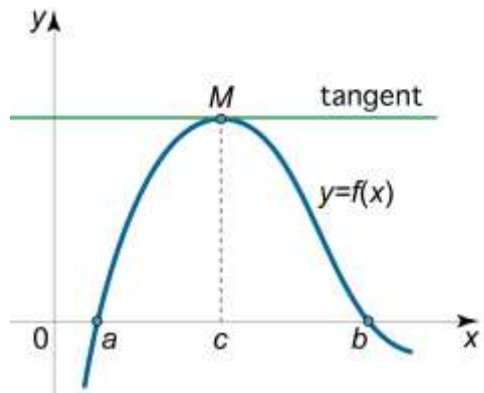
1. The Mean-Value Theorems

Fermat's Theorem: Let a function $f(x)$ be defined in a neighborhood of the point x_0 and differentiable at this point. Then, if the function $f(x)$ has a local extremum at x_0 then

$$f'(x_0) = 0.$$

Rolle's Theorem: Suppose that a function $f(x)$ is continuous on the closed interval [a, b] and differentiable on the open interval (a, b). Then if $f(a) = f(b)$, then there exists at least one point c in the open interval (a, b) for which $f'(c) = 0$.



Geometric interpretation: There is a point c on the interval (a, b) where the tangent to the graph of the function is horizontal.

Example 1: Let $f(x) = x^2 + 2x$. Find all values of c in the interval $[-2,0]$ such that $f'(c) = 0$.

First of all, we need to check that the function $f(x)$ satisfies all the conditions of Rolle's theorem

1. $f(x) = x^2 + 2x$ is continuous in $[-2,0]$ as a quadratic function;
2. It is differentiable everywhere over the open interval $(-2,0)$;
3. Finally,

$$f(-2) = (-2)^2 + 2 \cdot (-2) = 0,$$
$$f(0) = 0^2 + 2 \cdot 0 = 0,$$
$$\Rightarrow f(-2) = f(0).$$

So we can use Rolle's theorem.

To find the point c we calculate the derivative

$$f'(x) = (x^2 + 2x)' = 2x + 2$$

and solve the equation $f\,'(c) = 0$

$$f'(c) = 2c + 2 = 0, \Rightarrow c = -1.$$

Thus, $f\,'(c) = 0$ at $c = -1$.

Example 2: Given an interval $[a, b]$ that satisfies hypothesis of Rolle's theorem for the function

$$f(x) = x^4 + x^2 - 2.$$

It is known that $a = -1$. Find the value of b.

We factorize the polynomial:

$$x^4 + x^2 - 2 = (x^2 + 2)(x^2 - 1) = (x^2 + 2)(x - 1)(x + 1).$$

It is now easy to see that the function has two zeros: $x_1 = -1$ (coincides with the value of a) and $x_2 = 1$.

Since the function is a polynomial, it is everywhere continuous and differentiable. So this function satisfies Rolle's theorem on the interval $[-1,1]$. Hence, $b = 1$.

Example 3: Given an interval $[a, b]$ that satisfies hypothesis of Rolle's theorem for the function

$$f(x) = x^3 - 2x^2 + 3.$$

It is known that $a = 0$. Find the value of b.

We factorize the polynomial:

$$f_1(x) = x^3 - 2x^2 = x^2(x - 2).$$

It is now easy to see that the function has two zeros: $x_1 = 0$ (coincides with the value of a) and $x_2 = 2$.

The original function differs from this function in that it is shifted 3 units up. Therefore, we can write that

$$f(0) = f(2) = 3.$$

It is obvious that the function $f(x)$ is everywhere continuous and differentiable as a cubic polynomial. Consequently, it satisfies all the conditions of Rolle's theorem on the interval $[0,2]$. Hence, $b = 2$.

Example 4: Check the validity of Rolle's theorem for the function

$$f(x) = x^2 - 6x + 8.$$

The given quadratic function has roots $x = 2$ and $x = 4$ that is

$$f(2) = f(4) = 0.$$

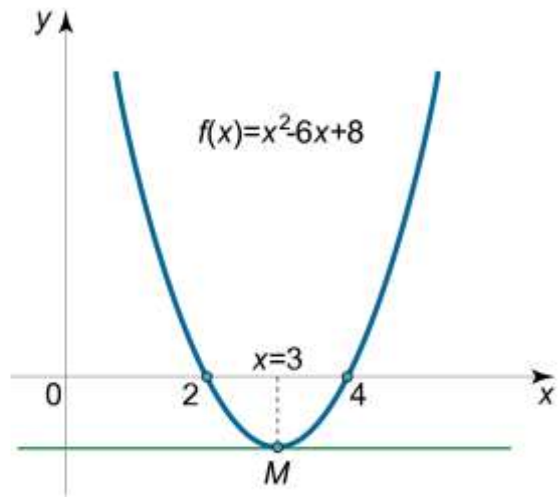
The by Rolle's theorem, there is a point ξ in the interval $(2,4)$ where the derivative of the function $f(x)$ equals zero.

$$f'(x) = (x^2 - 6x + 8)' = 2x - 6.$$

It is equal to zero at the following point $x = \xi$:

$$f'(x) = 0, \Rightarrow 2x - 6 = 0, \Rightarrow x = \xi = 3.$$

It can be seen that the resulting stationary point $\xi = 3$ belongs to the interval $(2,4)$.



2. *The Mean Value Theorems (MVT)*

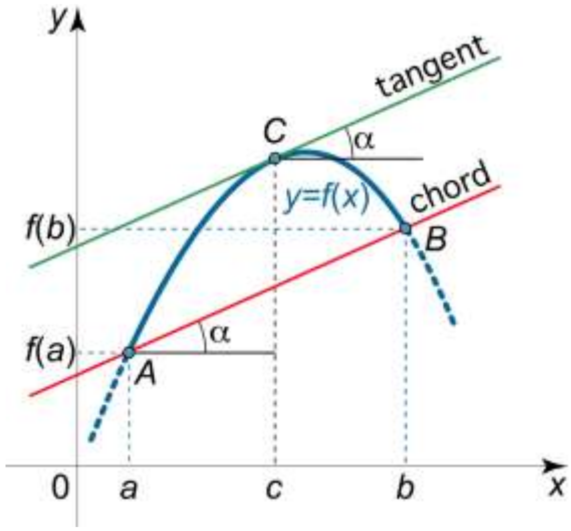
Lagrange's mean value theorem (MVT) states that if a function $f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b), then there is at least one point $x = c$ on this interval, such that

$$f(b) - f(a) = f'(c)(b - a).$$

Lagrange's mean value theorem has a simple *geometrical meaning*. The chord passing through the points of the graph corresponding to the ends of the segment a and b has the slope equal to

$$k = \tan\alpha = \frac{f(b) - f(a)}{b - a}.$$

Then there is a point *c* inside the interval $[a, b]$, where the tangent to the graph is parallel to the chord



Example 1: Check the validity of Lagrange's mean value theorem for the function $f(x) = x^2 - 3x + 5$ on the interval [1,4]. If the theorem holds, find a point *c* satisfying the conditions of the theorem.

The given quadratic function is continuous and differentiable on the entire set of real numbers. Hence, we can apply Lagrange's mean value theorem. The derivative of the function has the form

$$f'(x) = (x^2 - 3x + 5)' = 2x - 3.$$

Find the coordinates of the point *c*:

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \Rightarrow 2c - 3 = \frac{(4^2 - 3 \cdot 4 + 5) - (1^2 - 3 \cdot 1 + 5)}{4 - 1},$$

$$\Rightarrow 2c - 3 = \frac{9 - 3}{3} = 2, \Rightarrow 2c = 5, \Rightarrow c = 2{,}5.$$

You can see that the point $c = 2{,}5$ lies in the interval (1,4).

Example 2: Find all points *c* satisfying the conditions of the MVT for the

function $f(x) = x^3 - x$ in the interval $[-2,1]$.
We have here a cubic function which is continuous on the closed interval $[-2,1]$ and diferentiable on the open interval $(-2,1)$.
The MVT states that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Calculate the quotient in the right-hand side:

$$\frac{f(b) - f(a)}{b - a} = \frac{(1^3 - 1) - ((-2)^3 - (-2))}{1 - (-2)} = \frac{6}{3} = 2.$$

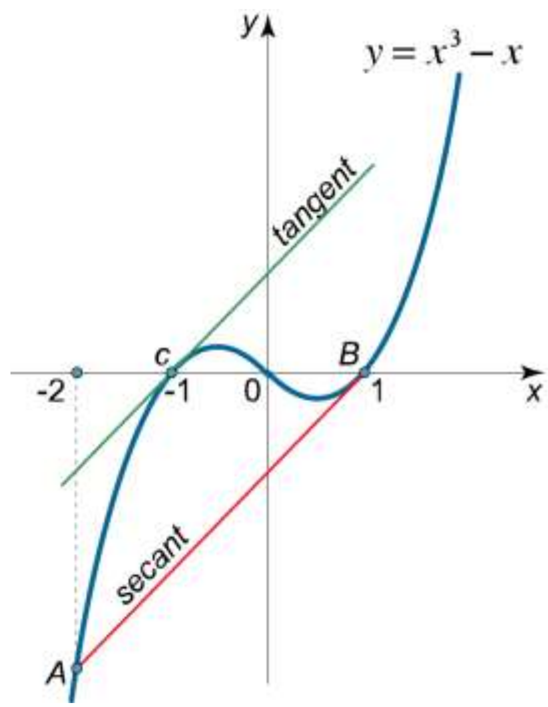
Next, we take the derivative

$$f'(x) = (x^3 - x)' = 3x^2 - 1.$$

Equate both expressions to find the value of c :

$$3c^2 - 1 = 2, \Rightarrow c^2 = 1, \Rightarrow c = \pm 1.$$

It is obvious that only one root $c = -1$ falls within the open interval $(-2,1)$
So, the answer is $c = -1$.



3. Cauchy's Mean Value Theorem

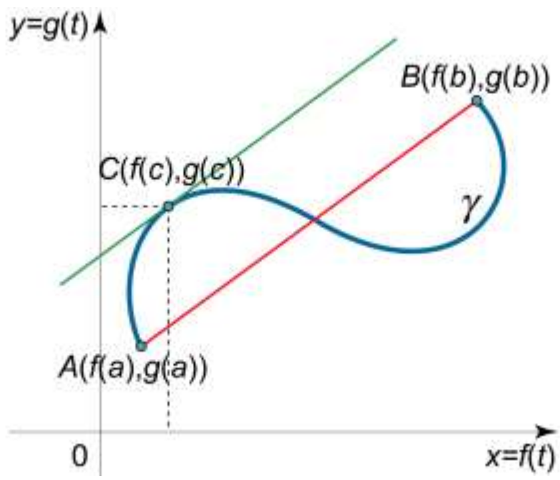
Cauchy's Mean Value Theorem generalizes Lagrange's Mean Value Theorem.

This theorem is also called the Extended or Second Mean Value Theorem. It establishes the relationship between the derivatives of two functions and changes in these functions on a finite interval.

Let the functions $f(x)$ and $g(x)$ be continuous on an interval $[a, b]$, differentiable on (a, b), and $g'(x) \neq 0$ for all $x \in (a, b)$. Then there is a point $x = c$ in this interval such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Cauchy's mean value theorem has the following geometric meaning. Suppose that a curve is described by the parametric equations $x = f(t)$, $y = g(t)$, where the parameter t ranges in the interval $[a, b]$. When changing the parameter t the point of the curve in Figure runs from $A(f(a), g(a))$ to $B(f(b), g(b))$. According to the theorem, there is a point $(f(c), g(c))$ on the curve where the tangent is parallel to the chord joining the ends A and B of the curve.



Example 1: Check the validity of Cauchy's mean value theorem for the functions $f(x) = x^4$ and $g(x) = x^2$ on the interval [1,2]

The derivatives of these functions are

$$f'(x) = (x^4) = 4x^3, g'(x) = (x^2) = 2x.$$

Substituting the functions and their derivatives in the Cauchy formula, we get

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}, \Rightarrow \frac{b^4 - a^4}{b^2 - a^2} = \frac{4c^3}{2c}, \Rightarrow \frac{\cancel{(b^2 - a^2)}(b^2 + a^2)}{\cancel{b^2 - a^2}} = 2c^2,$$

$$\Rightarrow c^2 = \frac{a^2+b^2}{2}, \Rightarrow c = \pm\sqrt{\frac{a^2+b^2}{2}}.$$

We take into account that the boundaries of the segment are $a = 1$ and $b = 2$. Consequently,

$$c = \pm\sqrt{\frac{1^2+2^2}{2}} = \pm\sqrt{\frac{5}{2}} \approx \pm 1{,}58.$$

In this case, the positive value of the square root $c = \sqrt{\frac{5}{2}} \approx 1{,}58$ is relevant. It is evident that this number lies in the interval $(1,2)$, i.e. satisfies the Cauchy theorem.

4. L'Hopital's Rule

Let a be either a finite number or infinity.

If $\lim\limits_{x\to a} f(x) = 0$ and $\lim\limits_{x\to a} g(x) = 0$, then $\lim\limits_{x\to a}\frac{f(x)}{g(x)} = \lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$;

If $\lim\limits_{x\to a} f(x) = \infty$ and $\lim\limits_{x\to a} g(x) = \infty$, then $\lim\limits_{x\to a}\frac{f(x)}{g(x)} = \lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$;

We can apply L'Hopital's rule to indeterminate forms of type $0\cdot\infty$, $\infty - \infty$, 0^0, 1^∞, ∞^0 as well. The first two indeterminate forms $0\cdot\infty$ and $\infty-\infty$ can be reduced to the forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$ using algebraic transformations. The indeterminate forms 0^0, 1^∞ and ∞^0 can be reduced to the form with the help of identity

$$f(x)^{g(x)} = e^{g(x)\ln f(x)}.$$

Example 1: Find the limit

$$\lim_{x\to 2}\frac{\sqrt{7+x}-3}{x-2}.$$

Because direct substitution leads to an indeterminate form $\frac{0}{0}$ we can use L'Hopital's rule:

$$\lim_{x\to 2}\frac{\sqrt{7+x}-3}{x-2} = [\frac{0}{0}] = \lim_{x\to 2}\frac{(\sqrt{7+x}-3)'}{(x-2)'} = \lim_{x\to 2}\frac{\frac{1}{2\sqrt{7+x}}}{1} = \frac{1}{2}\lim_{x\to 2}\frac{1}{\sqrt{7+x}}$$

$$= \frac{1}{2}\cdot\frac{1}{3} = \frac{1}{6}.$$

Example 2: Calculate the limit

$$\lim_{x\to 2}(\frac{4}{x^2-4} - \frac{1}{x-2}).$$

Here we deal with an inderminate form of type $\infty - \infty$. After simple transformations, we have

$$\lim_{x\to 2}(\frac{4}{x^2-4} - \frac{1}{x-2}) = \lim_{x\to 2}\frac{4-(x+2)}{x^2-4} = \lim_{x\to 2}\frac{2-x}{x^2-4} = [\frac{0}{0}]$$

$$= \lim_{x\to 2}\frac{(2-x)'}{(x^2-4)'} = \lim_{x\to 2}(\frac{-1}{2x}) = -\frac{1}{4}.$$

Example 3: Find the limit

$$\lim_{x\to\infty}\frac{x^2}{2^x}.$$

Using L'Hopital's rule, we can write

$$\lim_{x\to\infty}\frac{x^2}{2^x} = \left[\frac{\infty}{\infty}\right] = \lim_{x\to\infty}\frac{(x^2)'}{(2^x)'} = \lim_{x\to\infty}\frac{2x}{2^x\ln 2} = \frac{2}{\ln 2}\lim_{x\to\infty}\frac{x}{2^x} = \left[\frac{\infty}{\infty}\right] =$$

$$\frac{2}{\ln 2}\lim_{x\to\infty}\frac{(x)'}{(2^x)'} = \frac{2}{\ln 2}\lim_{x\to\infty}\frac{1}{2^x\ln 2} = \frac{2}{(\ln 2)^2}\lim_{x\to\infty}\frac{1}{2^x} = \frac{2}{(\ln 2)^2}\cdot 0 = 0.$$

Example 4: Find the limit

$$\lim_{x\to 1} x^{\frac{1}{1-x}}.$$

We have an indeterminate form of type 1^∞. Let $y = x^{\frac{1}{1-x}}$ Then

$$\ln y = \ln x^{\frac{1}{1-x}} = \frac{\ln x}{1-x}.$$

Using L'Hopital's rule, we get

$$\lim_{x\to 1}\ln y = \lim_{x\to 1}\frac{\ln x}{1-x} = \lim_{x\to 1}\frac{(\ln x)'}{(1-x)'} = \lim_{x\to 1}\frac{1/x}{-1} = -\lim_{x\to 1}\frac{1}{x} = -1.$$

Hence

$$\lim_{x\to 1} y = e^{-1} = \frac{1}{e}.$$

Example 5: Find the limit

$$\lim_{x\to \pi/2}(\sin x)^{\tan x}.$$

Direct substitution leads to the indeterminate form of type 1^{∞}. Let $y = (\sin x)^{\tan x}$. Take logarithms of both sides.

$$\ln y = \ln(\sin x)^{\tan x} = \tan x \ln\sin x.$$

Apply L'Hopital's rule:

$$\lim_{x\to \pi/2}\ln y = \lim_{x\to \frac{\pi}{2}}(\tan x\ln\sin x) = \lim_{x\to \frac{\pi}{2}}\frac{\ln\sin x}{\cot x} = [\frac{\infty}{\infty}] = \lim_{x\to \frac{\pi}{2}}\frac{(\ln\sin x)'}{(\cot x)'}$$

$$= \lim_{x\to \frac{\pi}{2}}\frac{\frac{1}{\sin x}\cdot\cos x}{-\frac{1}{\sin^2 x}} = -\lim_{x\to \frac{\pi}{2}}\frac{\sin^2 x\cdot\cos x}{\sin x} = -\lim_{x\to \frac{\pi}{2}}(\sin x\cos x)$$

$$= -\lim_{x\to \pi/2}\sin x\cdot\lim_{x\to \pi/2}\cos x = -1\cdot 0 = 0.$$

Then the final answer is

$$\lim_{x\to \pi/2} y = e^0 = 1.$$

Example 6: Find the limit

$$\lim_{x\to \pi/2}\frac{\tan x}{\tan 3x}.$$

According to L'Hopital's rule, we differentiate both the numerator and denominator a few times until the indeterminate form disappears:

$$\lim_{x\to \pi/2}\frac{\tan x}{\tan 3x} = [\frac{\infty}{\infty}] = \lim_{x\to \pi/2}\frac{(\tan x)'}{(\tan 3x)'} = \lim_{x\to \pi/2}\frac{\sec^2 x}{3\sec^2 3x} = \frac{1}{3}\lim_{x\to \pi/2}\frac{\frac{1}{\cos^2 x}}{\frac{1}{\cos^2 3x}}$$

$$= \frac{1}{3} \lim_{x \to \pi/2} \frac{\cos^2 3x}{\cos^2 x} = \left[\frac{0}{0}\right] = \frac{1}{3} \lim_{x \to \frac{\pi}{2}} \frac{(\cos^2 3x)'}{(\cos^2 x)'} = \frac{1}{3} \lim_{x \to \frac{\pi}{2}} \frac{2 \cos 3x \cdot (-3 \sin 3x)}{2 \cos x \cdot (-\sin x)}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\cos 3x \sin 3x}{\cos \mathrm{x} \sin x} = \lim_{x \to \frac{\pi}{2}} \frac{\cos 3x}{\cos x} \cdot \lim_{x \to \frac{\pi}{2}} \frac{\sin 3x}{\sin x} = \lim_{x \to \frac{\pi}{2}} \frac{\cos 3x}{\cos x} \cdot \frac{\sin \frac{3\pi}{2}}{\sin \frac{\pi}{2}} =$$

$$\lim_{x \to \pi/2} \frac{\cos 3x}{\cos x} \cdot \frac{(-1)}{1} = - \lim_{x \to \pi/2} \frac{\cos 3x}{\cos x} = [\frac{0}{0}] = - \lim_{x \to \pi/2} \frac{(\cos 3x)'}{(\cos x)'} =$$

$$- \lim_{x \to \pi/2} \frac{(-3 \sin 3x)}{(-\sin x)} = -3 \lim_{x \to \pi/2} \frac{\sin 3x}{\sin x} = -3 \cdot \frac{(-1)}{1} = 3$$

Example 7: Find the limit

$$\lim_{x \to \pi/2} (\tan x)^{\cos x}.$$

Here we deal with an indeterminate form of type ∞^0. Let $y = (\tan x)^{\cos x}$. Then

$$\ln y = \ln(\tan x)^{\cos x} = \cos x \ln \tan x.$$

Then the limit becomes

$$L = \lim_{x \to \pi/2} \ln y = \lim_{x \to \frac{\pi}{2}} (\cos x \ln \tan x) = \lim_{x \to \frac{\pi}{2}} \frac{\ln \tan x}{\sec x} = \left[\frac{\infty}{\infty}\right] =$$

$$\lim_{x \to \pi/2} \frac{(\ln \tan x)'}{(\sec x)'} = \lim_{x \to \pi/2} \frac{\frac{1}{\tan x} \cdot \sec^2 x}{\sec x \cdot \tan x} = \lim_{x \to \pi/2} \frac{\sec x}{\tan^2 x} = [\frac{\infty}{\infty}].$$

As it can be seen, we still have an indeterminate form, so we differentiate the numerator and denominator one more time:

$$L = \lim_{x \to \pi/2} \frac{(\sec x)'}{(\tan^2 x)'} = \lim_{x \to \pi/2} \frac{\sec x \tan x}{\tan x \sec^2 x} = \frac{1}{2} \lim_{x \to \pi/2} \frac{1}{\sec x} = \frac{1}{2} \lim_{x \to \pi/2} \cos x$$

$$= \frac{1}{2} \cdot 0 = 0.$$

Hence,

$$\lim_{x \to \pi/2} y = e^0 = 1.$$