1. The Chain Rule

If f and g are differentiable functions, then the composite function $f(g(x))$ is also differentiable in x and its derivative is given by

$$\frac{dy}{dx} = \frac{d}{dx}(f(x)) = \frac{d}{dx}f(g(x)) \cdot g'(x) = \frac{df}{du} \cdot \frac{du}{dx}.$$

where $u = g(x)$ is an inner function, and $y = f(u)$ is an outer function.

Example 1. $y = \ln x^2$

$$y'(x) = (\ln x^2)' = \frac{1}{x^2} \cdot (x^2)' = \frac{1}{x^2} \cdot 2x = \frac{2x}{x^2} = \frac{2}{x}(x \neq 0).$$

Example 2. $y = \ln^2 x$

$$y'(x) = (\ln^2 x)' = 2\ln x \cdot (\ln x)' = 2\ln x \cdot \frac{1}{x} = \frac{2\ln x}{x}(x > 0).$$

Example 3. $y = \cos(3x + 2)$

$$y'(x) = [\cos(3x + 2)]' = -\sin(3x + 2) \cdot (3x + 2)' = -3\sin(3x + 2).$$

Example 4. $y = \cos^4 x$

$$y'(x) = (\cos^4 x)' = 4\cos^3 x \cdot (\cos x)' = 4\cos^3 x \cdot (-\sin x)$$
$$= -4\cos^3 x \sin x.$$

Example 5. $y = 3^{\cos x}$

$$y'(x) = (3^{\cos x})' = 3^{\cos x} \cdot \ln 3 \cdot (\cos x)' = -3^{\cos x} \ln 3 \sin x.$$

Example 6. $y = (x\ln x)^2$

$$y' = [(x\ln x)^2]' = 2(x\ln x) \cdot (x\ln x)' = 2x\ln x \cdot [x'\ln x + x(\ln x)']$$
$$= 2x\ln x \cdot [\ln x + x \cdot \frac{1}{x}] = 2x\ln x(\ln x + 1).$$

Example 8. $y = \sin x^3 \cos x^2$

$$y'(x) = (\sin x^3 \cos x^2)' = (\sin x^3)' \cos x^2 + \sin x^3 (\cos x^2)'$$
$$= \cos x^3 \cdot (x^3)' \cdot \cos x^2 + \sin x^3 \cdot (-\sin x^2) \cdot (x^2)'$$
$$= \cos x^3 \cdot 3x^2 \cdot \cos x^2 - \sin x^3 \cdot \sin x^2 \cdot 2x$$
$$= 3x^2 \cos x^3 \cos x^2 - 2x \sin x^3 \sin x^2.$$

Example 9. $y = \log_5 \sin 2x$

$$y'(x) = (\log_5 \sin 2x)' = \frac{1}{\ln 5 \sin 2x} \cdot (\sin 2x)' = \frac{1}{\ln 5 \sin 2x} \cdot \cos 2x \cdot 2$$
$$= \frac{2\cos 2x}{\ln 5 \sin 2x} = \frac{2\cot 2x}{\ln 5}.$$

2. *Implicit Differentiation*

In many problems, however, the function can be defined in implicit form, that is by the equation $F(x, y) = 0$.

1. Differentiate both sides of the equation with respect to *x*, assuming that *y* is a differentiable function of *x* and using the chain rule. The derivative of zero (in the right side) will also be equal to zero.
2. Solve the resulting equation for the derivative $y'(x)$

Example 1. $x^3 + 2y^3 + yx^2 = 3$

Differentiate both sides term-by-term with respect to *x*

$$(x^3)' + (2y^3)' + (yx^2)' = 3', \Rightarrow 3x^2 + 6y^2 y' + y' x^2 + 2yx = 0.$$

Solve this equation for y'

$$6y^2 y' + y' x^2 = -(3x^2 + 2yx), \Rightarrow y'(x^2 + 6y^2) = -(3x^2 + 2yx), \Rightarrow$$
$$y' = -\frac{3x^2 + 2yx}{x^2 + 6y^2}.$$

Example 2. Calculate the derivative at the point (0, 0) of the function given by the equation $x^5+y^5-2x+2y=0$.

We differentiate this equation with respect to *x* and solve for y'

$$(x^5)'+(y^5)'-(2x)'+(2y)'=0',\Rightarrow$$
$$5x^4+5y^4y'-2+2y'=0,\Rightarrow$$
$$(5y^4+2)y'=2-5x^4,\Rightarrow$$
$$y'=\frac{2-5x^4}{2+5y^4}.$$

Substitute the coordinates $x=0$, $y=0$,

$$y'(0,0)=\frac{2-5\cdot 0^4}{2+5\cdot 0^4}=\frac{2}{2}=1.$$

Example 3. $3^x+3^y=3^{x+y}$

Differentiate both sides and solve the resulting equation for y'

$$(3^x+3^y)'=(3^{x+y})',\Rightarrow$$
$$3^x\ln 3+3^y\ln 3\cdot y'=3^{x+y}\ln 3\cdot(x+y)',\Rightarrow$$
$$3^x+3^y y'=3^{x+y}(1+y'),\Rightarrow$$
$$3^x+3^y y'=3^{x+y}+3^{x+y}y',\Rightarrow$$
$$3^y y'-3^{x+y}y'=3^{x+y}-3^x,\Rightarrow$$
$$3^y y'(1-3^x)=3^x(3^y-1),\Rightarrow$$
$$y'=-\frac{3^x(3^y-1)}{3^y(3^x-1)}.$$

Example 4. $x\sin y+y\cos x=0$

As usual, we differentiate both sides with respect to *x*

$$(x\sin y+y\cos x)'=0,\Rightarrow (x\sin y)'+(y\cos x)'=0.$$

Using the product rule for the derivative, we have:

$$x'\sin y+x(\sin y)'+y'\cos x+y(\cos x)'=0,\Rightarrow$$
$$1\cdot\sin y+x\cdot\cos y\cdot y'+y'\cdot\cos x+y\cdot(-\sin x)=0,\Rightarrow$$

$$\sin y + x\cos y \cdot y' + \cos x \cdot y' - y\sin x = 0, \Rightarrow$$
$$y'(x\cos y + \cos x) = y\sin x - \sin y, \Rightarrow$$
$$y' = \frac{y\sin x - \sin y}{x\cos y + \cos x}.$$

Example 5. $2^{\frac{x}{y}} = \frac{x^2}{y^2} (y \neq 0)$.

$$(2^{\frac{x}{y}})' = (\frac{x^2}{y^2})', \Rightarrow$$
$$2^{\frac{x}{y}} \ln 2 \cdot (\frac{x}{y})' = \frac{(x^2)'y^2 - x^2(y^2)'}{y^4}, \Rightarrow$$
$$2^{\frac{x}{y}} \ln 2 \cdot \frac{x'y - xy'}{y^2} = \frac{2x \cdot y^2 - x^2 \cdot 2y \cdot y'}{y^4}, \Rightarrow$$
$$2^{\frac{x}{y}} \ln 2 \cdot \frac{y - xy'}{y^2} = \frac{2xy(y - xy')}{y^4}, \Rightarrow$$
$$2^{\frac{x}{y}} \ln 2 (y - xy') = \frac{2x}{y}(y - xy'), \Rightarrow$$
$$(2^{\frac{x}{y}} \ln 2 - \frac{2x}{y}) \cdot (y - xy') = 0.$$

From the last equation we find the derivative:

$$y - xy' = 0, \Rightarrow y' = \frac{y}{x}, \quad \text{where} \quad x \neq 0.$$

3. Logarithmic Differentiation

Consider this method in more detail. Let $y = f(x)$. Take natural logarithms of both sides:

$$\ln y = \ln f(x).$$

Next, we differentiate this expression using the chain rule and keeping in mind that *y* is a function of *x*.

Example 1. $y = x^{\cos x}, x > 0.$

Take the logarithm of the given function:

$$\ln y = \ln(x^{\cos x}), \Rightarrow \ln y = \cos x \ln x.$$

Differentiating the last equation with respect to *x* we obtain:

$$(\ln y)' = (\cos x \ln x)', \Rightarrow$$

$$\frac{1}{y} \cdot y' = (\cos x)' \ln x + \cos x (\ln x)', \Rightarrow$$

$$\frac{y'}{y} = (-\sin x) \cdot \ln x + \cos x \cdot \frac{1}{x}, \Rightarrow$$

$$\frac{y'}{y} = -\sin x \ln x + \frac{\cos x}{x}, \Rightarrow$$

$$y' = y(\frac{\cos x}{x} - \sin x \ln x).$$

Example 2. $y = (\sin x)^{\arctan x}$

$$\ln y = \ln[(\sin x)^{\arctan x}], \Rightarrow \ln y = \arctan x \ln \sin x.$$

Differentiating both sides in *x* we find the derivative:

$$(\ln y)' = (\arctan x \ln \sin x)', \Rightarrow$$

$$\frac{1}{y} \cdot y' = (\arctan x)' \cdot \ln \sin x + \arctan x \cdot (\ln \sin x)', \Rightarrow$$

$$\frac{y'}{y} = \frac{1}{1 + x^2} \cdot \ln \sin x + \arctan x \cdot \frac{1}{\sin x} \cdot (\sin x)', \Rightarrow$$

$$\frac{y'}{y} = \frac{\ln \sin x}{1 + x^2} + \arctan x \cot x, \Rightarrow$$

$$y' = y \cdot (\frac{\ln \sin x}{1 + x^2} + \arctan x \cot x) \Rightarrow$$

$$\text{or } y' = (\sin x)^{\arctan x} (\frac{\ln \sin x}{1 + x^2} + \arctan x \cot x).$$

Example 3. $y = (x-1)^2(x-3)^5$

First we take logarithms of both sides:

$$\ln y = \ln[(x-1)^2(x-3)^5], \Rightarrow$$

$$\ln y = \ln(x-1)^2 + \ln(x-3)^5, \Rightarrow$$

$$\ln y = 2\ln(x-1) + 5\ln(x-3).$$

Now it is easy to find the logarithmic derivative:

$$(\ln y)' = [2\ln(x-1) + 5\ln(x-3)]', \Rightarrow$$

$$\frac{1}{y} \cdot y' = 2 \cdot \frac{1}{x-1} + 5 \cdot \frac{1}{x-3}, \Rightarrow$$

$$y' = y(\frac{2}{x-1} + \frac{5}{x-3}), \Rightarrow$$

$$y' = (x-1)^2(x-3)^5 \cdot (\frac{2}{x-1} + \frac{5}{x-3}).$$

Example 4. $y(x) = \frac{(x+1)^2}{(x+2)^3(x+3)^4}$,

Take logarithms of both sides:

$$\ln y = \ln \frac{(x+1)^2}{(x+2)^3(x+3)^4}, \Rightarrow$$

$$\ln y = \ln(x+1)^2 - \ln(x+2)^3 - \ln(x+3)^4, \Rightarrow$$

$$\ln y = 2\ln(x+1) - 3\ln(x+2) - 4\ln(x+3).$$

Now differentiate the left and right sides:

$$\frac{y'}{y} = \frac{2}{x+1} - \frac{3}{x+2} - \frac{4}{x+3}, \Rightarrow$$

$$y' = y \cdot (\frac{2}{x+1} - \frac{3}{x+2} - \frac{4}{x+3}), \Rightarrow$$

$$y' = \frac{(x+1)^2}{(x+2)^3(x+3)^4} \cdot (\frac{2}{x+1} - \frac{3}{x+2} - \frac{4}{x+3}).$$

4. *Derivatives of Parametric Functions*

The relationship between the variables x and y can be defined in parametric form using two equations:

$$\begin{cases} x & = x(t) \\ y & = y(t)' \end{cases}$$

Then its derivative is given by

$$y'_x = y'_t \cdot t'_x = y'_t \cdot \frac{1}{x'_t} = \frac{y'_t}{x'_t}.$$

Example 1. $x = e^{2t}, y = e^{3t}$.

$$x'_t = (e^{2t})' = 2e^{2t}, y'_t = (e^{3t})' = 3e^{3t}.$$

Hence,

$$\frac{dy}{dx} = y'_x = \frac{y'_t}{x'_t} = \frac{3e^{3t}}{2e^{2t}} = \frac{3}{2}e^{3t-2t} = \frac{3}{2}e^t.$$

Example 2. $x = \sin^2 t, y = \cos^2 t$.

Differentiate with respect to the parameter

$$x'_t = (\sin^2 t)' = 2\sin t \cdot \cos t = \sin 2t,$$

$$y'_t = (\cos^2 t)' = 2\cos t \cdot (-\sin t) = -2\sin t\cos t = -\sin 2t.$$

Then

$$\frac{dy}{dx} = y'_x = \frac{y'_t}{x'_t} = \frac{-\sin 2t}{\sin 2t} = -1, \text{where } t \neq \frac{\pi n}{2}, n \in \mathbb{Z}.$$

Example 3. Find the derivative $\frac{dy}{dx}$ for the function $x = \sin 2t$, $y = -\cos t$ at the point $t = \frac{\pi}{6}$.

Compute the derivatives with respect to t

$$x'_t = (\sin 2t)' = 2\cos 2t, y'_t = (-\cos t)' = \sin t.$$

So, the derivative is given by

$$\frac{dy}{dx} = \frac{y'_t}{x'_t} = \frac{2\cos 2t}{\sin t}.$$

Compute the derivative at $t = \frac{\pi}{6}$.

$$\frac{dy}{dx}\left(t = \frac{\pi}{6}\right) = \frac{2\cos\left(2 \cdot \frac{\pi}{6}\right)}{\sin\frac{\pi}{6}} = \frac{2\cos\frac{\pi}{3}}{\sin\frac{\pi}{6}} = \frac{2 \cdot \frac{1}{2}}{\frac{1}{2}} = 2.$$

5. The Chain Rule

For approximate calculations one sometimes uses the approximate equation

$$\Delta y \approx dy$$

or in expanded form

$$f(x+\Delta x) - f(x) \approx f'(x)\Delta x.$$

or

$$f(x+\Delta x) \approx f(x) + f'(x)\Delta x.$$

Example 1. Let us calculate the approximate value of $\sin 46^{\circ}$.

Let $f(x) = \sin x$, then $f'(x) = \cos x$.

In this case the approximate equation takes the form

$$\sin(x+\Delta x) \approx \sin x + \cos x \Delta x.$$

Setting $x = 45^{\circ} = \frac{\pi}{4}$, $\Delta x = 1^{\circ} = \frac{\pi}{180}$, and $x + \Delta x = \frac{\pi}{4} + \frac{\pi}{180}$.

Substituting all these into the equation we get

$$\sin 46^{\circ} \approx \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\frac{\pi}{180} \approx 0.7071 + 0.7071 \cdot 0.0175 = 0.7191.$$