1. *Differential of a Function*

The differential of a function $y = f(x)$ has the following form:

$$dy = y'dx = f'(x)dx$$

Example 1: Find the differential of the function $y = \cot\frac{\pi x}{4}$ at the point $x = 1$.

Determine the derivative of the given function:

$$y' = (\cot\frac{\pi x}{4})' = -\frac{1}{\sin^2(\frac{\pi x}{4})} \cdot \frac{\pi}{4} = -\frac{\pi}{4\sin^2(\frac{\pi x}{4})},$$

$$\Rightarrow y'(1) = -\frac{\pi}{4\sin^2(\frac{\pi}{4})} = -\frac{\pi}{4(\frac{\sqrt{2}}{2})^2} = -\frac{\pi}{2}.$$

The differential has the following form:

$$dy = y'dx = -\frac{\pi}{2}dx.$$

Example 2: Find the differential of the function $y = x^3 - 3x^2 + 4x$ at the point $x = 1$ when $dx = 0.1$.

$$f'(x) = (x^3 - 3x^2 + 4x)' = 3x^2 - 6x + 4.$$

$$dy = f'(x)dx = (3x^2 - 6x + 4)dx.$$

Substituting the given values, we calculate the differential:

$$dy = (3 \cdot 1^2 - 6 \cdot 1 + 4) \cdot 0{,}1 = 0{,}1$$

For approximate calculations one sometimes uses the approximate equation

$$\Delta y \approx dy$$

or in expanded form

$$f(x + \Delta x) - f(x) \approx f'(x)\Delta x.$$

or

$$f(x + \Delta x) \approx f(x) + f'(x)\Delta x.$$

Example 3: Use differential to approximate the change in $y = \frac{1}{\sin x}$ as x changes from $\frac{\pi}{4}$ to $\frac{3\pi}{10}$.

The differential dy is defined by the formula

$$dy = y'dx = y'(\frac{\pi}{4})dx.$$

Take the derivative

$$y' = (\frac{1}{\sin x})' = -\frac{1}{(\sin x)^2} \cdot (\sin x)' = -\frac{\cos x}{\sin^2 x}.$$

So,

$$y'(\frac{\pi}{4}) = -\frac{\cos\frac{\pi}{4}}{\sin^2\frac{\pi}{4}} = -\frac{\frac{\sqrt{2}}{2}}{(\frac{\sqrt{2}}{2})^2} = -\frac{2}{\sqrt{2}} = -\sqrt{2}.$$

Calculate the differential dx:

$$dx = \frac{3\pi}{10} - \frac{\pi}{4} = \frac{6\pi - 5\pi}{20} = \frac{\pi}{20}.$$

Hence,

$$dy = y'(\frac{\pi}{4})dx = -\sqrt{2} \cdot \frac{\pi}{20} = -\frac{\sqrt{2}\pi}{20}.$$

The approximate value of the function at $x = \frac{3\pi}{10}$ is

$$y(\frac{3\pi}{10}) \approx y(\frac{\pi}{4}) + dy = \frac{1}{\sin\frac{\pi}{4}} - \frac{\sqrt{2}\pi}{20} = \frac{1}{\frac{\sqrt{2}}{2}} - \frac{\sqrt{2}\pi}{20} = \sqrt{2} - \frac{\sqrt{2}\pi}{20}$$

$$= \frac{\sqrt{2}}{20}(20 - \pi).$$

Example 4. Find the differential of the function $y = \sqrt{x^3 + 4x}$ at a point $x = 2$

Differentiate the given function:

$$y' = (\sqrt{x^3 + 4x})' = \frac{1}{2\sqrt{x^3 + 4x}} \cdot (x^3 + 4x)' = \frac{3x^2 + 4}{2\sqrt{x^3 + 4x}}.$$

At the point $x = 2$ the derivative is equal to

$$y'(2) = \frac{3 \cdot 2^2 + 4}{2\sqrt{2^3 + 4 \cdot 2}} = \frac{16}{2\sqrt{16}} = 2.$$

Hence, the differential of the function at this point is

$$dy = y'(2)dx = 2dx.$$

Example 5. Let us calculate the approximate value of $\sin 46^{\circ}$.
Let $f(x) = \sin x$, then $f'(x) = \cos x$.
In this case the approximate equation takes the form

$$\sin(x + \Delta x) \approx \sin x + \cos x \Delta x.$$

Setting $x = 45^{\circ} = \frac{\pi}{4}$, $\Delta x = 1^{\circ} = \frac{\pi}{180}$, and $x + \Delta x = \frac{\pi}{4} + \frac{\pi}{180}$.
Substituting all these into the equation we get

$$\sin 46^{\circ} \approx \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\frac{\pi}{180} \approx 0.7071 + 0.7071 \cdot 0.0175 = 0.7191.$$

Example 6. The function $y(x)$ is defined by the parametric equations

$$\begin{cases} x = t^2 + t + 1 \\ y = t^3 - 2t \end{cases}.$$

Find the differential of the function at the point $(-3,1)$

We calculate the corresponding values of the parameter t from the equation: $3 = t^2 + t + 1$:

$$t^2 + t - 2 = 0, \Rightarrow D = 9, \Rightarrow t_{1,2} = \frac{-1 \pm 3}{2} = 1, -2.$$

Make sure that the value $t = 1$ satisfies the condition $y = -1$.

$$y'_x = \frac{y'_t}{x'_t} = \frac{(t^3 - 2t)'}{(t^2 + t + 1)'} = \frac{3t^2 - 2}{2t + 1}.$$

When $t = 1$ the derivative has the following value:

$$y'_x(t = 1) = \frac{3 \cdot 1^2 - 2}{2 \cdot 1 + 1} = \frac{1}{3}.$$

Thus, the differential of the function at the point $(3, -1)$ is expressed by the formula

$$dy = y'_x dx = \frac{dx}{3}.$$

Example 7. Given the composite function $y = \ln u, u = \cos x$. Express the differential of y in an invariant form.

We write the differential of the "outer" function:

$$dy = y'_u du = (\ln u)' du = \frac{1}{u} du.$$

Similarly, we find the differential of the "inner" function:

$$du = u'_x dx = (\cos x)' dx = -\sin x dx.$$

Substituting the expression for du in the previous formula, we obtain the differential dy in invariant form:

$$dy = \frac{1}{u} du = \frac{1}{u}(-\sin x) dx = -\frac{\sin x}{\cos x} dx = -\tan x dx.$$

2. Higher-Order Derivatives

$$\frac{d^n f}{dx^n} = \frac{d^n y}{dx^n} \text{ (in Leibnitz's notation)},$$

$$f^{(n)}(x) = y^{(n)}(x) \text{(in Lagrange's notation)}.$$

$$y^{(n)} = (y^{(n-1)})'.$$

Example 1. Find the fourth derivative of the polynomial function

$$y = 3x^4 - 2x^3 + 4x^2 - 5x + 1.$$

Take the first derivative using the power rule and the basic differentiation rules:

$$y' = 12x^3 - 6x^2 + 8x - 5.$$

Differentiate once more to find the second derivative:

$$y'' = 36x^2 - 12x + 8.$$

Also,

$$y''' = 72\text{x} - 12$$

Finally, $y^{IV} = 72$

Example 2. Find y'' if $y = \cot x$.

The first derivative of the cotangent function is given by

$$y' = (\cot x)' = -\frac{1}{\sin^2 x}.$$

Differentiate it again using the power and chain rules:

$$y'' = (-\frac{1}{\sin^2 x})' = -((\sin x)^{-2})' = (-1) \cdot (-2) \cdot (\sin x)^{-3} \cdot (\sin x)'$$

$$= \frac{2}{\sin^3 x} \cdot \cos x = \frac{2\cos x}{\sin^3 x}.$$

Example 3. Find y'' if $y = x\ln x$.

Calculate the first derivative using the product rule:

$$y' = (x\ln x)' = x' \cdot \ln x + x \cdot (\ln x)' = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1.$$

Now we can find the second derivative:

$$y'' = (\ln x + 1)' = \frac{1}{x} + 0 = \frac{1}{x}.$$

Example 4. The function $y = f(x)$ is given in parametric form by the equations

$$x = t^3, y = t^2 + 1,$$

where $t > 0$ Find y''_{xx}.

Determine the first derivative y'_x:

$$y'_x = \frac{y'_t}{x'_t} = \frac{(t^2 + 1)'_t}{(t^3)'_t} = \frac{2t}{3t^2} = \frac{2}{3t}.$$

Differentiate y'_x again with respect to x

$$y''_{xx} = (y'_x)'_x = (y'_x)'_t \cdot t'_x = (\frac{2}{3t})'_t \cdot \frac{1}{x'_t} = \frac{2}{3}(t^{-1})'_t \cdot \frac{1}{x'_t} = \frac{2}{3} \cdot (-1)t^{-2} \cdot \frac{1}{3t^2}$$

$$= -\frac{2}{3t^2} \cdot \frac{1}{3t^2} = -\frac{2}{9t^4}.$$

Example 5. The function $y = f(x)$ is given in parametric form by the equations

$$x = t + \cos t, y = 1 + \sin t,$$

where $t \in (0,2\pi)$. Find y''_{xx}.

Taking the first derivative of the parametric function, we have

$$y'_x = \frac{y'_t}{x'_t} = \frac{(1 + \sin t)'_t}{(t + \cos t)'_t} = \frac{\cos t}{1 - \sin t}.$$

Now we differentiate both sides of the expression for y'_x with respect to x This yields:

$$y''_{xx} = (y'_x)'_x = (y'_x)'_t \cdot t'_x = (\frac{\cos t}{1 - \sin t})'_t \cdot t'_x = (\frac{\cos t}{1 - \sin t})'_t \cdot \frac{1}{x'_t}$$

$$= \frac{(-\sin t)(1 - \sin t) - \cos t(-\cos t)}{(1 - \sin t)^2} \cdot \frac{1}{1 - \sin t}$$

$$= \frac{-\sin t + \sin^2 t + \cos^2 t}{(1 - \sin t)^3} = \frac{1 - \sin t}{(1 - \sin t)^3} = \frac{1}{(1 - \sin t)^2}.$$

Example 6. Find the second derivative of the function given by the equation

$$x^3 + y^3 = 1.$$

We use implicit differentiation:

$$x^3 + y^3 = 1, \Rightarrow (x^3)' + (y^3)' = 1', \Rightarrow 3x^2 + 3y^2y' = 0, \Rightarrow x^2 + y^2y' = 0,$$

$$\Rightarrow y' = -\frac{x^2}{y^2}.$$

Differentiate again the equation $x^2 + y^2y' = 0$:

$$x^2 + y^2y' = 0, \Rightarrow (x^2)' + (y^2y')' = 0, \Rightarrow 2x + 2yy'y' + y^2y'' = 0,$$

$$\Rightarrow 2x + 2y(y')^2 + y^2y'' = 0, \Rightarrow y'' = -\frac{2x + 2y(y')^2}{y^2}.$$

Substitute the expression for the first derivative y' found above

$$y'' = -\frac{2x + 2y(y')^2}{y^2} = -\frac{2x + 2y(-\frac{x^2}{y^2})^2}{y^2} = -\frac{2x + 2y \cdot \frac{x^4}{y^4}}{y^2}$$

$$= -\frac{2x + \frac{2x^4}{y^3}}{y^2} = -\frac{\frac{2xy^3 + 2x^4}{y^3}}{y^2} = -\frac{2x^4 + 2xy^3}{y^5} = -\frac{2x(x^3 + y^3)}{y^5}$$

$$= -\frac{2x \cdot 1}{y^5} = -\frac{2x}{y^5}.$$

Example 7. Find the second derivative of the function given by the equation

$$x + y = e^{x-y}.$$

Differentiating both sides in *x* we obtain:

$$(x + y)' = (e^{x-y})', \Rightarrow 1 + y' = e^{x-y} \cdot (x - y)', \Rightarrow 1 + y' = e^{x-y}(1 - y')$$

$$= e^{x-y} - e^{x-y}y', \Rightarrow y' + e^{x-y}y' = e^{x-y} - 1, \Rightarrow$$

$$y' = \frac{e^{x-y} - 1}{e^{x-y} + 1}.$$

Continuing the differentiation, we find the second derivative:

$$y'' = (\frac{e^{x-y} - 1}{e^{x-y} + 1})' = \frac{2e^{x-y}(1 - y')}{(e^{x-y} + 1)^2}.$$

Substitute the expression for the first derivative:

$$y'' = \frac{2e^{x-y}(1 - y')}{(e^{x-y} + 1)^2} = \frac{2e^{x-y}(1 - \frac{e^{x-y} - 1}{e^{x-y} + 1})}{(e^{x-y} + 1)^2}$$

$$= \frac{2e^{x-y} \cdot \frac{\cancel{e^{x-y}} + 1 - \cancel{e^{x-y}} + 1}{e^{x-y} + 1}}{(e^{x-y} + 1)^2} = \frac{4e^{x-y}}{(e^{x-y} + 1)^3}.$$

We now use the original equation, according to which

$$e^{x-y} = x + y.$$

As a result, we obtain the following expression for the derivative y''

$$y'' = \frac{4e^{x-y}}{(e^{x-y}+1)^3} = \frac{4(x+y)}{(x+y+1)^3}.$$

3. Leibniz Formula

$$(uv)^{(n)} = \sum_{i=0}^{n} \binom{n}{i} u^{(n-i)} v^{(i)},$$

where $\binom{n}{i}$ denotes the number of *i*-combinations of *n* elements: $\binom{n}{i} = \frac{n!}{i!(n-i)!}$

Example 1. Find the 3rd derivative of the function

$$y = e^x \cos x.$$

Let $u = \cos x$, $v = e^x$. Using the Leibniz formula, we have

$$y''' = (e^x \cos x)''' = \sum_{i=0}^{3} \binom{3}{i} (\cos x)^{(3-i)} (e^x)^{(i)}$$

$$= \binom{3}{0}(\cos x)''' e^x + \binom{3}{1}(\cos x)''(e^x)' + \binom{3}{2}(\cos x)'(e^x)''$$

$$+ \binom{3}{3}\cos x (e^x)'''.$$

The derivatives of cosine are

$(\cos x)' = -\sin x$; $(\cos x)'' = (-\sin x)' = -\cos x$; $(\cos x)''' = (-\cos x)' = \sin x$.

All derivatives of the exponential function $v = e^x$ are e^x

Hence,

$$y''' = 1 \cdot \sin x \cdot e^x + 3 \cdot (-\cos x) \cdot e^x + 3 \cdot (-\sin x) \cdot e^x + 1 \cdot \cos x \cdot e^x$$

$$= e^x(-2\sin x - 2\cos x) = -2e^x(\sin x + \cos x).$$

Example 2. Find all derivatives of the function

$$y = e^x x^2.$$

Let $u = e^x$ and $v = x^2$. Then

$$u' = (e^x)' = e^x, v' = (x^2)' = 2x, u'' = (e^x)' = e^x, v'' = (2x)' = 2.$$

It is easy to find the general formulas for the derivatives of order *n*:

$$u^{(n)} = e^x, v''' = v^{IV} = \cdots = v^{(n)} = 0.$$

Using the Leibniz formula, we obtain

$$y^{(n)} = e^x x^2 + ne^x \cdot 2x + \frac{n(n-1)}{1 \cdot 2} e^x \cdot 2$$

or

$$y^{(n)} = e^x[x^2 + 2nx + n(n-1)].$$

Example 3. Find the 10th-order derivative of the function

$$y = (x^2 + 4x + 1)\sqrt{e^x}$$

at the point $x = 0$

We denote $u = \sqrt{e^x}$, $v = x^2 + 4x + 1$. The derivatives of these functions have the following form:

$$u' = (\sqrt{e^x})' = \frac{1}{2\sqrt{e^x}} \cdot (e^x)' = \frac{e^x}{2\sqrt{e^x}} = \frac{\sqrt{e^x}}{2}, u'' = (\frac{\sqrt{e^x}}{2})' = \frac{\sqrt{e^x}}{4}, \ldots$$

$$\Rightarrow u^{(k)} = \frac{\sqrt{e^x}}{2^k},$$

$$v' = (x^2 + 4x + 1)' = 2x + 4, v'' = (2x + 4)' = 2.$$

The derivatives of the function v of order $i > 2$ are obviously zero. Therefore, the expansion of the derivative $y^{(10)}$ is limited to only a few terms:

$$y^{(10)} = \sum_{i=0}^{10} \binom{10}{i} u^{(10-i)} v^{(i)}$$

$$= \binom{10}{0} \frac{\sqrt{e^x}}{2^{10}} (x^2 + 4x + 1) + \binom{10}{1} \frac{\sqrt{e^x}}{2^9} (2x + 4) + \binom{10}{2} \frac{\sqrt{e^x}}{2^8} \cdot 2 =$$

$$\frac{10!}{10!0!} \cdot \sqrt{e^x} \cdot \frac{1}{2^{10}} \cdot (x^2 + 4x + 1) + \frac{10!}{9!1!} \cdot \sqrt{e^x} \cdot \frac{2}{2^{10}} \cdot (2x + 4) + \frac{10!}{8!2!} \cdot \sqrt{e^x} \cdot \frac{4}{2^{10}} \cdot$$

$$2 = \frac{\sqrt{e^x}}{2^{10}} \cdot [x^2 + 4x + 1 + 20(2x + 4) + 360] = \frac{\sqrt{e^x}}{2^{10}} (x^2 + 44x + 441)$$

When $x = 0$, the 10th-order derivative is respectively equal to

$$y^{(10)}(0) = \frac{441}{2^{10}} = \frac{441}{1024} = (\frac{21}{32})^2.$$

Example 4. Find the *n*th-order derivative of the function

$$y = x^3 \sin 2x.$$

Let $u = \sin 2x$, $v = x^3$. Write the *n*th-order derivative by the Leibniz formula:

$$(x^3 \sin 2x)^{(n)} = \sum_{i=0}^{n} \binom{n}{i} u^{(n-i)} v^{(i)} = \sum_{i=0}^{n} \binom{n}{i} (\sin 2x)^{(n-i)} (x^3)^{(i)}$$

$$= \binom{n}{0} (\sin 2x)^{(n)} x^3 + \binom{n}{1} (\sin 2x)^{(n-1)} (x^3)'$$

$$+ \binom{n}{2} (\sin 2x)^{(n-2)} (x^3)'' + \binom{n}{3} (\sin 2x)^{(n-3)} (x^3)''' + \cdots$$

Obviously, the remaining terms in the series expansion are zero since $(x^3)^{(i)} = 0$ for $i > 3$.

The *n*th-order derivative of the sine function was found on the Higher-Order Derivatives lecture. It is written in the form

$$(\sin x)^{(n)} = \sin(x + \frac{\pi n}{2}).$$

It can be shown that the derivative of $\sin 2x$ is defined by the similar formula:

$$(\sin 2x)^{(n)} = 2^n \sin(2x + \frac{\pi n}{2}).$$

Consequently, the remaining derivatives of $\sin 2x$ are given by

$$(\sin 2x)^{(n-1)} = 2^{n-1} \sin(2x + \frac{\pi(n-1)}{2}) = 2^{n-1} \sin(2x + \frac{\pi n}{2} - \frac{\pi}{2})$$

$$= -2^{n-1} \cos(2x + \frac{\pi n}{2}),$$

$$(\sin 2x)^{(n-2)} = 2^{n-2} \sin(2x + \frac{\pi(n-2)}{2}) = 2^{n-2} \sin(2x + \frac{\pi n}{2} - \pi)$$

$$= -2^{n-2} \sin(2x + \frac{\pi n}{2}),$$

$$(\sin 2x)^{(n-3)} = 2^{n-3}\sin(2x + \frac{\pi(n-3)}{2}) = 2^{n-3}\sin(2x + \frac{\pi n}{2} - \frac{3\pi}{2})$$
$$= 2^{n-3}\cos(2x + \frac{\pi n}{2}).$$

Substituting this into the formula for the *n*th derivative of the given function, we obtain:

$$(x^3\sin 2x)^{(n)}$$
$$= \binom{n}{0}x^3 2^n\sin(2x + \frac{\pi n}{2}) - \binom{n}{1}\cdot 3x^2 2^{n-1}\cos(2x + \frac{\pi n}{2}) - \binom{n}{2}$$
$$\cdot 6x2^{n-2}\sin(2x + \frac{\pi n}{2}) + \binom{n}{3}\cdot 6\cdot 2^{n-3}\cos(2x + \frac{\pi n}{2}).$$

Take into account that the combinations can be represented in the following form:

$$\binom{n}{0} = 1, \binom{n}{1} = n, \binom{n}{2} = \frac{n(n-1)}{2}, \binom{n}{3} = \frac{n(n-1)(n-2)}{6}.$$

Then

$$(x^3\sin 2x)^{(n)}$$
$$= x^3 2^n\sin(2x + \frac{\pi n}{2}) - 3x^2 n2^{n-1}\cos(2x + \frac{\pi n}{2}) - 6x$$
$$\cdot \frac{n(n-1)}{2}\cdot 2^{n-2}\sin(2x + \frac{\pi n}{2}) + 6\cdot\frac{n(n-1)(n-2)}{6}$$
$$\cdot 2^{n-3}\cos(2x + \frac{\pi n}{2})$$
$$= 2^n[x^3 - \frac{3xn(n-1)}{4}]\sin(2x + \frac{\pi n}{2}) + 2^n[\frac{n(n-1)(n-2)}{8}$$
$$- \frac{3x^2 n}{2}]\cos(2x + \frac{\pi n}{2}).$$

Example 5. Find the *n*th-order derivative of the function

$$y = x\ln x.$$

Let $u = \ln x$, $v = x$. Then

$$y^{(n)} = (x\ln x)^{(n)} = \sum_{i=0}^{n} \binom{n}{i} u^{(n-i)} v^{(i)} = \sum_{i=0}^{n} \binom{n}{i} (\ln x)^{(n-i)} x^{(i)}$$
$$= \binom{n}{0} (\ln x)^{(n)} x + \binom{n}{1} (\ln x)^{(n-1)} x' + \cdots$$

The other terms of the series are equal to zero as $x^{(i)} \equiv 0$ for $i > 1$.

Write the derivatives of $v = x$:

$$v' = x' = 1, v'' = v''' = \cdots = v^{(n)} \equiv 0.$$

Compute the derivatives of $u = \ln x$:

$$u' = (\ln x)' = \frac{1}{x}, u'' = \left(\frac{1}{x}\right)' = -\frac{1}{x^2}, u''' = \left(-\frac{1}{x^2}\right)' = \frac{2}{x^3}, u^{(4)} = \left(\frac{2}{x^3}\right)'$$
$$= -\frac{6}{x^4}, \ldots$$

So, the *n*th derivative of the natural logarithm is written in the form

$$u^{(n)} = \frac{(-1)^{n-1}(n-1)!}{x^n}.$$

Hence, the series expansion for $y^{(n)}$ is given by

$$y^{(n)} = \binom{n}{0} (\ln x)^{(n)} \cdot x + \binom{n}{1} (\ln x)^{(n-1)} \cdot 1$$
$$= \frac{(-1)^{n-1}(n-1)!}{x^n} \cdot x + n \cdot \frac{(-1)^{n-2}(n-2)!}{x^{n-1}}$$
$$= \frac{(-1)^{n-1} n!}{n x^{n-1}} + \frac{(-1)^{n-1}(-1)^{-1} n!}{(n-1)x^{n-1}} = \frac{(-1)^{n-1} n!}{x^{n-1}} \left(\frac{1}{n} - \frac{1}{n-1}\right)$$
$$= \frac{(-1)^{n-1} n!}{x^{n-1}} \cdot \frac{\cancel{n} - 1 - \cancel{n}}{n(n-1)} = \frac{(-1)^n (n-2)! \cancel{(n-1)} \cancel{n}}{x^{n-1} \cancel{n} \cancel{(n-1)}}$$
$$= \frac{(-1)^n (n-2)!}{x^{n-1}}.$$

4. *Higher-Order Differentials*

$$d^n y = f^{(n)}(x) dx^n,$$

Let the functions u and v have the *n*th order derivatives. Then the following properties are valid:

$$d^n(\alpha u + \beta v) = \alpha d^n u + \beta d^n v;$$

$$d^n(uv) = \sum_{i=0}^{n} C_n^i d^{n-i} u d^i v.$$

Consider now the composition of two functions such that $y = f(u)$ and $u = g(x)$. In this case, *y* is a composite function of the independent variable *x*

$$y = f(g(x)).$$

$$dy = [f(g(x))]' dx = f'(g(x))g'(x)dx.$$

$$d^2y = f''(u)du^2 + f'(u)d^2u$$

$$d^3y = f'''(u)du^3 + 3f''(u)dud^2u + f'(u)d^3u.$$

It follows from the above that the higher order differentials are generally not invariant.

Example 1. Find the differential d^4y of the function $y = x^5$.

The 4th order differential is given by

$$d^4y = f^{(4)}(x)dx^4 = (x^5)^{(4)}dx^4.$$

We find the fourth derivative of this function by successive differentiation:

$$(x^5)' = 5x^4, (x^5)'' = (5x^4)' = 20x^3, (x^5)''' = (20x^3)' = 60x^2,$$

$$(x^5)^{(4)} = (60x^2)' = 120x.$$

Hence,

$$d^4y = 120xdx^4.$$

Example 2. Find the second differential of the function

$$y = x^2 \cos 2x.$$

Determine the second derivative of this function:

$$y' = (x^2 \cos 2x)' = (x^2)' \cos 2x + x^2 (\cos 2x)'$$

$$= 2x \cos 2x + x^2 \cdot (-2 \sin 2x) = 2x \cos 2x - 2x^2 \sin 2x,$$

$$y'' = (2x\cos 2x - 2x^2 \sin 2x)' = 2(x\cos 2x - x^2 \sin 2x)'$$
$$= 2[x'\cos 2x + x(\cos 2x)'(x^2)'\sin 2x - x^2(\sin 2x)']$$
$$= 2[\cos 2x - 2x\sin 2x - 2x\sin 2x - 2x^2\cos 2x]$$
$$= (2 - 2x^2)\cos 2x - 4x\sin 2x.$$

Then the second-order differential is written in the form:

$$d^2y = y''dx^2 = [(2 - 2x^2)\cos 2x - 4x\sin 2x]dx^2.$$

Example 3. Find d^3y of the function $y = x\ln\frac{1}{x}$.

The third order differential is given by

$$d^3y = y'''(x)dx^3.$$

We differentiate the given function successively:

$$y' = (x\ln\frac{1}{x})' = x \cdot (\frac{1}{\frac{1}{x}}) \cdot (\frac{1}{x})' + 1 \cdot \ln\frac{1}{x} = x^2 \cdot (-\frac{1}{x^2}) + \ln\frac{1}{x} = \ln\frac{1}{x} - 1;$$

$$y'' = (\ln\frac{1}{x} - 1)' = (\frac{1}{\frac{1}{x}}) \cdot (\frac{1}{x})' = x \cdot (-\frac{1}{x^2}) = -\frac{1}{x};$$

$$y''' = (-\frac{1}{x})' = \frac{1}{x^2}.$$

Hence,

$$d^3y = y'''(x)dx^3 = \frac{dx^3}{x^2}.$$

Example 4. Find the second differential d^2y of the function (graph of which is known as asteroid) defined by the equation

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = R^{\frac{2}{3}}, R \text{ – is constant}$$

We compute successively the first and second derivatives of the function $y(x)$ describing the astroid. Differentiating both sides of the implicit equation with respect to x, we have:

$$(x^{\frac{2}{3}}+y^{\frac{2}{3}})' = (R^{\frac{2}{3}})', \Rightarrow \frac{2}{3}x^{-\frac{1}{3}}+\frac{2}{3}y^{-\frac{1}{3}}y' = 0, \Rightarrow x^{-\frac{1}{3}}+y^{-\frac{1}{3}}y' = 0, \Rightarrow y^{-\frac{1}{3}}y'$$
$$= -x^{-\frac{1}{3}}, \Rightarrow y' = -(\frac{x}{y})^{-\frac{1}{3}} = -(\frac{y}{x})^{\frac{1}{3}}.$$

Differentiate again, given that *y* is the function of *x*

$$y'' = [-(\frac{y}{x})^{\frac{1}{3}}]' = -\frac{1}{3}(\frac{y}{x})^{-\frac{2}{3}} \cdot (\frac{y}{x})' = -\frac{1}{3}(\frac{x}{y})^{\frac{2}{3}} \cdot \frac{y'x - yx'}{x^2}$$
$$= -\frac{x^{\frac{2}{3}}}{3y^{\frac{2}{3}}} \cdot \frac{y'x - y}{x^2}.$$

Substitute the expression for the first derivative y' found above:

$$y'' = -\frac{x^{\frac{2}{3}}}{3y^{\frac{2}{3}}} \cdot \frac{(-(\frac{y}{x})^{\frac{1}{3}})x - y}{x^2} = -\frac{x^{\frac{2}{3}}}{3y^{\frac{2}{3}}} \cdot \frac{(-y^{\frac{1}{3}}x^{\frac{2}{3}} - y)}{x^2} = \frac{x^{\frac{2}{3}}y^{\frac{1}{3}}(x^{\frac{2}{3}}+y^{\frac{2}{3}})}{3y^{\frac{2}{3}}x^2}$$
$$= \frac{R^{\frac{2}{3}}}{3y^{\frac{1}{3}}x^{\frac{4}{3}}}.$$

Then the second differential is given by

$$d^2y = y''dx^2 = \frac{R^{\frac{2}{3}}dx^2}{3y^{\frac{1}{3}}x^{\frac{4}{3}}}.$$

Example 5. The function is given in parametric form by the equations

$$\begin{cases} x & = t^2 + t - 1 \\ y & = t^3 - 2t \end{cases}.$$

Find the second-order differential d^2y.

We determine the second-order differential by the formula

$$d^2y = y''(x)dx^2.$$

Find the second derivative $y''(x)$. The first derivative is given by

$$y'(x) = y'_x = \frac{y'_t}{x'_t} = \frac{(t^3 - 2t)'}{(t^2 + t - 1)'} = \frac{3t^2 - 2}{2t + 1}.$$

Then the second derivative can be expressed as follows:

$$y''(x) = y''_{xx} = (y'_x)'_x = \frac{(y'_x)'_t}{x'_t} = \frac{(\frac{3t^2 - 2}{2t + 1})'}{(t^2 + t - 1)'}$$

$$= \frac{\frac{(3t^2 - 2)'(2t + 1) - (3t^2 - 2)(2t + 1)'}{(2t + 1)^2}}{2t + 1}$$

$$= \frac{6t \cdot (2t + 1) - (3t^2 - 2) \cdot 2}{(2t + 1)^3} = \frac{12t^2 + 6t - 6t^2 + 4}{(2t + 1)^3}$$

$$= \frac{6t^2 + 6t + 4}{(2t + 1)^3}.$$

Calculate the differential dx^2:

$$dx^2 = (dx)^2 = (d(t^2 + t - 1))^2 = ((2t + 1)dt)^2 = (2t + 1)^2(dt)^2$$

$$= (2t + 1)^2 dt^2.$$

Thus, the differential of the 2nd order of the original function is given by

$$d^2y = y''(x)dx^2 = \frac{6t^2 + 6t + 4}{(2t + 1)^3} \cdot (2t + 1)^2 dt^2 = \frac{6t^2 + 6t + 4}{2t + 1} dt^2.$$