

1. The Chain Rule

If f and g are differentiable functions, then the composite function $f(g(x))$ is also differentiable in x and its derivative is given by

$$\frac{dy}{dx} = \frac{d}{dx}(f(x)) = \frac{d}{dx}f(g(x)) \cdot g'(x) = \frac{df}{du} \cdot \frac{du}{dx}$$

where $u = g(x)$ is an inner function, and $y = f(u)$ is an outer function.

Example 1. $y = \ln x^2$

$$y'(x) = (\ln x^2)' = \frac{1}{x^2} \cdot (x^2)' = \frac{1}{x^2} \cdot 2x = \frac{2x}{x^2} = \frac{2}{x} \quad (x \neq 0).$$

Example 2. $y = \ln^2 x$

$$y'(x) = (\ln^2 x)' = 2 \ln x \cdot (\ln x)' = 2 \ln x \cdot \frac{1}{x} = \frac{2 \ln x}{x} \quad (x > 0).$$

Example 3. $y = \cos(3x + 2)$

$$y'(x) = [\cos(3x + 2)]' = -\sin(3x + 2) \cdot (3x + 2)' = -3 \sin(3x + 2).$$

Example 4. $y = \cos^4 x$

$$\begin{aligned} y'(x) &= (\cos^4 x)' = 4 \cos^3 x \cdot (\cos x)' = 4 \cos^3 x \cdot (-\sin x) \\ &= -4 \cos^3 x \sin x. \end{aligned}$$

Example 5. $y = 3^{\cos x}$

$$y'(x) = (3^{\cos x})' = 3^{\cos x} \cdot \ln 3 \cdot (\cos x)' = -3^{\cos x} \ln 3 \sin x.$$

Example 6. $y = (x \ln x)^2$

$$\begin{aligned} y' &= [(x \ln x)^2]' = 2(x \ln x) \cdot (x \ln x)' = 2x \ln x \cdot [x' \ln x + x(\ln x)'] \\ &= 2x \ln x \cdot \left[\ln x + x \cdot \frac{1}{x} \right] = 2x \ln x (\ln x + 1). \end{aligned}$$

Example 8. $y = \sin x^3 \cos x^2$

$$\begin{aligned}y'(x) &= (\sin x^3 \cos x^2)' = (\sin x^3)' \cos x^2 + \sin x^3 (\cos x^2)' \\&= \cos x^3 \cdot (x^3)' \cdot \cos x^2 + \sin x^3 \cdot (-\sin x^2) \cdot (x^2)' \\&= \cos x^3 \cdot 3x^2 \cdot \cos x^2 - \sin x^3 \cdot \sin x^2 \cdot 2x \\&= 3x^2 \cos x^3 \cos x^2 - 2x \sin x^3 \sin x^2.\end{aligned}$$

Example 9. $y = \log_5 \sin 2x$

$$\begin{aligned}y'(x) &= (\log_5 \sin 2x)' = \frac{1}{\ln 5 \sin 2x} \cdot (\sin 2x)' = \frac{1}{\ln 5 \sin 2x} \cdot \cos 2x \cdot 2 \\&= \frac{2 \cos 2x}{\ln 5 \sin 2x} = \frac{2 \cot 2x}{\ln 5}.\end{aligned}$$

2. Implicit Differentiation

In many problems, however, the function can be defined in implicit form, that is by the equation $F(x, y) = 0$.

1. Differentiate both sides of the equation with respect to x , assuming that y is a differentiable function of x and using the chain rule. The derivative of zero (in the right side) will also be equal to zero.
2. Solve the resulting equation for the derivative $y'(x)$

Example 1. $x^3 + 2y^3 + yx^2 = 3$

Differentiate both sides term-by-term with respect to x

$$(x^3)' + (2y^3)' + (yx^2)' = 3', \Rightarrow 3x^2 + 6y^2 y' + y'x^2 + 2yx = 0.$$

Solve this equation for y'

$$\begin{aligned}6y^2 y' + y'x^2 &= -(3x^2 + 2yx), \Rightarrow y'(x^2 + 6y^2) = -(3x^2 + 2yx), \Rightarrow \\y' &= -\frac{3x^2 + 2yx}{x^2 + 6y^2}.\end{aligned}$$

Example 2. Calculate the derivative at the point $(0, 0)$ of the function given by

the equation $x^5 + y^5 - 2x + 2y = 0$.

We differentiate this equation with respect to x and solve for y'

$$\begin{aligned}(x^5)' + (y^5)' - (2x)' + (2y)' &= 0', \Rightarrow \\ 5x^4 + 5y^4y' - 2 + 2y' &= 0, \Rightarrow \\ (5y^4 + 2)y' &= 2 - 5x^4, \Rightarrow \\ y' &= \frac{2 - 5x^4}{2 + 5y^4}.\end{aligned}$$

Substitute the coordinates $x = 0, y = 0$,

$$y'(0,0) = \frac{2 - 5 \cdot 0^4}{2 + 5 \cdot 0^4} = \frac{2}{2} = 1.$$

Example 3. $3^x + 3^y = 3^{x+y}$

Differentiate both sides and solve the resulting equation for y'

$$\begin{aligned}(3^x + 3^y)' &= (3^{x+y})', \Rightarrow \\ 3^x \ln 3 + 3^y \ln 3 \cdot y' &= 3^{x+y} \ln 3 \cdot (x + y)', \Rightarrow \\ 3^x + 3^y y' &= 3^{x+y} (1 + y'), \Rightarrow \\ 3^x + 3^y y' &= 3^{x+y} + 3^{x+y} y', \Rightarrow \\ 3^y y' - 3^{x+y} y' &= 3^{x+y} - 3^x, \Rightarrow \\ 3^y y' (1 - 3^x) &= 3^x (3^y - 1), \Rightarrow \\ y' &= -\frac{3^x (3^y - 1)}{3^y (3^x - 1)}.\end{aligned}$$

Example 4. $x \sin y + y \cos x = 0$

As usual, we differentiate both sides with respect to x

$$(x \sin y + y \cos x)' = 0, \Rightarrow (x \sin y)' + (y \cos x)' = 0.$$

Using the product rule for the derivative, we have:

$$\begin{aligned}x' \sin y + x(\sin y)' + y' \cos x + y(\cos x)' &= 0, \Rightarrow \\ 1 \cdot \sin y + x \cdot \cos y \cdot y' + y' \cdot \cos x + y \cdot (-\sin x) &= 0, \Rightarrow\end{aligned}$$

$$\begin{aligned}\sin y + x \cos y \cdot y' + \cos x \cdot y' - y \sin x &= 0, \Rightarrow \\ y'(x \cos y + \cos x) &= y \sin x - \sin y, \Rightarrow \\ y' &= \frac{y \sin x - \sin y}{x \cos y + \cos x}.\end{aligned}$$

Example 5. $2^{\frac{x}{y}} = \frac{x^2}{y^2} (y \neq 0).$

$$\begin{aligned}\left(2^{\frac{x}{y}}\right)' &= \left(\frac{x^2}{y^2}\right)', \Rightarrow \\ 2^{\frac{x}{y}} \ln 2 \cdot \left(\frac{x}{y}\right)' &= \frac{(x^2)'y^2 - x^2(y^2)'}{y^4}, \Rightarrow \\ 2^{\frac{x}{y}} \ln 2 \cdot \frac{x'y - xy'}{y^2} &= \frac{2x \cdot y^2 - x^2 \cdot 2y \cdot y'}{y^4}, \Rightarrow \\ 2^{\frac{x}{y}} \ln 2 \cdot \frac{y - xy'}{y^2} &= \frac{2xy(y - xy')}{y^4}, \Rightarrow \\ 2^{\frac{x}{y}} \ln 2 (y - xy') &= \frac{2x}{y} (y - xy'), \Rightarrow \\ \left(2^{\frac{x}{y}} \ln 2 - \frac{2x}{y}\right) \cdot (y - xy') &= 0.\end{aligned}$$

From the last equation we find the derivative:

$$y - xy' = 0, \Rightarrow y' = \frac{y}{x}, \quad \text{where } x \neq 0.$$

3. Logarithmic Differentiation

Consider this method in more detail. Let $y = f(x)$. Take natural logarithms of both sides:

$$\ln y = \ln f(x).$$

Next, we differentiate this expression using the chain rule and keeping in mind that y is a function of x .

Example 1. $y = x^{\cos x}, x > 0.$

Take the logarithm of the given function:

$$\ln y = \ln(x^{\cos x}), \Rightarrow \ln y = \cos x \ln x.$$

Differentiating the last equation with respect to x we obtain:

$$(\ln y)' = (\cos x \ln x)', \Rightarrow$$

$$\frac{1}{y} \cdot y' = (\cos x)' \ln x + \cos x (\ln x)', \Rightarrow$$

$$\frac{y'}{y} = (-\sin x) \cdot \ln x + \cos x \cdot \frac{1}{x}, \Rightarrow$$

$$\frac{y'}{y} = -\sin x \ln x + \frac{\cos x}{x}, \Rightarrow$$

$$y' = y \left(\frac{\cos x}{x} - \sin x \ln x \right).$$

Example 2. $y = (\sin x)^{\arctan x}$

$$\ln y = \ln[(\sin x)^{\arctan x}], \Rightarrow \ln y = \arctan x \ln \sin x.$$

Differentiating both sides in x we find the derivative:

$$(\ln y)' = (\arctan x \ln \sin x)', \Rightarrow$$

$$\frac{1}{y} \cdot y' = (\arctan x)' \cdot \ln \sin x + \arctan x \cdot (\ln \sin x)', \Rightarrow$$

$$\frac{y'}{y} = \frac{1}{1+x^2} \cdot \ln \sin x + \arctan x \cdot \frac{1}{\sin x} \cdot (\sin x)', \Rightarrow$$

$$\frac{y'}{y} = \frac{\ln \sin x}{1+x^2} + \arctan x \cot x, \Rightarrow$$

$$y' = y \cdot \left(\frac{\ln \sin x}{1+x^2} + \arctan x \cot x \right) \Rightarrow$$

$$\text{or } y' = (\sin x)^{\arctan x} \left(\frac{\ln \sin x}{1+x^2} + \arctan x \cot x \right).$$

Example 3. $y = (x - 1)^2(x - 3)^5$

First we take logarithms of both sides:

$$\begin{aligned}\ln y &= \ln[(x - 1)^2(x - 3)^5], \Rightarrow \\ \ln y &= \ln(x - 1)^2 + \ln(x - 3)^5, \Rightarrow \\ \ln y &= 2\ln(x - 1) + 5\ln(x - 3).\end{aligned}$$

Now it is easy to find the logarithmic derivative:

$$\begin{aligned}(\ln y)' &= [2\ln(x - 1) + 5\ln(x - 3)]', \Rightarrow \\ \frac{1}{y} \cdot y' &= 2 \cdot \frac{1}{x - 1} + 5 \cdot \frac{1}{x - 3}, \Rightarrow \\ y' &= y \left(\frac{2}{x - 1} + \frac{5}{x - 3} \right), \Rightarrow \\ y' &= (x - 1)^2(x - 3)^5 \cdot \left(\frac{2}{x - 1} + \frac{5}{x - 3} \right).\end{aligned}$$

Example 4. $y(x) = \frac{(x+1)^2}{(x+2)^3(x+3)^4}$

Take logarithms of both sides:

$$\begin{aligned}\ln y &= \ln \frac{(x + 1)^2}{(x + 2)^3(x + 3)^4}, \Rightarrow \\ \ln y &= \ln(x + 1)^2 - \ln(x + 2)^3 - \ln(x + 3)^4, \Rightarrow \\ \ln y &= 2\ln(x + 1) - 3\ln(x + 2) - 4\ln(x + 3).\end{aligned}$$

Now differentiate the left and right sides:

$$\begin{aligned}\frac{y'}{y} &= \frac{2}{x + 1} - \frac{3}{x + 2} - \frac{4}{x + 3}, \Rightarrow \\ y' &= y \cdot \left(\frac{2}{x + 1} - \frac{3}{x + 2} - \frac{4}{x + 3} \right), \Rightarrow \\ y' &= \frac{(x + 1)^2}{(x + 2)^3(x + 3)^4} \cdot \left(\frac{2}{x + 1} - \frac{3}{x + 2} - \frac{4}{x + 3} \right).\end{aligned}$$

4. Derivatives of Parametric Functions

The relationship between the variables x and y can be defined in parametric form using two equations:

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

Then its derivative is given by

$$y'_x = y'_t \cdot t'_x = y'_t \cdot \frac{1}{x'_t} = \frac{y'_t}{x'_t}$$

Example 1. $x = e^{2t}, y = e^{3t}$.

$$x'_t = (e^{2t})' = 2e^{2t}, y'_t = (e^{3t})' = 3e^{3t}.$$

Hence,

$$\frac{dy}{dx} = y'_x = \frac{y'_t}{x'_t} = \frac{3e^{3t}}{2e^{2t}} = \frac{3}{2}e^{3t-2t} = \frac{3}{2}e^t.$$

Example 2. $x = \sin^2 t, y = \cos^2 t$.

Differentiate with respect to the parameter

$$x'_t = (\sin^2 t)' = 2\sin t \cdot \cos t = \sin 2t,$$

$$y'_t = (\cos^2 t)' = 2\cos t \cdot (-\sin t) = -2\sin t \cos t = -\sin 2t.$$

Then

$$\frac{dy}{dx} = y'_x = \frac{y'_t}{x'_t} = \frac{-\sin 2t}{\sin 2t} = -1, \text{ where } t \neq \frac{\pi n}{2}, n \in \mathbb{Z}.$$

Example 3. Find the derivative $\frac{dy}{dx}$ for the function $x = \sin 2t, y = -\cos t$ at the point $t = \frac{\pi}{6}$.

Compute the derivatives with respect to t

$$x'_t = (\sin 2t)' = 2\cos 2t, y'_t = (-\cos t)' = \sin t.$$

So, the derivative is given by

$$\frac{dy}{dx} = \frac{y'_t}{x'_t} = \frac{2\cos 2t}{\sin t}.$$

Compute the derivative at $t = \frac{\pi}{6}$.

$$\frac{dy}{dx} \left(t = \frac{\pi}{6} \right) = \frac{2\cos\left(2 \cdot \frac{\pi}{6}\right)}{\sin \frac{\pi}{6}} = \frac{2\cos \frac{\pi}{3}}{\sin \frac{\pi}{6}} = \frac{2 \cdot \frac{1}{2}}{\frac{1}{2}} = 2.$$