(x)$ is called *increasing* (or *non-decreasing*) on this interval (a, b) if

$$\forall x_1, x_2 \in (a, b): x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2);$$

A function $f(x)$ is called *strictly increasing* on this interval (a, b) if

$$\forall x_1, x_2 \in (a, b): x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$

Similarly, we define

Definition 2: Let $y = f(x)$ be a differentiable function on an interval (a, b). If for any two points x_1, $x_2 \in (a, b)$ such that $x_1 < x_2$, there holds the inequality $f(x_1) \geq f(x_2)$, the function is called *decreasing* (or *non-increasing*) in this interval, i.e.

$$\forall x_1, x_2 \in (a, b): x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2);$$

If this inequality is strict, i.e. $f(x_1) > f(x_2)$, then the function $y = f(x)$ is said to be *strictly decreasing* on the interval (a, b), i.e.

$$\forall x_1, x_2 \in (a, b): x_1 < x_2 \Rightarrow f(x_1) > f(x_2).$$

Note: If a function $f(x)$ is differentiable on the interval (a, b) and belongs to one of the four considered types (i.e. it is increasing, strictly increasing, decreasing, or strictly decreasing), this function is called *monotonic* on this interval.

Alternatively, the concept of increasing and decreasing functions can also be defined for a single point x_0. In this case, we consider a small δ-neighborhood $(x_0 - \delta, x_0 + \delta)$ of this point.

Definition 3: A function $y = f(x)$ is *strictly increasing* at x_0 if there exists a number $\delta > 0$ such that

$$\forall x \in (x_0 - \delta, x_0) \Rightarrow f(x) < f(x_0);$$
$$\forall x \in (x_0, x_0 + \delta) \Rightarrow f(x) > f(x_0).$$

Similarly, we can define

Definition 4: A function $y = f(x)$, which is *strictly decreasing* at the point x_0 if there exists a number $\delta > 0$ such that:

$$\forall x \in (x_0 - \delta, x_0) \Rightarrow f(x) > f(x_0);$$
$$\forall x \in (x_0, x_0 + \delta) \Rightarrow f(x) < f(x_0).$$

Criteria for Increasing and Decreasing Functions

Again consider a function $y = f(x)$ assuming it is differentiable on an interval (a, b). To determine if the function is increasing or decreasing on the interval, we use *the sign of the first derivative* of the function.

Theorem 1. In order for the function $y = f(x)$ to be increasing on the interval (a, b) it is necessary and sufficient that the first derivative of the function be non-negative everywhere in this interval:

$$f'(x) \geq 0 \forall x \in (a,b).$$

A similar criterion applies to the case of a function that is decreasing on the interval:

$$f'(x) \leq 0 \quad \forall x \in (a,b).$$

We prove both (necessary and sufficient) parts of the theorem for the case of an increasing function.

<u>Necessary condition</u>. Consider an arbitrary point $x_0 \in (a,b)$. If the function $y = f(x)$ is increasing on (a,b) then by definition, we can write:

$$\forall x \in (a,b): x > x_0 \Rightarrow f(x) > f(x_0);$$

$$\forall x \in (a,b): x < x_0 \Rightarrow f(x) < f(x_0).$$

Thus, there exists the sign preservation of the fraction:

$$\frac{f(x) - f(x_0)}{x - x_0} \geq 0, \quad \text{where} \quad x \neq x_0.$$

In the limit as $x \to x_0$, the left-hand side of the inequality is equal to the derivative of the function at the point x_0 that is by the limit sign preservation property:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \geq 0.$$

This relationship is valid for any $x_0 \in (a,b)$.

<u>Sufficient condition</u>. Consider the sufficient condition, that is the converse statement. Suppose that the derivative $f'(x)$ of a function $y = f(x)$ is non-negative in the interval (a,b)

$$f'(x_0) \geq 0 \quad \forall x \in (a,b).$$

If x_1 and x_2 are two arbitrary points of the interval such that $x_1 < x_2$, then by Lagrange's theorem we can write:

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1),$$

where $c \in [x_1, x_2], \Rightarrow c \in (a,b)$.

Since $f'(c) \geq 0$, then the right-hand side of the equality is non-negative. Consequently,

$$f(x_2) \geq f(x_1).$$

i.e. the function $y = f(x)$ is increasing in the interval (a,b).

Consider now the cases of a *strictly* increasing and *strictly* decreasing function. There exists a similar theorem that describes the necessary and sufficient conditions. Omitting the proof, we state it for the case of a strictly increasing function.

Theorem 2. Suppose that a function $y = f(x)$ is differentiable on an interval (a, b). In order for the function to be strictly increasing in this interval, it is necessary and sufficient that the following conditions are satisfied:

1) $f'(x) \geq 0 \;\; \forall x \in (a, b)$;
2) $f'(x)$ is not identically equal to zero at any interval $[x_1, x_2] \in (a, b)$.

Remark: The condition 1 is contained in Theorem 1 and is an indication of a non-decreasing function. The additional condition 2 is required in order to exclude the intervals of constancy, in which the derivative of $f(x)$ is identically zero.

In practice (when finding the intervals of monotonicity), the sufficient condition for a strictly increasing or a strictly decreasing function is commonly used. Theorem 2 implies the following sufficient criterion:

If the condition $f'(x) > 0$ is satisfied for all $x \in (a, b)$, except perhaps only a few distinct points where $f'(x) = 0$, then the function $f(x)$ is *strictly increasing* in this interval.

Accordingly, the condition $f'(x) < 0$ defines a strictly decreasing function for all $x \in (a, b)$, except perhaps only a few distinct points where $f'(x) = 0$.

The number of points where $f'(x) = 0$ is usually finite. According to Theorem 2 they cannot tightly fill any subinterval of the interval

We also give a criterion for increasing/decreasing functions at a point:

Theorem 3. Let $x_0 \in (a, b)$.

If $f'(x_0) > 0$, then the function $f(x)$ is strictly increasing at the point x_0

If $f'(x_0) < 0$, then the function $f(x)$ is strictly decreasing at the point x_0

25.2 Local Extrema of Functions

25.2.1 Definition of Local Maximum and Local Minimum

Definition 1. Let a function $y = f(x)$ be defined in a δ-neighborhood of a point x_0, where $\delta > 0$. The function $f(x)$ is said to have a *local* (or *relative*) *maximum* at the point x_0, if for all points $x \neq x_0$ belonging to the neighborhood $(x_0 - \delta, x_0 + \delta)$ the following inequality holds:

$$f(x) \leq f(x_0).$$

If the strict inequality holds for all points $x \neq x_0$ in some neighborhood of x_0:

$$f(x) < f(x_0),$$

then the point x_0 is a *strict local maximum point.*
Similarly,
Definition 2. We define a *local* (or *relative*) *minimum* of the function $y = f(x)$. In this case, the following inequality is valid for all points $x \neq x_0$ of the δ -neighborhood $(x_0 - \delta, x_0 + \delta)$ of the point x_0

$$f(x) \geq f(x_0).$$

Accordingly, a *strict local minimum* at the point x_0 is described by the inequality

$$f(x) > f(x_0),$$

25.2.2 Stationary and Critical Points

Definition. The points at which the derivative of the function $f(x)$ is equal to zero are called the *stationary points*.

Definition. Let $f(x)$ be a function and let x_0 be a point in the domain of the function. The point x_0 is called a *critical point* of $f(x)$ if either $f'(x_0) = 0$ or $f'(x_0)$ does not exist.
Consequently, the stationary points are a subset of the set of critical points.

25.2.3 Necessary Condition for an Extremum

Theorem. A necessary condition for an extremum is formulated as follows: If the point x_0 is an extremum point of the function $f(x)$ then the derivative at this point either is zero or does not exist. In other words, the extrema of a function are contained among its critical points.

The proof of the necessary condition follows from *Fermat's theorem*.

Note that the necessary condition does not guarantee the existence of an extremum. A classic illustration here is the cubic function $f(x) = x^3$ Despite the fact that the derivative of the function at the point $x = 0$ is zero: $f'^{(0)} = 3 \cdot 0^2 = 0$ this point is not an extremum.

Thereby, local extrema of differentiable functions exist when the sufficient conditions are satisfied.

These conditions are based on the use of the first-, second-, or higher-order derivative. Respectively, three sufficient conditions for local extrema are considered. Now we turn to their formulation and proof.

25.2.4 First Derivative Test

Theorem 1. Let the function $f(x)$ be differentiable in a neighborhood of the point x_0, except perhaps at the point x_0 itself, in which, however, the function is continuous. Then:

1) If the derivative $f'(x)$ changes sign from minus to plus when passing through the point x_0 (from left to right), then x_0 is a *strict minimum point* (Figure 1). In other words, in this case there exists a number $\delta > 0$ such that

$$\forall x \in (x_0 - \delta, x_0) \Rightarrow f'(x) < 0, \text{ and } \forall x \in (x_0, x_0 + \delta) \Rightarrow f'(x) > 0.$$

2) If the derivative $f'(x)$ on the contrary, changes sign from plus to minus when passing through the point x_0 then x_0 is a *strict maximum point* (Figure 2). In other words, there exists a number $\delta > 0$ such that

$$\forall x \in (x_0 - \delta, x_0) \Rightarrow f'(x) > 0, \text{ and } \forall x \in (x_0, x_0 + \delta) \Rightarrow f'(x) < 0.$$

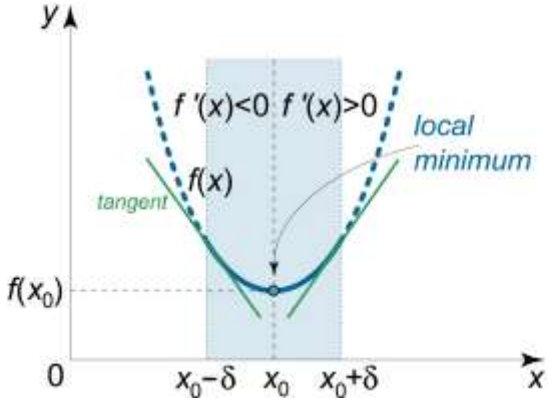


Figure 1

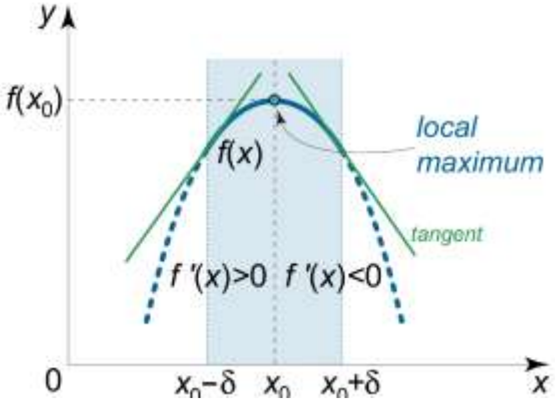


Figure 2

Proof. We confine ourselves to the case of the *minimum*. Suppose that the derivative $f'(x)$ changes sign from minus to plus when passing through the point x_0. To the left from the point x_0 the following condition is satisfied:

$$\forall x \in (x_0 - \delta, x_0) \Rightarrow f'(x) < 0.$$

By Lagrange's theorem, the difference of the values of the function at the points *x* and x_0 is written as

$$f(x) - f(x_0) = f'(c)(x - x_0),$$

where the point *c* belongs to the interval $(x_0 - \delta, x_0)$, in which the derivative is negative, i.e. $f'(c) < 0$. Since $x - x_0 < 0$ to the left of the point x_0 then

$$f(x) - f(x_0) > 0 \quad \text{for all} \quad x \in (x_0 - \delta, x_0).$$

Likewise, it is established that

$$f(x) - f(x_0) > 0 \quad \text{for all} \quad x \in (x_0, x_0 + \delta).$$

(to the right of the point x_0).

Based on the definition, we conclude that x_0 is a strict minimum point of the function

Similarly, we can prove the first derivative test for a strict *maximum*.

Note that the first derivative test does not require the function to be differentiable at the point x_0. If the derivative at this point is infinite or does not exist (i.e. the point x_0 is critical, but not stationary), the first derivative test can still be used to investigate the local extrema of the function.

25.2.5 Second Derivative Test

Theorem 2. Let the first derivative of a function $f(x)$ at the point x_0 be equal to zero: $f(x_0) = 0$, that is x_0 is a stationary point of Suppose also that there exists the second derivative at this point. Then

1) If $f''(x_0) > 0$, then x_0 is a *strict minimum point* of the function ;
2) If $f''(x_0) < 0$, then x_0 is a *strict maximum point* of the function

Proof. In the case of a strict minimum $f''(x_0) > 0$. Then the first derivative is an increasing function at the point x_0. Consequently, there exists a number $\delta > 0$ such that

$$\forall x \in (x_0 - \delta, x_0) \Rightarrow f'(x) < f'(x_0),$$
$$\forall x \in (x_0, x_0 + \delta) \Rightarrow f'(x) > f'(x_0).$$

Since $f''(x_0) = 0$ (because x_0 is a stationary point), therefore the first derivative is negative in the δ -neighborhood to the left of the point , and is positive to the right, i.e. the derivative changes sign from minus to plus when passing through the point x_0. By the first derivative test, this means that x_0 is a strict minimum point.

The case of the maximum can be considered in a similar way.

Note. The second derivative test is convenient to use when calculation of the first derivatives in the neighborhood of a stationary point is difficult. On the other hand, the second test may be used only for stationary points (where the first derivative is zero) − in contrast to the first derivative test, which is applicable to any critical points.

***Example* 1.** Investigate the function $y = \frac{3}{8}x^4 - 2x^3 + 3x^2 + 1$ for extremum.

1. Let us find the first derivative

$$y' = \frac{3}{2}x^3 - 6x^2 + 6x = \frac{3}{2}x\left(x^2 - 4x + 4\right) = \frac{3}{2}x(x-2)^2.$$

2. Find the real roots of the derivative. The derivative vanishes at two points*): $x_1 = 0$ and $x_2 = 2$, Fig. It means that these points are critical ones. The derivative is everywhere continuous and so there are no other critical points.

3. Investigate the character of the critical points and record the results.

Since $(x-2)^2 \geq 0$ for all x, then value y' changes its sign with «–» on «+» in moving from left to right through the point $x = 0$. Hence the function has minimum at the point $x = 0$ ($y(0) = 1$). In moving from left to right through the point $x = 2$ sign of derivative y' doesn't change, because on the left and on the right from this point it will be $y' > 0$. Consequently the function has no extremum at the point $x = 2$, Fig.

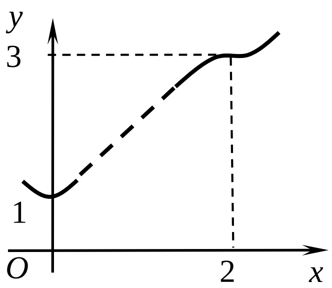


Fig.

*) A Point is called stationary one if derivative of a function vanishes at this point.

***Example* 2.** Investigate the function $y = 2x^3 - 3x^2$ for extremum. Let us find a derivative and stationary points

$$y' = 6x^2 - 6x = 6x(x-1),$$

thus the stationary points are: $x_1 = 0, x_2 = 1$. Further find the second derivative

$$y'' = 12x - 6$$

and calculate its values at these points

$$y''(0) = -6 < 0,\ y''(1) = 6 > 0.$$

Consequently the function has maximum at the point $x = 0$ and minimum at $x = 1$ (Fig.)

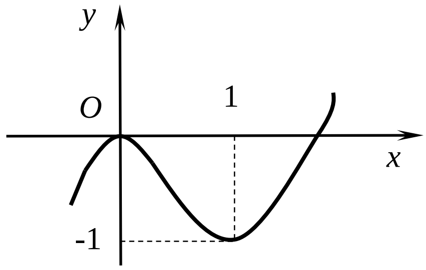


Fig.

25.2.6 Third Derivative Test

Theorem 3. Let the function $f(x)$ have derivatives at the point x_0 up to the *n*th order inclusively. Then if

$$f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0 \text{ and } f^{(n)}(x_0) \neq 0,$$

the point x_0 for even *n* is

1) a *strict minimum point* if $f^{(n)}(x_0) > 0$, and

2) a *strict maximum point* if $f^{(n)}(x_0) < 0$.

For odd *n* the extremum at x_0 does not exist.

It is clear that for $n = 2$, we obtain as a special case the second derivative test for local extrema considered above. To avoid such a transition, the third derivative test implies that $n > 2$.

25.3 Convex Functions

25.3.1 Definition of Convexity of a Function

Definition 1. Consider a function $y = f(x)$, which is assumed to be continuous on the closed interval $[a, b]$. The function $y = f(x)$ is called *convex downward* (or *concave upward*) if for any two points x_1 and x_2 in $[a, b]$, the following inequality holds:

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{f(x_1) + f(x_2)}{2}$$

If this inequality is strict for any $x_1, x_2 \in [a, b]$, such that $x_1 \neq x_2$, then the function $f(x)$ is called *strictly convex downward* on the interval $[a, b]$.

Similarly, we define a *concave function*.

Definition 2. A function $f(x)$ is called *convex upward* (or *concave downward*) if for any two points x_1 and x_2 in the interval $[a, b]$, the following inequality is valid:

$$f(\frac{x_1 + x_2}{2}) \geq \frac{f(x_1) + f(x_2)}{2}$$

If this inequality is strict for any $x_1, x_2 \in [a, b]$, such that $x_1 \neq x_2$, then the function $f(x)$ is called *strictly convex upward* on the interval $[a, b]$.

The introduced concept of convexity has a simple *geometric interpretation*.

If a function is *convex downward* (Figure 1), the midpoint B of each A_1A_2 chord lies above the corresponding point A_0 of the graph of the function or coincides with this point. Similarly,

If a function is *convex upward* (Figure 2), the midpoint B of each chord A_1A_2 is located below the corresponding point A_0 of the graph of the function or coincides with this point.

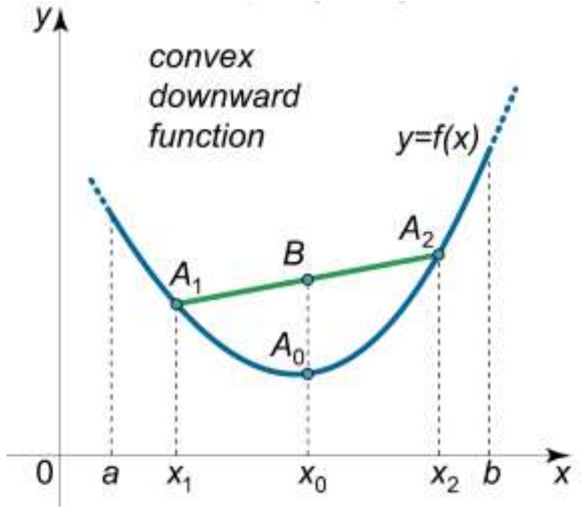


Figure 1

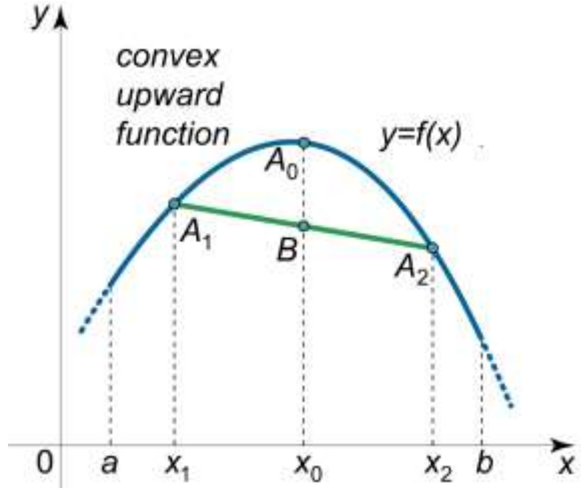


Figure 2

Also, convex functions have another obvious property, which is related to the location of the tangent to the graph of the function.

Definition 3. The function $f(x)$ is *convex downward* on the interval $[a, b]$ if and only if its graph does not lie below the tangent drawn to it at any point x_0 of the segment $[a, b]$ (Fig. 3).

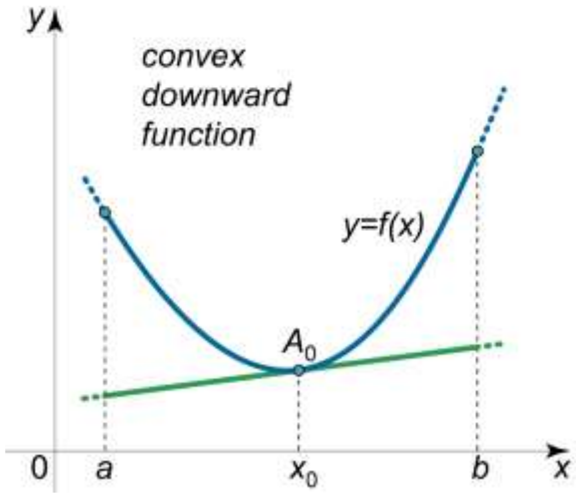


Figure 3

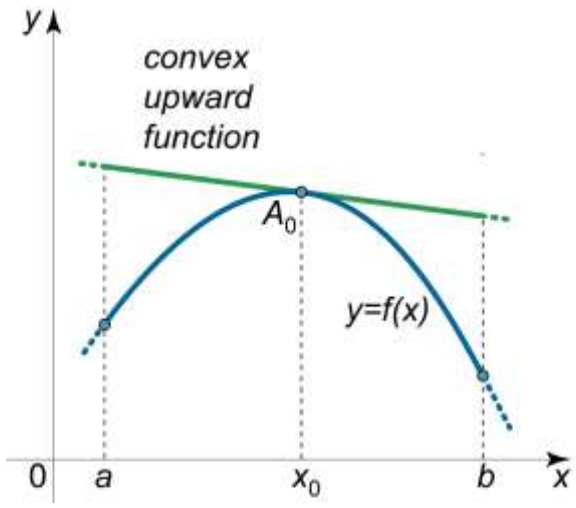


Figure 4

Accordingly,

Definition 4. The function $f(x)$ is *convex upward* (or *concave downward*) on the interval $[a, b]$ if and only if its graph does not lie above the tangent drawn to it at any point x_0 of the segment $[a, b]$ (Figure 4).

These properties represent a theorem and can be proved using the definition of convexity:

25.3.2 Sufficient Conditions for Convexity/Concavity

Theorem. Suppose that the first derivative $f'(x)$ of a function $f(x)$ exists in a closed interval $[a, b]$, and the second derivative $f''(x)$ exists in an open interval (a, b). Then the following *sufficient conditions* for convexity/concavity are valid:

1) If $f''(x) \geq 0$ for all $x \in (a, b)$, then the function $f(x)$ is *convex downward* (or *concave upward*) on the interval $[a, b]$;
2) If $f''(x) \leq 0$ for all $x \in (a, b)$, then the function is convex upward (or concave downward) on the interval $[a, b]$.

In the cases where the second derivative is strictly greater (or less) than zero, we say, respectively, about the *strict convexity downward* (or *strict convexity upward*).

We prove the theorem for the case of convexity downward. Let the function $f(x)$ have a non-negative second derivative on the interval (a, b): $f''(x) \geq 0$. Let x_0 be the midpoint of the interval $[x_1, x_2]$. Suppose that the length of this interval is equal to $2h$. Then the coordinates x_1 and x_2 can be written as

$$x_1 = x_0 - h, x_2 = x_0 + h.$$

Expand the function $f(x)$ at x_0 in the Taylor series with the remainder in the Lagrange form. We obtain the following expressions:

$$f(x_1) = f(x_0 - h) = f(x_0) - f'(x_0)h + \frac{f''(\xi_1)h^2}{2!},$$

$$f(x_2) = f(x_0 + h) = f(x_0) + f'(x_0)h + \frac{f''(\xi_2)h^2}{2!},$$

where $x_0 - h < \xi_1 < x_0, x_0 < \xi_2 < x_0 + h$.

Add the two equations:

$$f(x_1) + f(x_2) = 2f(x_0) + \frac{h^2}{2}[f''(\xi_1) + f''(\xi_2)].$$

Since $\xi_1, \xi_2 \in (a, b)$, then the second derivatives in the right-hand side are non-negative. Consequently,

$$f(x_1) + f(x_2) \geq 2f(x_0)$$

or

$$f(\frac{x_1 + x_2}{2}) \leq \frac{f(x_1) + f(x_2)}{2},$$

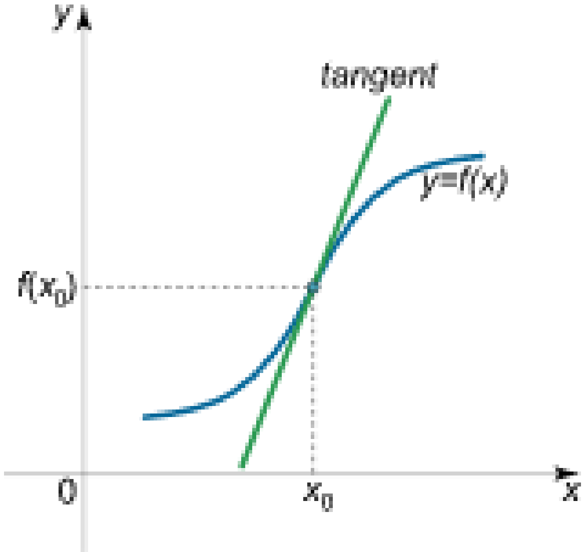
that is according to the definition, the function $f(x)$ is convex downward.

Note that the necessary condition for convexity (for example, the implication when from the convexity downwards it follows that $f''(x) \geq 0$) holds only for *the non-strict inequality*. In the case of strict convexity, the necessary condition is generally ***not valid***. For example, the function $f(x) = x^4$ is strictly convex downward. However, its second derivative is zero at $x = 0$, that is the strict inequality $f''(x) > 0$ does not hold in this case.

25.3.3 Inflection Points

Definition: Consider a function $y = f(x)$, which is continuous at a point x_0. The function $f(x)$ can have a finite or infinite derivative $f'(x_0)$ at this point. If, when passing through x_0, the function changes the direction of convexity, i.e. there exists a number $\delta > 0$ such that the function is convex upward on one of the intervals $(x_0 - \delta, x_0)$ or $(x_0, x_0 + \delta)$, and is convex downward on the other, then x_0 is called a *point of inflection* of the function $y = f(x)$.

The geometric meaning of an inflection point is that the graph of the function $f(x)$ passes from one side of the tangent line to the other at this point, i.e. the curve and the tangent line intersect (see Figure).



Necessary Condition for an Inflection Point

Theorem. If x_0 is a point of inflection of the function $f(x)$, and this function has a second derivative in some neighborhood of x_0 which is continuous at the point x_0 itself, then

$$f''(x_0) = 0.$$

Proof. Suppose that the second derivative at the inflection point x_0 is not zero: $f''(x_0) \neq 0$. Since it is continuous at x_0 then there exists a δ neighborhood of the point x_0 where the second derivative preserves its sign, that is

$$f''(x_0) < 0 \text{ or } f''(x_0) < 0 \quad \forall x \in (x_0 - \delta, x_0 + \delta).$$

In this case, the function is either strictly convex upward (when $f''(x) < 0$) or strictly convex downward (when $f''(x) > 0$). But then the point x_0 is not an inflection point. Hence, the assumption is wrong and the second derivative of the inflection point must be equal to zero.

1st Sufficient Condition for an Inflection Point (Second Derivative Test)

Theorem 1. If the function $f(x)$ is continuous and differentiable at a point x_0, has a second derivative $f''(x_0)$ in some deleted δ-neighborhood of the point x_0 and if the second derivative changes sign when passing through the point x_0 then x_0 is a point of inflection of the function $f(x)$

Proof. Suppose, for example, that the second derivative $f''(x)$ changes sign from plus to minus when passing through the point x_0. Hence, in the left δ -neighborhood $(x_0 - \delta, x_0)$, the inequality $f''(x) > 0$, holds, and in the right δ -neighborhood $(x_0, x_0 + \delta)$, the inequality $f''(x) < 0$ is valid.

In this case, according to the sufficient conditions for convexity, the function $f(x)$ is convex downward in the left δ-neighborhood of the point x_0 and is convex upward in the right δ-neighborhood.

Consequently, the function changes the direction of convexity at the point x_0 that is by definition, x_0 is a point of inflection.

2nd Sufficient Condition for an Inflection Point (Third Derivative Test)

Theorem 2. Let $f''(x_0) = 0$, $f'''(x_0) \neq 0$. Then x_0 is a point of inflection of the function $f(x)$

Proof. As $f'''(x_0) \neq 0$, the second derivative is either strictly increasing at x_0 (if $f'''(x_0) > 0$) or strictly decreasing at this point (if $f'''(x_0) < 0$). Because $f''(x_0) = 0$, then the second derivative for some $\delta > 0$ has different signs in the left and right δ-neighborhood of x_0. Hence, on the basis of the previous theorem, it follows that is a point of inflection of the function $f(x)$

25.4 Asymptotes

Definition: An asymptote of a curve $y = f(x)$ that has an infinite branch is called a line such that the distance between the point $(x, f(x))$ lying on the curve and the line approaches zero as the point moves along the branch to infinity.

Asymptotes can be *vertical*, *oblique* (*slant*) and *horizontal*. A horizontal asymptote is often considered as a special case of an oblique asymptote.

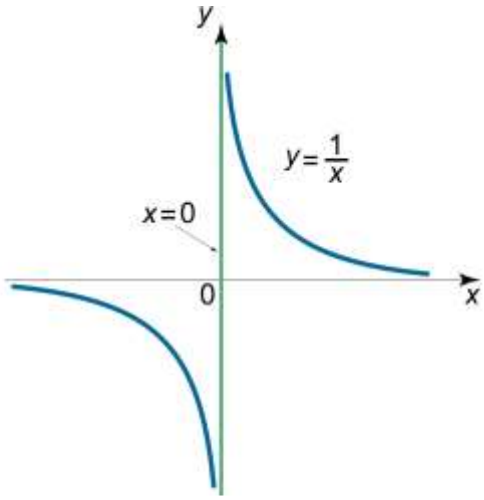
25.4.1 Vertical Asymptote

Definition: The straight line $x = a$ is a *vertical asymptote* of the graph of the function $y = f(x)$ if at least one of the following conditions is true:

$$\lim_{x \to a-0} f(x) = \pm\infty, \ \lim_{x \to a+0} f(x) = \pm\infty.$$

In other words, at least one of the one-sided limits at the point $x = a$ must be equal to infinity.

A vertical asymptote occurs in rational functions at the points when the denominator is zero and the numerator is not equal to zero (i.e. at the points of *discontinuity of the second kind*).

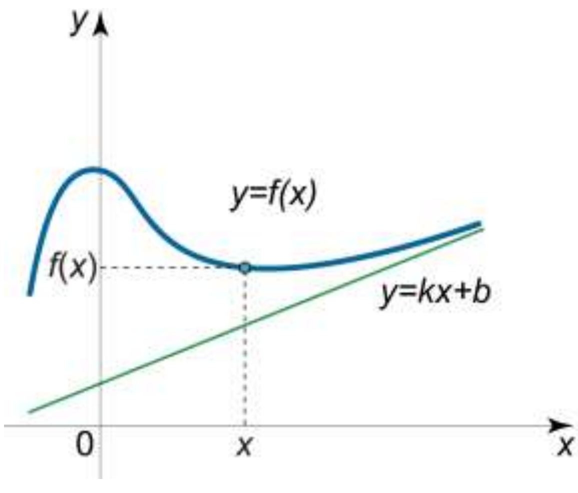


$$\lim_{x \to 0-0} \frac{1}{x} = -\infty, \ \lim_{x \to 0+0} \frac{1}{x} = +\infty.$$

25.4.2 Oblique Asymptote

Definition: The straight line $y = kx + b$ is called an oblique (slant) asymptote of the graph of the function $y = f(x)$ as $x \to +\infty$ (Figure) if

$$\lim_{x \to +\infty} [f(x) - (kx + b)] = 0.$$



Similarly, we introduce oblique asymptotes as $x \to -\infty$.

The oblique asymptotes of the graph of the function $y = f(x)$ may be different as $x \to +\infty$ and $x \to -\infty$

Therefore, when finding oblique (or horizontal) asymptotes, it is a good practice to compute them separately.

The coefficients k and b of an oblique asymptote $y = kx + b$ are defined by the following theorem:

Theorem. A straight line $y = kx + b$ is an asymptote of a function $y = f(x)$ as $x \to +\infty$ if and only if the following two limits are finite:

$$\lim_{x\to+\infty} \frac{f(x)}{x} = k \quad \text{and} \quad \lim_{x\to+\infty} [f(x) - kx] = b.$$

Proof. *Necessity*

A straight line $y = kx + b$ is an asymptote of a graph of a function $y = f(x)$ as $x \to +\infty$. Then the following condition is true:

$$\lim_{x\to+\infty} [f(x) - (kx + b)] = 0$$

or equivalently

$$f(x) = kx + b + \alpha(x), \quad \text{where} \quad \lim_{x\to+\infty} \alpha(x) = 0.$$

Dividing both sides of the equation by x we obtain:

$$\frac{f(x)}{x} = \frac{kx + b + \alpha(x)}{x}, \quad \Rightarrow \quad \frac{f(x)}{x} = k + \frac{b}{x} + \frac{\alpha(x)}{x}.$$

Consequently, in the limit as $x \to +\infty$ we have

$$\lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} [k + \frac{b}{x} + \frac{\alpha(x)}{x}] = k,$$

$$\lim_{x\to+\infty} [f(x) - kx] = \lim_{x\to+\infty} [b + \alpha(x)] = b.$$

Sufficiency. Suppose that there are finite limits

$$\lim_{x\to+\infty} \frac{f(x)}{x} = k \quad \text{and} \quad \lim_{x\to+\infty} [f(x) - kx] = b.$$

The second limit can be written as

$$\lim_{x\to+\infty} [f(x) - (kx + b)] = 0,$$

that meets the definition of an oblique asymptote. Thus, the straight line $y = kx + b$ is an asymptote of the function $y = f(x)$.

Similarly we can prove the theorem for the case of $x \to -\infty$

25.4.3 Horizontal Asymptote

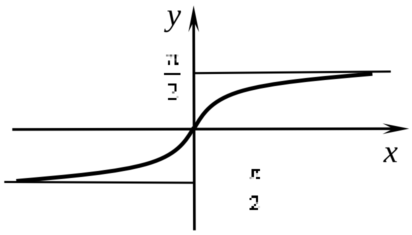
Definition: In particular, if $k = 0$, we obtain *a horizontal asymptote*, which is described by the equation $y = b$. The theorem on necessary and sufficient conditions for the existence of a horizontal asymptote is stated as follows:

Theorem. A straight line $y = b$ is an asymptote of a function $y = f(x)$ as $x \to +\infty$, if and only if the following limit is finite:

$$\lim_{x \to +\infty} f(x) = b.$$

The case $x \to -\infty$ is considered in the same way.

Example. Since $\lim\limits_{x\to-\infty} \arctan x = -\frac{\pi}{2}$, $\lim\limits_{x\to+\infty} \arctan x = \frac{\pi}{2}$, then curve



$y = \arctan x$ has two horizontal asymptotes: $y = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}$, (Fig.).

25.5 Curve Sketching

Now, we summarize all the items considered above together for sketching a function graph. The following steps are taken in the process of curve sketching:

1. *Domain:* Find the domain of the function and determine the points of discontinuity (if any).

2. *Intercepts:* Determine the $x-$ and $y-$intercepts of the function, if possible. To find the x-intercept, we set $y = 0$ and solve the equation for x. Similarly, we set $y = 0$ to find the y-intercept. Find the intervals where the function has a constant sign ($f(x) > 0$ and $f(x) < 0$).

3. *Symmetry:* Determine whether the function is even, odd, or neither, and check the periodicity of the function. If $f(-x) = f(x)$ for all x in the domain, then $f(x)$ is even and symmetric about the $y-$axis. If $f(-x) = -f(x)$ for all x in the domain, then $f(x)$ is odd and symmetric about the

origin.

4. *Asymptotes:* Find the vertical, horizontal and oblique (slant) asymptotes of the function.

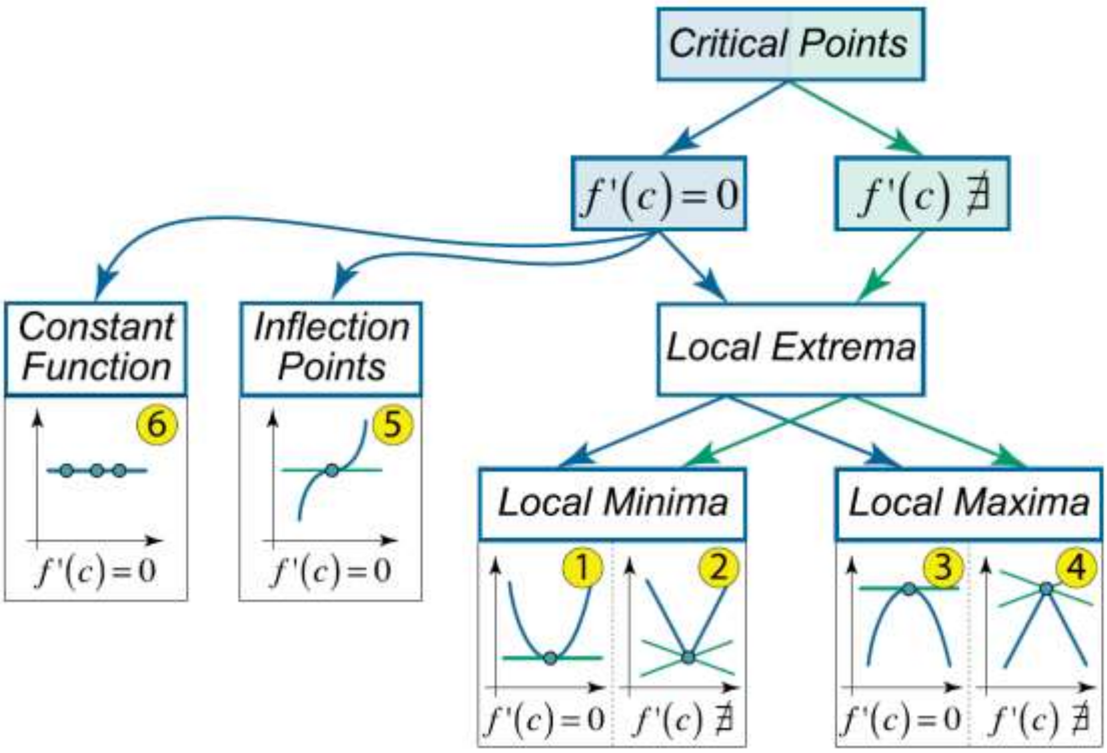
5. *Intervals of Increase and Decrease:* Calculate the first derivative $f'(x)$ and find the critical points of the function. (Remember that critical points are the points where the first derivative is zero or does not exist.) Determine the intervals where the function is increasing and decreasing using the First Derivative Test.

6. *Local Maximum and Minimum:* Use the First or Second Derivative Test to classify the critical points as local maximum or local minimum. Calculate the y −values of the local extrema points.

7. *Concavity/Convexity and Points of Inflection:* Using the Second Derivative Test, find the points of inflection (at which $f''(x) = 0$). Determine the intervals where the function is convex upward ($f''(x) < 0$) and convex downward ($f''(x) > 0$)

8. *Graph of the Function:* Sketch a graph of $f(x)$ using all the information obtained above.

Note. It is useful to classify points calculated during this procedure:



Further we use this algorithm for the investigation of functions.

Example. Let us investigate the function $y = \dfrac{x^3}{3 - x^2}$

1°.Obvious that The function is defined for all real values of x except the points $x = \pm\sqrt{3}$ where it has discontinuities.
2°. Since the equality $f(-x) = -f(x)$:

$$f(-x) = -\frac{x^3}{3 - x^2} = -f(x)$$

is valid, the function is odd, i.e. symmetric with respect to the origin.
3°. Find the y −intercept:

$$y(0) = \frac{0^3}{3 - 0^2} = 0.$$

Find the x −intercepts:

$$\frac{x^3}{3-x^2} = 0 \Rightarrow x = 0$$

4°. Look for vertical asymptote near $x = \pm\sqrt{3}$:

$$\lim_{x \to \sqrt{3}+0} \frac{x^3}{3 - x^2} = -\infty; \quad \lim_{x \to -\sqrt{3}+0} \frac{x^3}{3 - x^2} = -\infty;$$

$$\lim_{x \to \sqrt{3}-0} \frac{x^3}{3 - x^2} = \infty; \quad \lim_{x \to -\sqrt{3}-0} \frac{x^3}{3 - x^2} = \infty.$$

There are two vertical asymptotes at $x = \pm\sqrt{3}$.
5°. To find an oblique asymptote, we need to calculate the following limits:

$$k = \lim_{x \to \pm\infty} \frac{x^3}{(3 - x^2)x} = -1;$$

$$b = \lim_{x \to \pm\infty} \left(\frac{x^3}{(3 - x^2)} + x \right) = \lim_{x \to \pm\infty} \left(\frac{x^3 + 3x - x^3}{(3 - x^2)} \right) = 0.$$

Hence the function has an oblique asymptote $y = -x$.

5°. Let us investigate the function for extremum.

Take the first derivative:

$$y' = \frac{3x^2(3 - x^2) + 2x^4}{(3 - x^2)^2} = \frac{9x^2 - 3x^4 - 2x^4}{(3 - x^2)^2} = \frac{9x^2 - x^4}{(3 - x^2)^2}$$

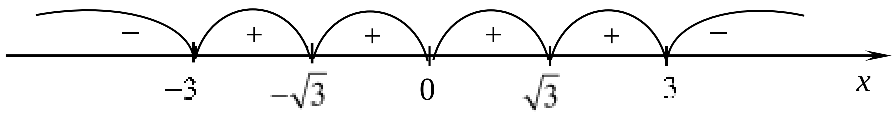
Determine the critical points:

$$y'(x) = 0, \Rightarrow \frac{9x^2 - x^4}{(3 - x^2)^2} = 0 \Rightarrow x^2(9 - x^2) = 0,$$

Thus, the function has two critical points:

$$x_1 = 0, \quad x_2 = 3, \quad x_3 = -3, \quad x_{4,5} = \pm\sqrt{3}$$

Draw a sign chart for the first derivative:



Calculating their y −coordinates, we can illustrate the obtained results in Table 1:

Table 1

x	$(-\infty,-3)$	-3	$(-3,-\sqrt{3})$	$-\sqrt{3}$	$(-\sqrt{3},0)$	0	$(0,\sqrt{3})$	$\sqrt{3}$	$(\sqrt{3},3)$	3	$(3,+\infty)$
y'	–	0	+	Does not exist	+	0	+	Does not exist	+	0	–
y	↘	min $y_{min} =$ 4.5	↗	Does not exist	↗	0	↗	Does not exist	↗	max $y_{max} =$ $-4{,}5$	↘

The point $x = -3$ is a local maximum, and the point $x = 3$ is a local minimum.

6°. To investigate a curve for convexity and concavity, the second derivative is written as

$$y'' = \frac{(18x - 4x^3)(3 - x^2)^2 + 2(3 - x^2)2x(9x^2 - x^4)}{(3 - x^2)^4} =$$

$$= \frac{2x((9 - 2x^2)(3 - x^2) + 2x^2(9 - x^2))}{(3 - x^2)^3} =$$

$$= \frac{2x(27 - 15x^2 + 2x^4 + 18x^2 - 2x^4)}{(3 - x^2)^3} = \frac{2x(27 + 3x^2)}{(3 - x^2)^3} = \frac{6x(9 + x^2)}{(3 - x^2)^3}.$$

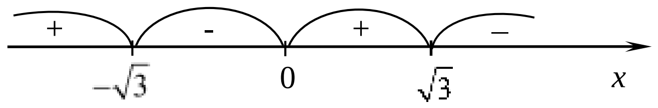
Determine the critical points:

$$y''(x) = 0, \Rightarrow \frac{6x(9 + x^2)}{(3 - x^2)^3} = 0$$

Solving the equation, leads to the points:

$$x_1 = 0, \quad x_{2,3} = \pm\sqrt{3}$$

Draw a sign chart for the second derivative:



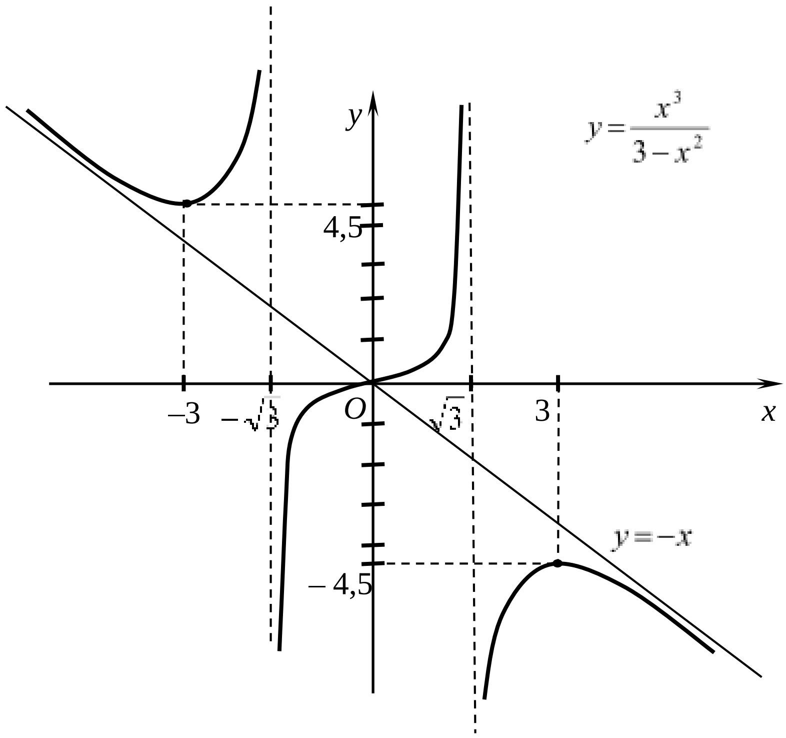
Calculating their y −coordinates, we can illustrate the obtained results in Table 2:

Table 2

x	$(-\infty,-\sqrt{3})$	$-\sqrt{3}$	$(-\sqrt{3},0)$	0	$(0,\sqrt{3})$	$\sqrt{3}$	$(\sqrt{3},\infty)$
y''	+	Does not	–	0	+	Does not	–
y	Concave	exist	Convex	Point of inflection $y=0$	Concave	exist	Convex

The point $x = 0$ is an inflection point,

7°. Now we can sketch a graph of the function:



25.6 Global Extrema of Functions: the greatest and the smallest values of a function on an interval

Given a particular function, we are often interested in determining the largest and smallest values of the function. This information is important in creating accurate graphs. Finding the maximum and minimum values of a function also has practical significance because we can use this method to solve optimization problem

Let a function $f(x)$ be continuous on an interval (a,b). Since it is opened interval then this function $f(x)$ could not reach the largest and smallest values on it. However the following theorem is valid.

Theorem. If a function $f(x)$ has only extremum on an interval (a,b), then corresponding value of this function is either the largest or the smallest value of $f(x)$ on this interval (a,b).

■ Suppose that a function has extremum at the point x_0. And this extremum is maximum (Fig.). Assume that there exists another point $x_1 \in (a,b)$ such that $f(x_1) > f(x_0)$. Let us consider the segment $[x_0, x_1] \subset (a,b)$. By the 2-nd Weierstrass' theorem the function $f(x)$ takes its the smallest value on the interval $[x_0, x_1]$ at some point $x_2 \in [x_0, x_1]$. It is obvious that the point x_2 does not coincide with x_0, and what is more with point x_1, consequently a point x_2 is interior one of the segment $[x_0, x_1]$. But then the function $f(x)$ has minimum at the point x_2, that contrary to extremum unique of this function on the interval (a,b). It proves that the value $f(x_0)$ is the largest value of the function $f(x)$ on the interval (a,b). □

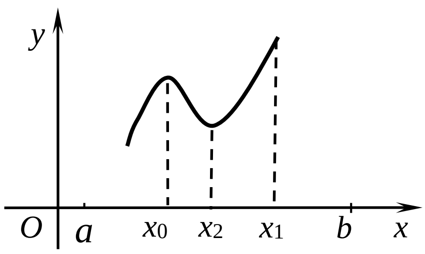


Fig.

Let a function $f(x)$ be continuous on closed interval $[a,b]$. In this case the function reaches its greatest and smallest values on an interval. If a function takes its largest (smallest) value at interior point of an interval, then this point is maximum (minimum) point. But the function could reach the largest (smallest) value at one of the end points of the interval as well.

Consequently, from the foregoing we get the following rule: if it is required to find the greatest $M = \max\limits_{[a,b]} f(x)$ and the smallest $m = \min\limits_{[a,b]} f(x)$ values of continuous function on closed interval $[a,b]$ the following steps should be done:

1. Find all maxima and minima of a function on the interval.
2. Calculate the values of the function $f(x)$ at the end points *a* and *b*, that is, calculate $f(a), f(b)$.
3. Choose the greatest and the smallest values of all the obtained ones.

Example. Determine the greatest and the smallest values of the function $y = 2x^3 + 3x^2 - 12x + 1$ on the interval $[-4,2]$.

Solution. The first let us find the stationary points

$$y' = 6x^2 + 6x - 12.$$

$$y' = 0,$$

or

$$x^2 + x - 2 = 0,$$

roots of this equation are: $x_1 = 1,\ x_2 = -2$.

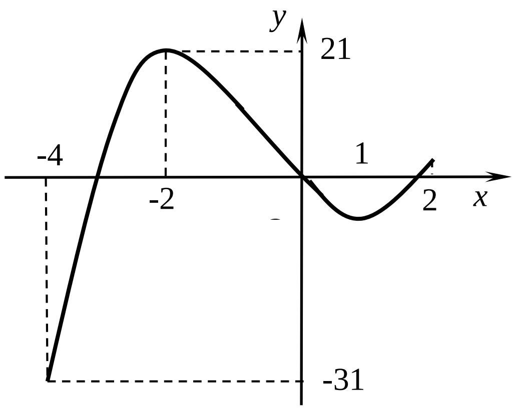


Fig.

Calculate the values:

$$f(1) = -6,\ f(-2) = 21,\ f(-4) = -31,\ f(2) = 5.$$

Hence $\max f(x)\big|_{x=-2} = 21$, $\min f(x)\big|_{x=-4} = -31$ (on Fig. for conveniently it is taken different scales on axis Ox and Oy).

Note. Let us indicate separate cases when the maximum (minimum) value at the stationary point is obviously the greatest (the smallest) one on the given interval (finite or infinite). Let the following information be known.[*)]

[*)] This and similar situation appears in many applied problems.

1. A function $f(x)$ vanishes at end points of the interval $[a,b]$.

2. $f(x)>0$ for all $x\in[a,b]$.

3. A function is differentiable at all interior points of the interval $[a,b]$.

4. There is only stationary point x_0 within of the interval $[a,b]$.

Then it is obvious that $f(x_0)$ is the greatest value $f(x)$ on the interval $[a,b]$ (Fig. 1). Another similar case is possible Fig. 2:

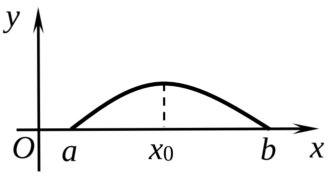


Fig. 1

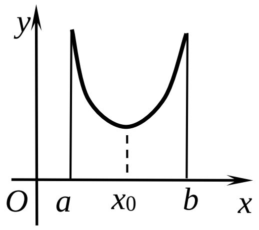


Fig. 2

$$\lim_{x\to a+0} f(x)=+\infty, \quad \lim_{x\to b-0} f(x)=+\infty;$$

conditions 3 и 4 are fulfilled from previous case. Then $f(x_0)$ is the smallest value of the function $f(x)$ on the interval $[a,b]$.