Using the Derivative to Graph Functions

1. Increasing and Decreasing Function

Definition 1: Let $y = f(x)$ be a differentiable function on an interval (a, b). If for any two points $x_1,\ x_2 \in (a, b)$ such that $x_1 < x_2$, there holds the inequality $f(x_1) \leq f(x_2)$, the function is called *increasing* (or *non-decreasing*) in this interval.

Definition 2: Let $y = f(x)$ be a differentiable function on an interval (a, b). If for any two points $x_1,\ x_2 \in (a, b)$ such that $x_1 < x_2$, there holds the inequality $f(x_1) \geq f(x_2)$, the function is called *decreasing* (or *non-increasing*) in this interval, i.e.

$$\forall x_1, x_2 \in (a, b): x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2);$$

Theorem 1. In order for the function $y = f(x)$ to be increasing on the interval (a, b) it is necessary and sufficient that the first derivative of the function be non-negative everywhere in this interval:

$$f'(x) \geq 0 \forall x \in (a, b).$$

A similar criterion applies to the case of a function that is decreasing on the interval:

$$f'(x) \leq 0 \quad \forall x \in (a, b).$$

Theorem 2. Suppose that a function $y = f(x)$ is differentiable on an interval (a, b). In order for the function to be strictly increasing in this interval, it is necessary and sufficient that the following conditions are satisfied:

1) $f'(x) \geq 0 \ \ \forall x \in (a, b)$;
2) $f'(x)$ is not identically equal to zero at any interval $[x_1, x_2] \in (a, b)$.

Theorem 3. Let $x_0 \in (a, b)$.
If $f'(x_0) > 0$, then the function $f(x)$ is strictly increasing at the point x_0
If $f'(x_0) < 0$, then the function $f(x)$ is strictly decreasing at the point x_0

Example 1. Show that the function $f(x) = x^3 - 3x^2 + 6x - 1$ is strictly increasing on $\mathbb{R}$

Find the derivative:

$$f'(x) = (x^3 - 3x^2 + 6x - 1)' = 3x^2 - 6x + 6.$$

Notice that the discriminant of the quadratic function is negative:

$$D = b^2 - 4ac = (-6)^2 - 4 \cdot 3 \cdot 6 = 36 - 72 = -36 < 0.$$

Therefore, the quadratic function has no zeros and has the same sign over the interval $(-\infty, \infty)$.

We choose $x = 0$ to evaluate the sign of the derivative:

$$f'(0) = 3 \cdot 0^2 - 6 \cdot 0 + 6 = 6 > 0.$$

Hence, the function is strictly increasing on $(-\infty, \infty)$.

Example 2. For what values of x is the function $f(x) = x^4 - 2x^2$ strictly increasing?

Calculate the derivative:

$$f'(x) = (x^4 - 2x^2)' = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x-1)(x+1).$$

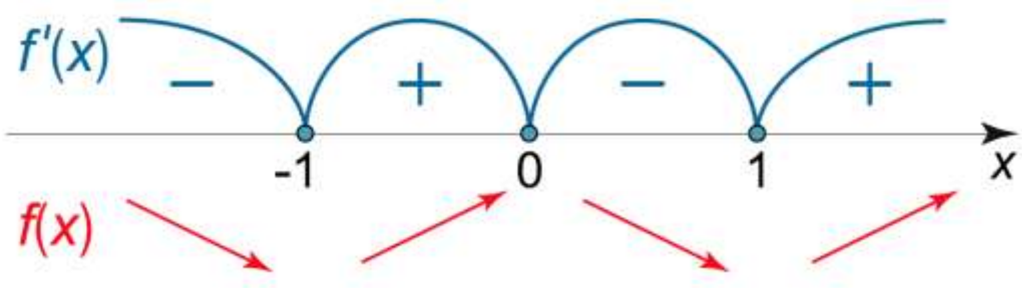
Equate the derivative to zero:

$$f'(x) = 0, \Rightarrow 4x(x-1)(x+1) = 0$$

The derivative is zero at the points

$$x_1 = -1, x_2 = 0, x_3 = 1$$

Using the interval method we find the intervals where the derivative has a constant sign (see the sign chart below).



Hence, the function is increasing on $(-1,0)$ and $(1, +\infty)$.

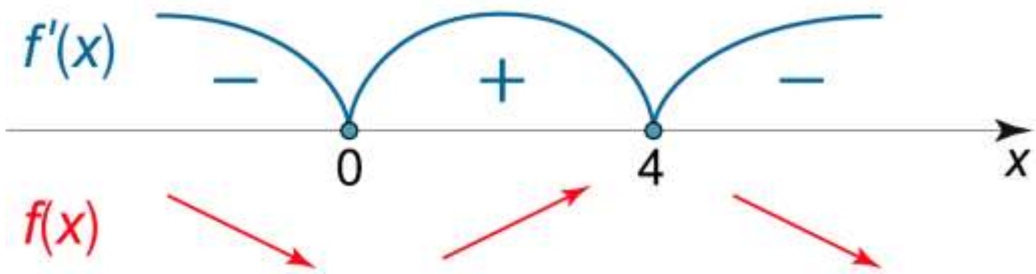
Example 3. What is the length L of the interval on which the function $f(x) =$

x^4e^{-x} is increasing?

Find the derivative using the product rule:

$$f'(x) = (x^4e^{-x})' = (x^4)' \cdot e^{-x} + x^4 \cdot (e^{-x})' = 4x^3e^{-x} - x^4e^{-x}$$
$$= x^3e^{-x}(4 - x).$$

Determine the sign of the derivative by the interval method.



We see in the figure above that the derivative is positive for $x \in (0,4)$, so the length of the interval on which the function is increasing is 4.

Example 4. Find the intervals of monotonicity of the function $f(x) = x^3 - 12x + 5$.

The derivative of this function is given by

$$f'(x) = (x^3 - 12x + 5)' = 3x^2 - 12 = 3(x^2 - 4).$$

Determine the intervals where the derivative is positive and negative. Solve the following inequality:

$$f'(x) > 0, \Rightarrow 3(x^2 - 4) > 0, \Rightarrow x^2 - 4 > 0, \Rightarrow (x - 2)(x + 2) > 0.$$

Using the interval method we find that

$$f'(x) > 0 \text{for} x \in (-\infty, -2) \cup (2, \infty),$$
$$f'(x) < 0 \text{for} x \in (-2,2).$$

Consequently, the function $f(x)$ is increasing (in the strict sense) in the intervals $(-\infty, -2)$ and $(2, \infty)$ and, accordingly, is strictly decreasing in the interval $(-2,2)$.

Example 5. Find the intervals of monotonicity of the function $f(x) = \frac{x}{x^2+1}$.

The function is defined and differentiable on the whole set of real numbers.

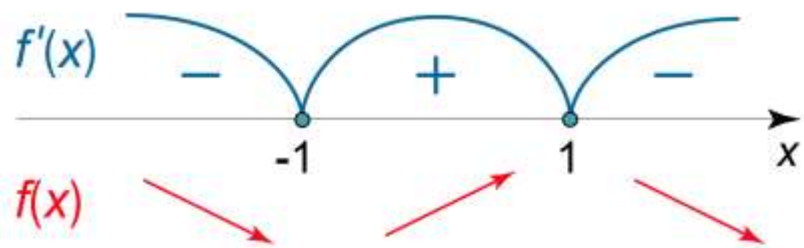
Calculate its derivative:

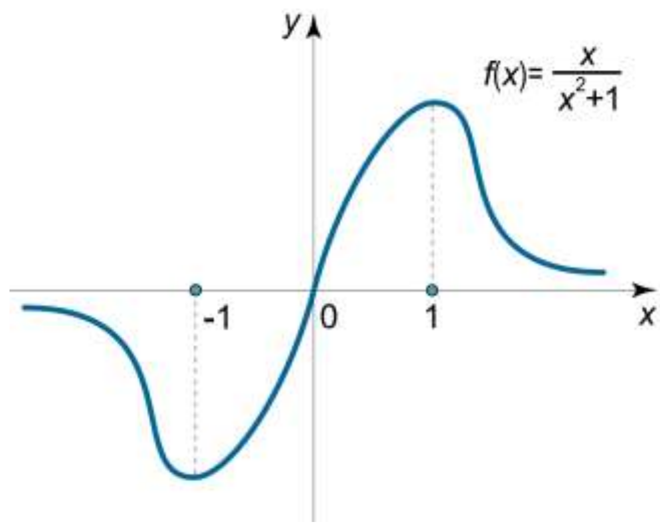
$$f'(x) = \left(\frac{x}{x^2+1}\right)' = \frac{x'(x^2+1) - x(x^2+1)'}{(x^2+1)^2} = \frac{1 \cdot (x^2+1) - x \cdot 2x}{(x^2+1)^2}$$
$$= \frac{1-x^2}{(x^2+1)^2}.$$

Determine the intervals where the derivative has a constant sign. Equate the derivative to zero and find the roots of the equation:

$$f'(x) = 0, \Rightarrow \frac{1-x^2}{(x^2+1)^2} = 0, \Rightarrow \begin{cases} 1-x^2 = 0 \\ (x^2+1)^2 \neq 0 \end{cases}, \Rightarrow 1-x^2 = 0,$$
$$\Rightarrow (1-x)(1+x) = 0.$$

Determine the intervals where the derivative has a constant sign. Equate the derivative to zero and find the roots of the equation:





Thus, the function is decreasing (in the strict sense) in the intervals $(-\infty, -1)$ and $(1, \infty)$ and increasing in the interval $(-1,1)$. Given that the root of the function is of the form $x = 0$, we can schematically draw its graph (Figure).

2. Local Extrema of Functions

Definition 1. Let a function $y = f(x)$ be defined in a δ-neighborhood of a point x_0, where $\delta > 0$. The function $f(x)$ is said to have a *local* (or *relative*) *maximum* at the point x_0, if for all points $x \neq x_0$ belonging to the neighborhood $(x_0 - \delta, x_0 + \delta)$ the following inequality holds:

$$f(x) \leq f(x_0).$$

If the strict inequality holds for all points $x \neq x_0$ in some neighborhood of x_0:

$$f(x) < f(x_0),$$

then the point x_0 is a *strict local maximum point.*

Similarly,

Definition 2. We define a *local* (or *relative*) *minimum* of the function $y = f(x)$. In this case, the following inequality is valid for all points $x \neq x_0$ of the δ -neighborhood $(x_0 - \delta, x_0 + \delta)$ of the point x_0

$$f(x) \geq f(x_0).$$

Accordingly, a *strict local minimum* at the point x_0 is described by the inequality

$$f(x) > f(x_0)$$

Definition. The points at which the derivative of the function $f(x)$ is equal to zero are called the *stationary points*.

Definition. Let $f(x)$ be a function and let x_0 be a point in the domain of the function. The point x_0 is called a *critical point* of $f(x)$ if either $f'(x_0) = 0$ or $f'(x_0)$ does not exist.

First Derivative Test: Let the function $f(x)$ be differentiable in a neighborhood of the point x_0, except perhaps at the point x_0 itself, in which, however, the function is continuous. Then:

1) If the derivative $f'(x)$ changes sign from minus to plus when passing through the point x_0 (from left to right), then x_0 is a *strict minimum point*

(Figure 1). In other words, in this case there exists a number $\delta > 0$ such that

$$\forall x \in (x_0 - \delta, x_0) \Rightarrow f'(x) < 0, \text{ and } \forall x \in (x_0, x_0 + \delta) \Rightarrow f'(x) > 0.$$

2) If the derivative $f'(x)$ on the contrary, changes sign from plus to minus when passing through the point x_0 then x_0 is a *strict maximum point* (Figure 2). In other words, there exists a number $\delta > 0$ such that

$$\forall x \in (x_0 - \delta, x_0) \Rightarrow f'(x) > 0, \text{ and } \forall x \in (x_0, x_0 + \delta) \Rightarrow f'(x) < 0.$$

Second Derivative Test: Let the first derivative of a function $f(x)$ at the point x_0 be equal to zero: $f(x_0) = 0$, that is x_0 is a stationary point of Suppose also that there exists the second derivative at this point. Then

1) If $f''(x_0) > 0$, then x_0 is a *strict minimum point* of the function ;
2) If $f''(x_0) < 0$, then x_0 is a *strict maximum point* of the function

Proof. In the case of a strict minimum $f''(x_0) > 0$. Then the first derivative is an increasing function at the point x_0. Consequently, there exists a number $\delta > 0$ such that

$$\forall x \in (x_0 - \delta, x_0) \Rightarrow f'(x) < f'(x_0),$$
$$\forall x \in (x_0, x_0 + \delta) \Rightarrow f'(x) > f'(x_0).$$

Example 1. Find the local (relative) extrema of the function $f(x) = -x^2 + 4x - 3$.

This function is differentiable everywhere on the set $(-\infty, +\infty)$. Consequently, the extrema of the function are contained among its stationary points. Solve the equation $f'(x) = 0$:

$$f'(x) = (-x^2 + 4x - 3)' = -2x + 4,$$
$$f'(x) = 0, \Rightarrow -2x + 4 = 0, \Rightarrow x = 2.$$

The function has one stationary point $x = 2$. Determine the sign of the derivative to the left and right of the point $x = 2$. The derivative is positive for $x < 2$ and negative for $x > 2$. Thus, when passing through the point $x = 2$, the derivative changes sign from plus to minus. By the first derivative test, this means that $x = 2$ is a maximum point.

The maximum value (that is the value of the function at the maximum point) is equal to

$$f_{max} = f(2) = -2^2 + 4 \cdot 2 - 3 = 1.$$

Example 2. Find the local (relative) extrema of the function $f(x) = x^3 - 12x$. This function is differentiable everywhere on the set $(-\infty, +\infty)$. Determine the critical points. The first derivative is given by

$$f'(x) = (x^3 - 12x)' = 3x^2 - 12.$$

It is equal to zero at the following points:

$$f'(x) = 0, \Rightarrow 3x^2 - 12 = 0, \Rightarrow x^2 = 4, \Rightarrow x_1 = -2, x_2 = 2.$$

These two points are critical since the function is defined and continuous over all x. The derivative also exists for all x so there are no other critical points.

We use the Second Derivative Test:

$$f''(x) = (3x^2 - 12)' = 6x,$$

$$f''(-2) = 6 \cdot (-2) = -12 < 0,$$

$$f''(2) = 6 \cdot 2 = 12 > 0.$$

Hence, $x = -2$ is a point of local maximum, and $x = 2$ is a point of local minimum.

Calculate the y $-$values for these points:

$$f_{max} = f(-2) = (-2)^3 - 12 \cdot (-2) = 16,$$

$$f_{min} = f(2) = 2^3 - 12 \cdot 2 = -16.$$

Answer:

$$\text{local max:}(-2{,}16); \text{local min:}(2, -16).$$

Example 3. Find the local (relative) extrema of the function

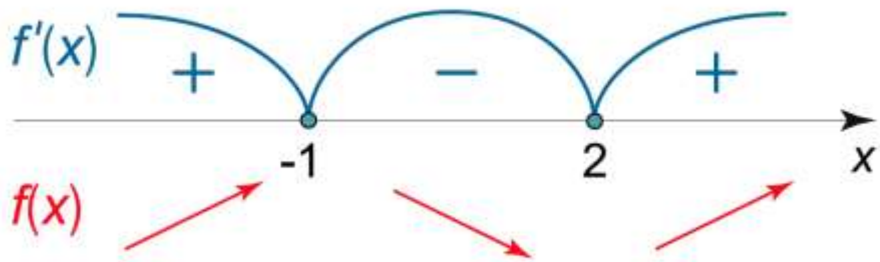
$$f(x) = x^3 - 3x^2 - 9x + 2.$$

The function is differentiable on the whole set of real numbers. Therefore, the extremum points are contained among the stationary points (where the derivative is equal to zero).

We find these points:

$$f'(x) = 0, \Rightarrow (x^3 - 3x^2 - 9x + 2) = 0, \Rightarrow 3x^2 - 6x - 9 = 0,$$
$$\Rightarrow x^2 - 2x - 3 = 0, \Rightarrow x_1 = -1, x_2 = 2.$$

Substituting test values of x, we determine the sign of the derivative $f'(x) = 3x^2 - 6x - 9$ in the corresponding intervals (Figure).



As seen, when passing through the point $x = -1$, the derivative changes sign from plus to minus. By the first derivative test, this point is a local maximum point. Similarly, we establish that $x = 2$ is a local minimum point.

We now determine the maximum and minimum values of the function:

$$f_{max} = f(-1) = (-1)^3 - 3 \cdot (-1)^2 - 9 \cdot (-1) + 2 = 7,$$
$$f_{min} = f(2) = 2^3 - 3 \cdot 2^2 - 9 \cdot 2 + 2 = -20.$$

Example 4. Using the second derivative test, find the local extrema of the function $f(x) = x^3 - 9x^2 + 24x - 7$.

The function is defined for all x. Take the first derivative and determine the critical points:

$$f'(x) = (x^3 - 9x^2 + 24x - 7)' = 3x^2 - 18x + 24.$$
$$f'(x) = 0, \Rightarrow 3x^2 - 18x + 24 = 0, \Rightarrow 3(x^2 - 6x + 8) = 0,$$
$$\Rightarrow 3(x - 2)(x - 4) = 0, \Rightarrow x_1 = 2, x_2 = 4.$$

The second derivative is given by

$$f''(x) = (3x^2 - 18x + 24)' = 6x - 18.$$

Determine the sign of the 2nd derivative at the critical points:

$$f''(x_1) = f''(2) = 6 \cdot 2 - 18 = -6 < 0.$$
$$f''(x_2) = f''(4) = 6 \cdot 4 - 18 = 6 > 0.$$

Hence, the point $x_1 = 2$ is a local maximum, and the point $x_2 = 4$ is a local minimum.

Compute the $y-$ coordinates:

$$f(x_1) = 2^3 - 9 \cdot 2^2 + 24 \cdot 2 - 7 = 13,$$
$$f(x_2) = 4^3 - 9 \cdot 4^2 + 24 \cdot 4 - 7 = 9.$$

The answer is

$$\text{local max:}(2{,}13);\ \text{local min:}(4{,}9).$$

Example 5. Find the local extrema of the function $f(x) = x^2e^{-x}$.

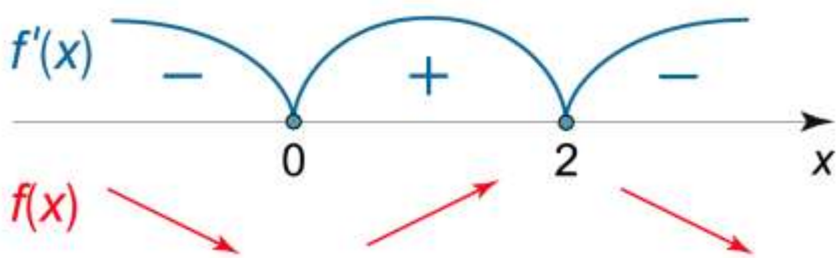
The function is defined and differentiable on the whole set $\mathbb{R}$ Calculate its derivative:

$$f'(x) = (x^2e^{-x})' = (x^2)'e^{-x} + x^2(e^{-x})' = 2xe^{-x} - x^2e^{-x}$$
$$= xe^{-x}(2-x).$$

Find the roots of the equation $f'(x) = 0$:

$$xe^{-x}(2-x) = 0, \Rightarrow x_1 - 0, x_2 = 2.$$

When passing through these points, the derivative changes sign as shown above in Figure



Hence, at the point $x = 0$, the function has a minimum, and at the point $x = 0$, it has a maximum. The minimum and maximum values, respectively, are equal to:

$$f_{min} = f(0) = 0^2e^{-0} = 0,$$
$$f_{max} = f(2) = 2^2e^{-2} = \frac{4}{e^2} \approx 0{,}541.$$

3. Convex Functions

Definition 1. Consider a function $y = f(x)$, which is assumed to be continuous on the closed interval $[a, b]$. The function $y = f(x)$ is called *convex downward* (or *concave upward*) if for any two points x_1 and x_2 in $[a,b]$, the following inequality holds:

$$f(\frac{x_1+x_2}{2}) \leq \frac{f(x_1)+f(x_2)}{2}$$

If this inequality is strict for any $x_1, x_2 \in [a,b]$, such that $x_1 \neq x_2$, then the function $f(x)$ is called *strictly convex downward* on the interval $[a,b]$.

Similarly, we define a *concave function*.

Definition 2. A function $f(x)$ is called *convex upward* (or *concave downward*) if for any two points x_1 and x_2 in the interval $[a,b]$, the following inequality is valid:

$$f(\frac{x_1+x_2}{2}) \geq \frac{f(x_1)+f(x_2)}{2}$$

If this inequality is strict for any $x_1, x_2 \in [a,b]$, such that $x_1 \neq x_2$, then the function $f(x)$ is called *strictly convex upward* on the interval $[a,b]$.

Theorem. Suppose that the first derivative $f'(x)$ of a function $f(x)$ exists in a closed interval $[a,b]$, and the second derivative $f''(x)$ exists in an open interval (a,b). Then the following *sufficient conditions* for convexity/concavity are valid:

1) If $f''(x) \geq 0$ for all $x \in (a,b)$, then the function $f(x)$ is *convex downward* (or *concave upward*) on the interval $[a,b]$;
2) If $f''(x) \leq 0$ for all $x \in (a,b)$, then the function is convex upward (or concave downward) on the interval $[a,b]$.

In the cases where the second derivative is strictly greater (or less) than zero, we say, respectively, about the *strict convexity downward* (or *strict convexity upward*).

Example 1. Find the intervals of convexity and concavity of the function $f(x) = -x^3 + 6x^2 - 2x + 1$.

Compute the derivatives:

$$f'(x) = (-x^3 + 6x^2 - 2x + 1)' = -3x^2 + 12x - 2;$$
$$f''(x) = (-3x^2 + 12x - 2)' = -6x + 12.$$

The second derivative is equal to zero at the following point:

$$f''(x) = 0, \Rightarrow -6x + 12 = 0, \Rightarrow x = 2.$$

The second derivative is positive to the left of this point and negative to the right. Hence, the function is convex downward on $(-\infty, 2)$ and convex upward on $(2, +\infty)$.

Example 2. Find the intervals of convexity and concavity of the function $f(x) = 2x^3 - 18x^2$.

Differentiating this function, we have

$$f'(x) = (2x^3 - 18x^2)' = 6x^2 - 36x;$$
$$f''(x) = (6x^2 - 26x)' = 12x - 36.$$

We set $f''(x)$ equal to zero and solve the equation:

$$f''(x) = 0, \Rightarrow 12x - 36 = 0, \Rightarrow x = 3.$$

The second derivative is negative if $x < 3$ and positive if $x > 3$ Hence, the function is convex downward on the interval $(3, +\infty)$ and convex upward on $(-\infty, 3)$.

Example 3. Find the intervals of convexity and concavity of the function $f(x) = \sqrt{2 + x^2}$.

This function is defined and differentiable for all $x \in \mathbb{R}$. Calculate the second derivative:

$$f'(x) = (\sqrt{2 + x^2})' = \frac{1}{2\sqrt{2 + x^2}} \cdot (2 + x^2)' = \frac{\cancel{2}x}{\cancel{2}\sqrt{2 + x^2}} = \frac{x}{\sqrt{2 + x^2}};$$

$$f''(x) = (\frac{x}{\sqrt{2 + x^2}})' = \frac{x'\sqrt{2 + x^2} - x(\sqrt{2 + x^2})'}{(\sqrt{2 + x^2})^2}$$

$$= \frac{\sqrt{2 + x^2} - x \cdot \frac{x}{\sqrt{2 + x^2}}}{2 + x^2} = \frac{(\sqrt{2 + x^2})^2 - x^2}{(2 + x^2)\sqrt{2 + x^2}} = \frac{2 + \cancel{x^2} - \cancel{x^2}}{\sqrt{(2 + x^2)^3}}$$

$$= \frac{2}{\sqrt{(2+x^2)^3}}.$$

It can be seen that the second derivative is always positive. Therefore, the function is convex downward for all values of x.

4. Inflection Points

Definition: Consider a function $y = f(x)$, which is continuous at a point x_0. The function $f(x)$ can have a finite or infinite derivative $f'(x_0)$ at this point. If, when passing through x_0, the function changes the direction of convexity, i.e. there exists a number $\delta > 0$ such that the function is convex upward on one of the intervals $(x_0 - \delta, x_0)$ or $(x_0, x_0 + \delta)$, and is convex downward on the other, then x_0 is called a *point of inflection* of the function $y = f(x)$.

Theorem. If x_0 is a point of inflection of the function $f(x)$, and this function has a second derivative in some neighborhood of x_0 which is continuous at the point x_0 itself, then

$$f''(x_0) = 0.$$

Second Derivative Test: If the function $f(x)$ is continuous and differentiable at a point x_0, has a second derivative $f''(x_0)$ in some deleted δ-neighborhood of the point x_0 and if the second derivative changes sign when passing through the point x_0 then x_0 is a point of inflection of the function $f(x)$

Third Derivative Test: Let $f''(x_0) = 0$, $f'''(x_0) \neq 0$. Then x_0 is a point of inflection of the function $f(x)$

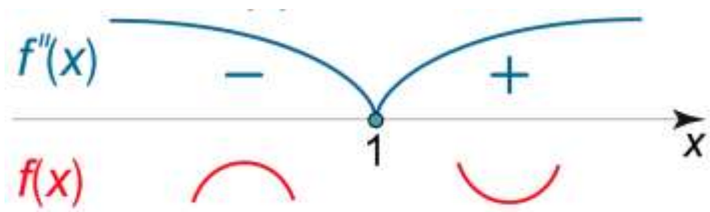
Example 1. Find the points of inflection of the function $f(x) = x^3 - 3x^2 - 1$.

Compute the first and second derivatives:

$$f'(x) = (x^3 - 3x^2 - 1)' = 3x^2 - 6x;$$

$$f''(x) = (3x^2 - 6x)' = 6x - 6.$$

We see that $f''(x) = 0$ at $x = 1$. The function changes concavity as shown in figure below.



Since

$$f(1) = 1^3 - 3 \cdot 1^2 - 1 = -3,$$

the inflection point is at $(1, -3)$.

Example 2. Find the inflection points of the function $f(x) = x^4 - 6x^2$.
Compute the first derivative:

$$f'(x) = (x^4 - 6x^2)' = 4x^3 - 12x.$$

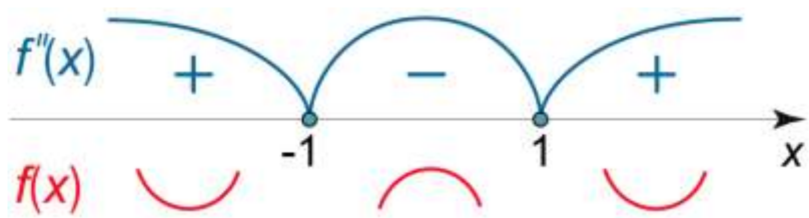
The second derivative is

$$f''(x) = (4x^3 - 12x)' = 12x^2 - 12 = 12(x^2 - 1).$$

Find the roots of the second derivative:

$$f''(x) = 0, \Rightarrow 12(x^2 - 1) = 0, \Rightarrow x_1 = -1, x_2 = 1.$$

We need to determine where the second derivative changes sign. Draw a sign $f''(x)$chart for (see below).



Clearly, the concavity changes at both points, $x = -1$ and $x = 1$. Hence, these points are points of inflection.

We can easily calculate their y −coordinates:

$$f(-1) = (-1)^4 - 6 \cdot (-1)^2 = -5;$$
$$f(1) = 1^4 - 6 \cdot 1^2 = -5.$$

So, the inflection points are $(-1, -5)$ and $(1, -5)$.

Example 3. Find the points of inflection of the function $f(x) = x^4 - 12x^3 + 48x^2 + 12x + 1$.

Find the derivatives:

$$f'(x) = (x^4 - 12x^3 + 48x^2 + 12x + 1)' = 4x^3 - 36x^2 + 96x + 12$$
$$= 4(x^3 - 9x^2 + 24x + 3);$$
$$f''(x) = (4(x^3 - 9x^2 + 24x + 3))' = 4(3x^2 - 18x + 24)$$
$$= 12(x^2 - 6x + 8).$$

Calculate the roots of the second derivative:

$f''(x) = 0, \Rightarrow 12(x^2 - 6x + 8) = 0, \Rightarrow x^2 - 6x + 8 = 0, \Rightarrow x_1 = 2, x_2 = 4.$

In this case it is convenient to use the second sufficient condition for the existence of an inflection point. The third derivative is written as

$$f'''(x) = (12(x^2 - 6x + 8))' = 12(2x - 6) = 24(x - 3).$$

From this we immediately see that the third derivative is not zero at the points $x_1 = 2$ and $x_2 = 4$. Therefore, these points are points of inflection.

5. *Asymptotes*

Definition: An asymptote of a curve $y = f(x)$ that has an infinite branch is called a line such that the distance between the point $(x, f(x))$ lying on the curve and the line approaches zero as the point moves along the branch to infinity.

Vertical Asymptote

Definition: The straight line $x = a$ is a *vertical asymptote* of the graph of the function $y = f(x)$ if at least one of the following conditions is true:

$$\lim_{x \to a-0} f(x) = \pm\infty, \lim_{x \to a+0} f(x) = \pm\infty.$$

Oblique Asymptote

Definition: The straight line $y = kx + b$ is called an oblique (slant) asymptote of the graph of the function $y = f(x)$ as $x \to +\infty$ (Figure) if

$$\lim_{x \to +\infty} [f(x) - (kx + b)] = 0.$$

Theorem. A straight line $y = kx + b$ is an asymptote of a function $y = f(x)$ as $x \to +\infty$ if and only if the following two limits are finite:

$$\lim_{x \to +\infty} \frac{f(x)}{x} = k \quad \text{and} \quad \lim_{x \to +\infty} [f(x) - kx] = b.$$

Horizontal Asymptote

Definition: In particular, if $k = 0$, we obtain *a horizontal asymptote*, which is described by the equation $y = b$. The theorem on necessary and sufficient conditions for the existence of a horizontal asymptote is stated as follows:

Theorem. A straight line $y = b$ is an asymptote of a function $y = f(x)$ as $x \to +\infty$, if and only if the following limit is finite:

$$\lim_{x \to +\infty} f(x) = b.$$

The case $x \to -\infty$ is considered in the same way.

Example 1. Find the asymptotes of the function $f(x) = \frac{x}{x+1}$.

When $x = -1$, the function has a discontinuity of the second kind. Indeed:

$$\lim_{x \to -1-0} f(x) = \lim_{x \to -1-0} \frac{x}{x+1} = \frac{-1}{(-1-0)+1} = \frac{-1}{-0} = +\infty,$$

$$\lim_{x \to -1+0} f(x) = \lim_{x \to -1+0} \frac{x}{x+1} = \frac{-1}{(-1+0)+1} = \frac{-1}{+0} = -\infty.$$

Hence, $x = -1$ is the equation of the vertical asymptote.

Find the horizontal asymptote. Compute the limit:

$$\lim_{x \to \pm\infty} \frac{x}{x+1} = \lim_{x \to \pm\infty} \frac{1}{1 + \frac{1}{x}} = 1.$$

Thus, there exists a horizontal asymptote for the curve, and its equation is $y = 1$.

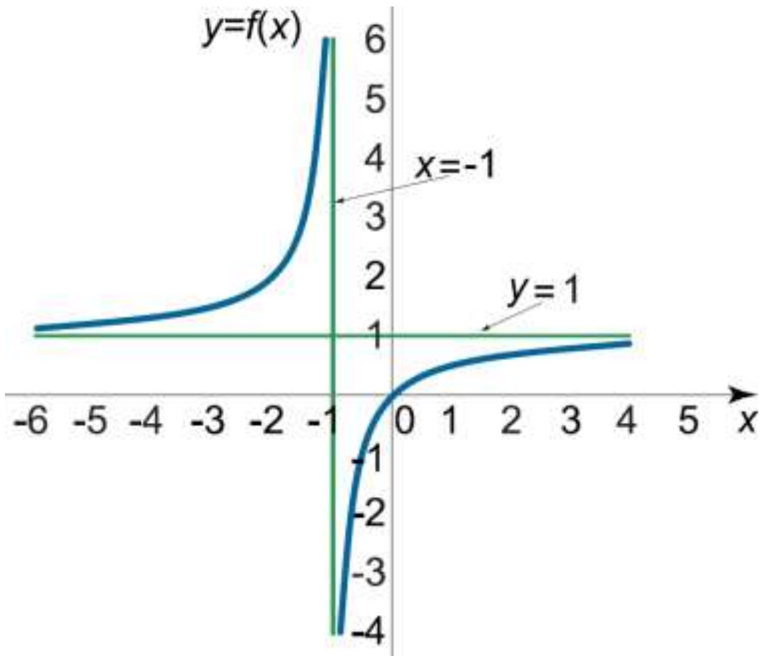
The function has no oblique asymptotes. This can be verified by calculating the coefficients k and b:

$$k = \lim_{x \to \pm\infty} \frac{y(x)}{x} = \lim_{x \to \pm\infty} \frac{x}{(x+1)x} = \lim_{x \to \pm\infty} \frac{x}{x^2 + x} = \lim_{x \to \pm\infty} \frac{\frac{1}{x}}{1 + \frac{1}{x}} = 0,$$

$$b = \lim_{x \to \pm\infty} [y(x) - kx] = \lim_{x \to \pm\infty} (\frac{x}{x+1} - 0) = \lim_{x \to \pm\infty} \frac{1}{1 + \frac{1}{x}} = 1.$$

It can be seen that actually we obtained the horizontal asymptote, which has already been defined above.

So, the graph of the function has the vertical asymptote $x = -1$ and the horizontal asymptote $y = 1$ (Figure).



Example 2. Find the asymptotes of the function $f(x) = \frac{x^2 - 2x - 3}{x}$.

The function has a discontinuity at $x = 0$. Since

$$\lim_{x \to 0-0} f(x) = \lim_{x \to 0-0} \frac{x^2 - 2x - 3}{x} = +\infty;$$

$$\lim_{x \to 0+0} f(x) = \lim_{x \to 0+0} \frac{x^2 - 2x - 3}{x} = -\infty,$$

the straight line $x = 0$ (the y –axis) is a vertical asymptote.

The function does not have a horizontal asymptote because the following limits are infinite:

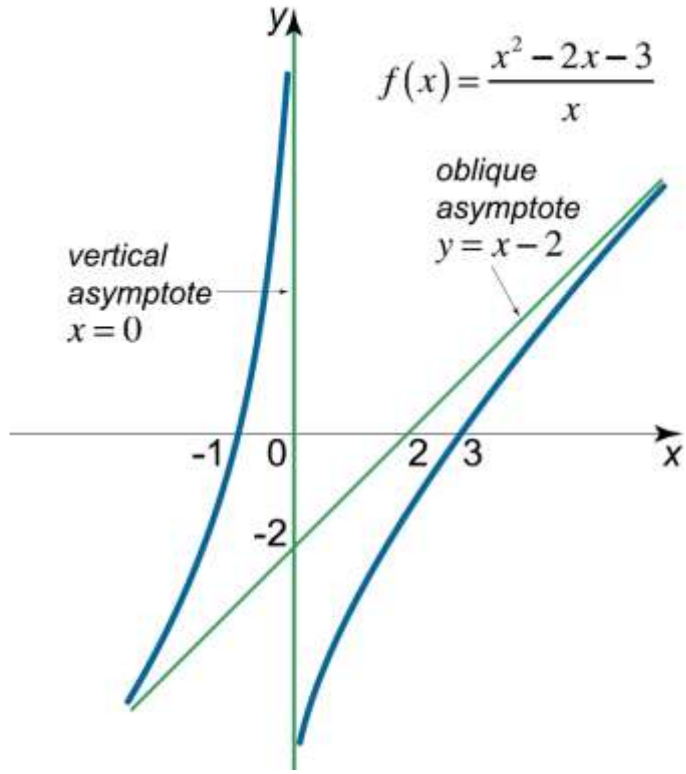
$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{x^2 - 2x - 3}{x} = +\infty;$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^2-2x-3}{x} = -\infty.$$

We write the function as follows:

$$f(x) = \frac{x^2-2x-3}{x} = x-2-\frac{3}{x}.$$

The term $\frac{3}{x}$ approaches zero as $x \to \pm\infty$. Hence, the function has the oblique asymptote $y = x - 2$.



Example 3. Find the asymptotes of the function $f(x) = \frac{3x^2-2x+1}{x-1}$.

It is clear that the line $x = 1$ is a vertical asymptote, because at this point the function has a discontinuity and the following conditions are true:

$$\lim_{x\to1-0} f(x) = \lim_{x\to1-0} \frac{3x^2-2x+1}{x-1} = \frac{3(1-0)^2-2(1-0)+1}{1-0-1} = -\infty,$$

$$\lim_{x\to1+0} f(x) = \lim_{x\to1+0} \frac{3x^2-2x+1}{x-1} = \frac{3(1+0)^2-2(1+0)+1}{1+0-1} = +\infty.$$

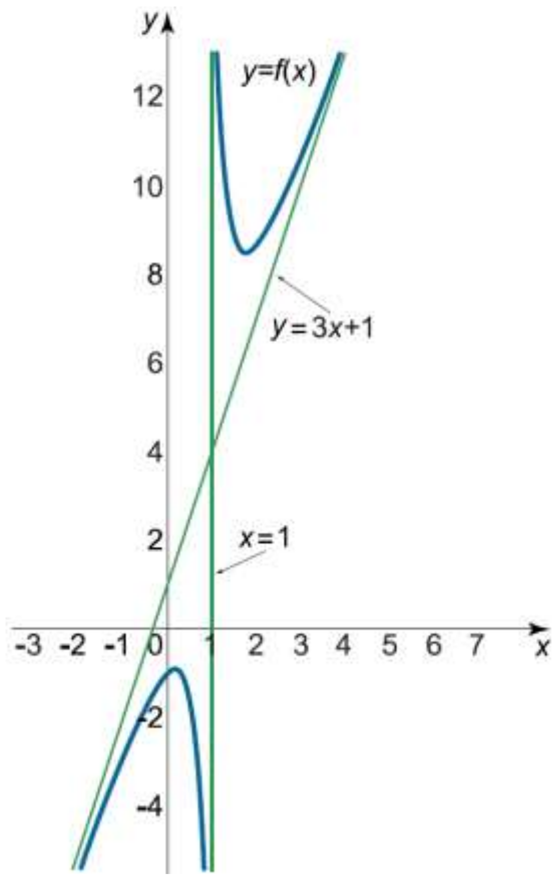
We write the function as

$$y = f(x) = \frac{3x^2 - 2x + 1}{x - 1} = \frac{3x^2 - 3x + x - 1 + 2}{x - 1}$$
$$= \frac{3x\cancel{(x-1)}}{\cancel{x-1}} + \frac{\cancel{x-1}}{\cancel{x-1}} + \frac{2}{x-1} = 3x + 1 + \frac{2}{x-1}$$
$$= 3x + 1 + \alpha(x),$$

where $\alpha(x) \to 0$ as $x \to \pm\infty$.

Thus, the function has the oblique asymptote $y = 3x + 1$.

Note that a rational function may have an oblique asymptote if the degree of the numerator is one greater than the degree of the denominator. A schematic view of this curve is shown in Figure



6. Curve Sketching

Example 1. $y = x^3 - 3x^2 + 2x$.

1. The function is defined for all $x \in \mathbb{R}$. Consequently, this function has no vertical asymptotes. Check for oblique (slant) asymptotes. Calculate the slope of the asymptote:

$$k = \lim_{x\to\pm\infty} \frac{y(x)}{x} = \lim_{x\to\pm\infty} \frac{x^3 - 3x^2 + 2x}{x} = \lim_{x\to\pm\infty} (x^2 - 3x + 2) = +\infty.$$

This indicates that the function has also no oblique asymptotes.

2. Determine the points of intersection of the graph with the coordinate axes:

$$y(0) = 0.$$

Next, solving the equation

$$x^3 - 3x^2 + 2x = 0,$$

we find:

$$x(x^2 - 3x + 2) = 0, \Rightarrow x_1 = 0, x_2 = 1, x_3 = 2,$$

that is the function has three real roots.

3. The intervals where the function is positive or negative can be found solving the following inequality (Figure):

$$x^3 - 3x^2 + 2x > 0, \Rightarrow x(x-1)(x-2) > 0.$$

4. The first derivative of the function is

$$y'(x) = (x^3 - 3x^2 + 2x)' = 3x^2 - 6x + 2.$$

We find the stationary points by setting the first derivative equal to zero:

$$y'(x) = 0, \Rightarrow 3x^2 - 6x + 2 = 0, \Rightarrow D = 36 - 4 \cdot 3 \cdot 2 = 12, \Rightarrow$$

$$x_{1,2} = \frac{6 \pm \sqrt{12}}{6} = 1 \pm \sqrt{3} \approx 0{,}42; 1{,}58$$

When passing through the point $x = 1 - \frac{\sqrt{3}}{3}$, the derivative changes sign from plus to minus (Figure). Hence, this point is the maximum point. Similarly, we establish that $x = 1 + \frac{\sqrt{3}}{3}$ is the minimum point. Calculate the approximate value of the function at the points of maximum and minimum:

$$y(1 - \frac{\sqrt{3}}{3}) = (1 - \frac{\sqrt{3}}{3})^3 - 3(1 - \frac{\sqrt{3}}{3})^2 + 2(1 - \frac{\sqrt{3}}{3}) =$$

$$1 - 3 \cdot \frac{\sqrt{3}}{3} + 3 \cdot (\frac{\sqrt{3}}{3})^2 - (\frac{\sqrt{3}}{3})^3 - 3[1 - \frac{2\sqrt{3}}{3} + (\frac{\sqrt{3}}{3})^2] + 2 - \frac{2\sqrt{3}}{3} =$$

$$\cancel{1}-\sqrt{3}+\cancel{1}-\frac{\sqrt{3}}{9}-\cancel{3}+2\sqrt{3}-\cancel{1}+\cancel{2}-\frac{2\sqrt{3}}{3}=\frac{9\sqrt{3}-\sqrt{3}-6\sqrt{3}}{9}=\frac{2\sqrt{3}}{9}$$
$$\approx 0{,}38;$$

Similarly, we find that

$$y(1+\frac{\sqrt{3}}{3})=-\frac{2\sqrt{3}}{9}\approx -0{,}38.$$

Thus, the function has a local maximum at the point

$$(1-\frac{\sqrt{3}}{3},\frac{2\sqrt{3}}{9})\approx (0{,}42;0{,}38).$$

Respectively, the local minimum is reached at the point

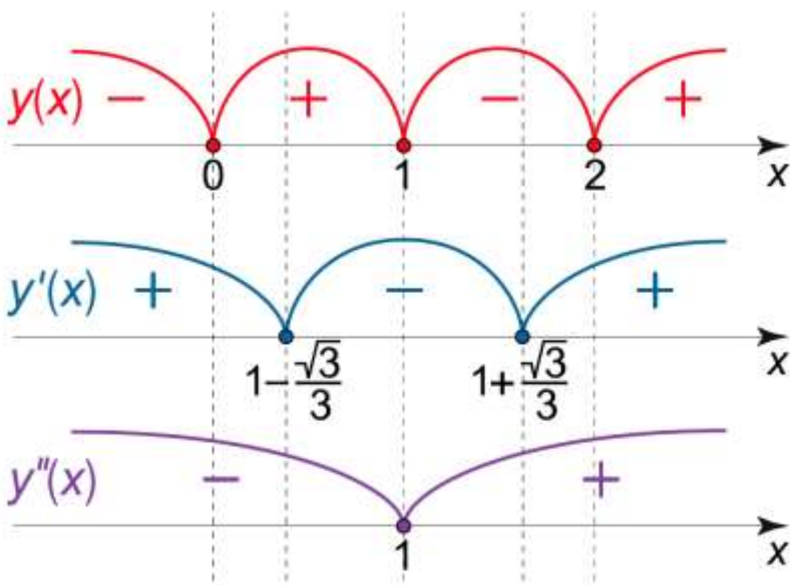
$$(1+\frac{\sqrt{3}}{3},-\frac{2\sqrt{3}}{9})\approx (1{,}58;-0{,}38)$$

The intervals of increasing/decreasing follow from Figure

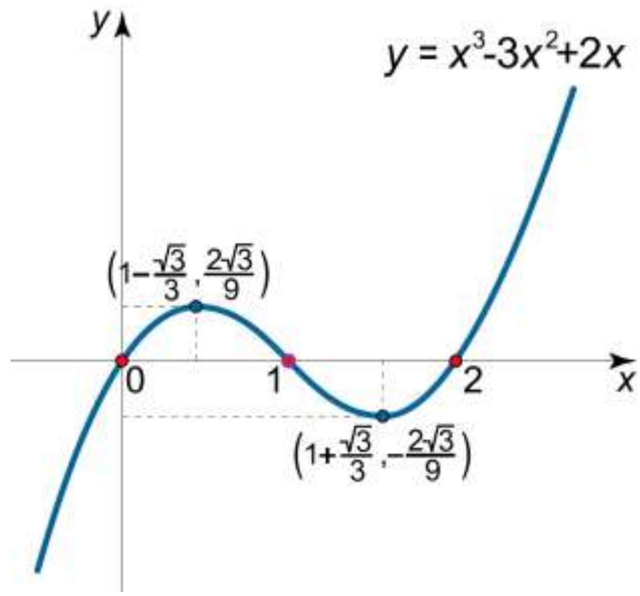
5. Consider the second derivative:

$$y''(x)=(3x^2-6x+2)'=6x-6;$$
$$y''(x)=0,\Rightarrow 6x-6=0,\Rightarrow x=1.$$

If $x\le 1$, the function is convex upward, and if $x\ge 1$, it is convex downward. Hence, the point $x=1$ is a point of inflection. At this point we have:



6. Given these results, we can draw a schematic graph of the function



Example 2. $y = \frac{1}{1+x^2}$.

The function is defined for all real values of x. Consequently, it has no vertical asymptotes. Since

$$\lim_{x\to\pm\infty} y(x) = \lim_{x\to\pm\infty} \frac{1}{1+x^2} = 0,$$

then the graph of the function has horizontal asymptote $y = 0$, that is the x-axis is the horizontal asymptote.

This function is even. Indeed,

$$y(-x) = \frac{1}{1+(-1)^2} = \frac{1}{1+x^2} = y(x).$$

It is obvious that the function has no roots and positive for all x. At the point $x = 0$, its value is

$$y(0) = \frac{1}{1+0^2} = 1.$$

Find the first derivative:

$$y'(x) = (\frac{1}{1+x^2})' = -\frac{1}{(1+x^2)^2} \cdot (1+x^2)' = -\frac{2x}{(1+x^2)^2}.$$

This shows that $x = 0$ is a stationary point. When passing through this point the derivative changes sign from plus to minus (Figure). Therefore, we have

a maximum at $x = 0$ Its value is $y(0) = 1$.
Calculate the second derivative:

$$y''(x) = (-\frac{2x}{(1+x^2)^2})' = -\frac{2(1+x^2)^2 - 2x \cdot 2(1+x^2)}{(1+x^2)^4}$$

$$= \frac{8x^2 - 2 - 2x^2}{(1+x^2)^3} = \frac{6x^2 - 2}{(1+x^2)^3}.$$

It is equal to zero at the following points:

$$y''(x) = 0, \Rightarrow \frac{6x^2 - 2}{(1+x^2)^3} = 0, \Rightarrow \frac{2(x-\sqrt{3})(x+\sqrt{3})}{(1+x^2)^3} = 0, \Rightarrow$$

$$x_1 = -\sqrt{3}, x_2 = \sqrt{3}.$$

When passing through these points the second derivative changes its sign. Therefore, both points are inflection points. The function is strictly convex downward in the intervals $(-\infty, -\sqrt{3})$ and $(\sqrt{3}, +\infty)$ and, accordingly, strictly convex upward in the interval $(-\sqrt{3}, \sqrt{3})$. Since the function is even, the found inflection points have the same values of$y$:

$$y(\pm\sqrt{3}) = \frac{1}{1+(\pm\sqrt{3})^2} = \frac{1}{1+3} = \frac{1}{4}.$$

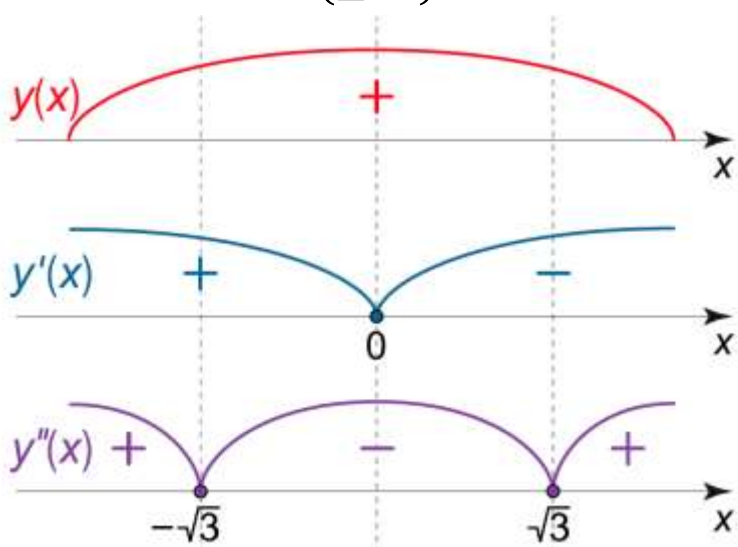
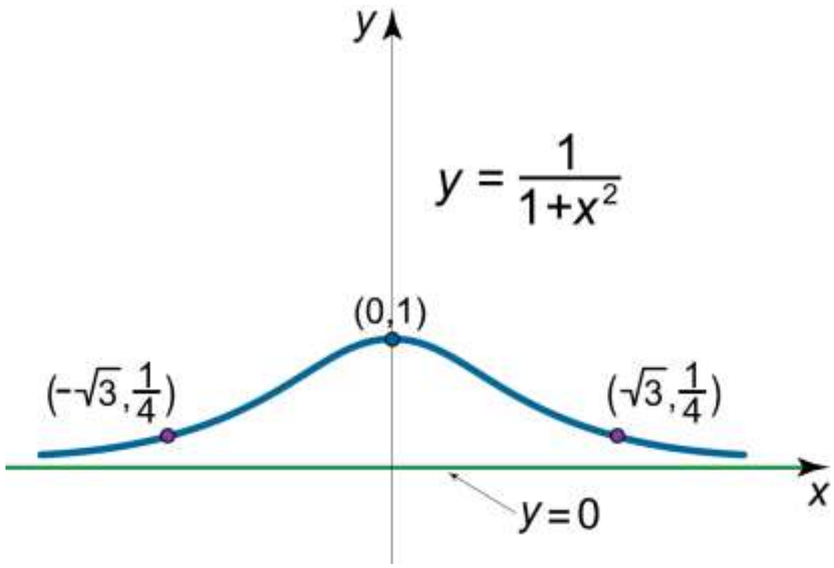


Figure presents a schematic graph of the function.



Example 3. $f(x) = \frac{x^2+1}{x-1}$.

The function is defined for all x except the point $x = 1$ where it has a discontinuity.

Find the intercept:

$$f(0) = \frac{0^2 + 1}{0 - 1} = -1.$$

The function is negative for $x < 1$ and positive for $x > 1$ but it has no x −intercepts.

Look for vertical asymptote near $x = 1$:

$$\lim_{x \to 1-0} f(x) = \lim_{x \to 1-0} \frac{x^2 + 1}{x - 1} = -\infty;$$

$$\lim_{x \to 1+0} f(x) = \lim_{x \to 1+0} \frac{x^2 + 1}{x - 1} = \infty.$$

There is a vertical asymptote at $x = 1$.

Rewrite the function in the form

$$f(x) = \frac{x^2 + 1}{x - 1} = \frac{x^2 - x + x - 1 + 2}{x - 1} = \frac{x(x - 1) + x - 1 + 2}{x - 1}$$
$$= x + 1 + \frac{2}{x - 1},$$

where $\frac{2}{x-1} \to 0$ as $x \to \infty$. Hence the function has an oblique asymptote $y =$

$x + 1$.

Take the first derivative:

$$f'(x) = (\frac{x^2+1}{x-1})' = \frac{2x \cdot (x-1) - (x^2+1) \cdot (x-1)}{(x-1)^2}$$
$$= \frac{2x^2 - 2x - x^2 - 1}{(x-1)^2} = \frac{x^2 - 2x - 1}{(x-1)^2}.$$

Determine the critical points:

$$f'(x) = 0, \Rightarrow \frac{x^2 - 2x - 1}{(x-1)^2} = 0, \Rightarrow \begin{cases} x^2 - 2x - 1 = 0 \\ x \neq 1 \end{cases}.$$

Solve the quadratic equation:

$$x^2 - 2x - 1 = 0, \Rightarrow D = (-2)^2 - 4 \cdot (-1) = 8, \Rightarrow$$
$$x_{1,2} = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}.$$

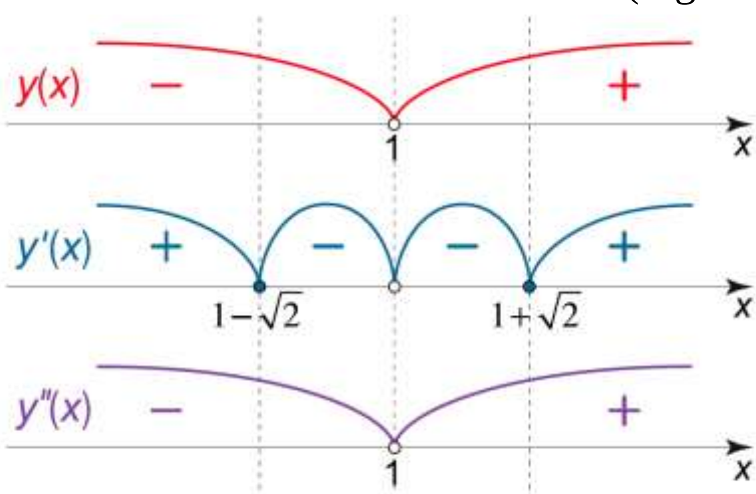
Thus, the function has two critical points: $x_1 = 1 - \sqrt{2} \approx -0.41$ and $x_2 = 1 + \sqrt{2} \approx 2.41$

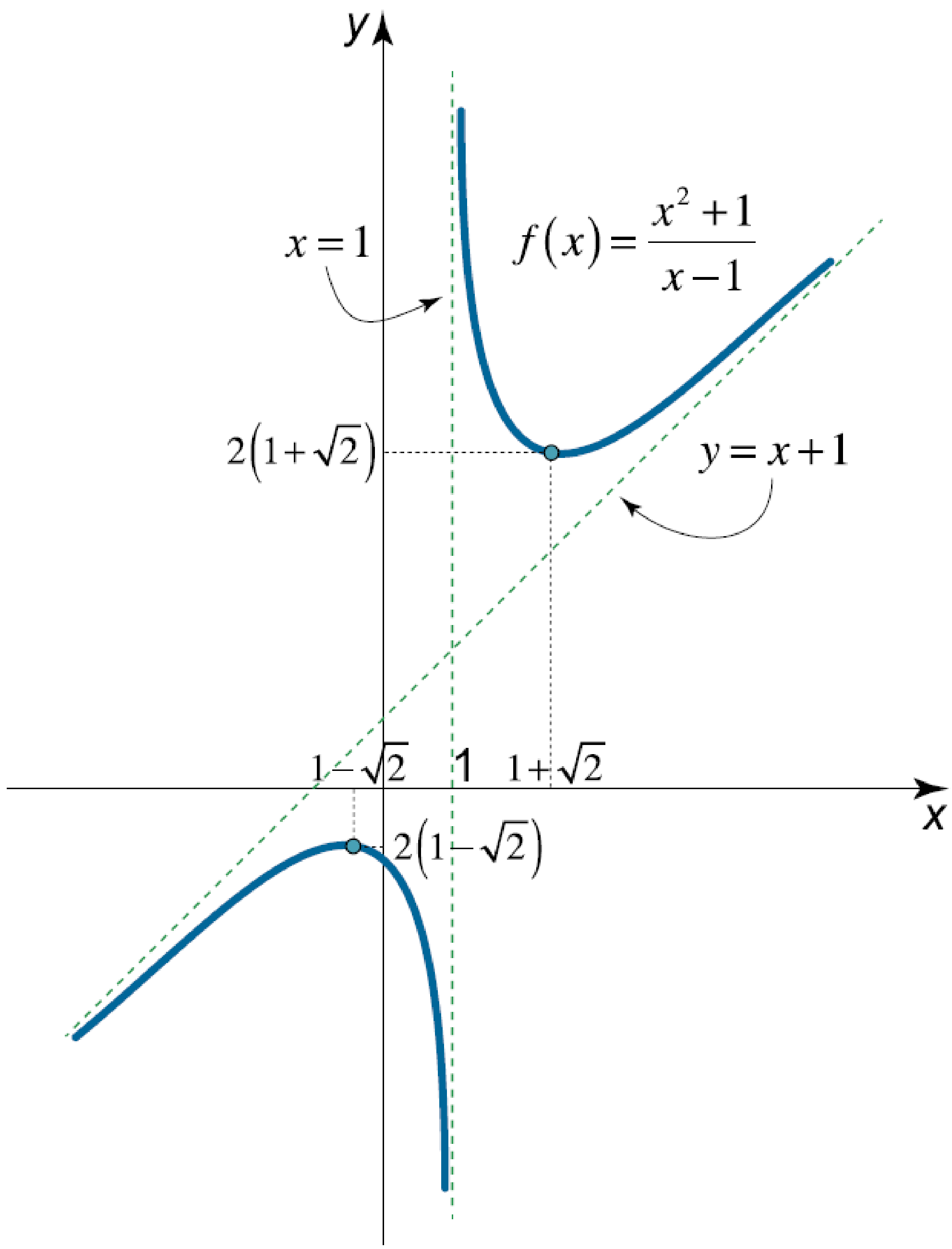
The second derivative is written as

$$f''(x) = (\frac{x^2 - 2x - 1}{(x-1)^2})' = \frac{4}{(x-1)^3}.$$

We see that the function is concave downward at $x < 1$ and concave upward at $x > 1$ though it has no inflection points.

Draw a sign chart for the function and its derivatives (Figure).





7. Global Extrema of Functions

Example 1. Find the global maximum and minimum of the function in the given interval: $f(x) = x^2 - 2x + 5$, where $x \in [-1,4]$.

The function is defined and differentiable for all $x \in \mathbb{R}$. Determine its stationary points:

$$f'(x) = 0, \Rightarrow (x^2 - 2x + 5)' = 0, \Rightarrow 2x - 2 = 0, \Rightarrow x = 1.$$

This local extremum point belongs to the interval $(-1,4)$. We compute the values of the function at $x = 1$ and at the endpoints of the interval:

$$f(1) = 1^2 - 2 \cdot 1 + 5 = 4, f(-1) = (-1)^2 - 2 \cdot (-1) + 5 = 8, f(4) = 4^2 - 2 \cdot 4 + 5 = 13.$$

Consequently, the maximum value of the function is equal $f(4) = 13$, and the minimum value is $f(1) = 4$.

Example 2. Calculate the difference *d* between the global maximum and global minimum values of $f(x) = x^2 - 4x + 6$ in the interval $[-2,4]$.
Find the derivative:

$$f'(x) = (x^2 - 4x + 6)' = 2x - 4.$$

Solve the equation $f'(c) = 0$ to determine the critical points:

$$f'(c) = 0, \Rightarrow 2c - 4 = 0, \Rightarrow c = 2.$$

We must evaluate $f(x)$ at the critical point $x = 2$ and at the endpoints $x = -2, x = 4$:

$$f(-2) = (-2)^2 - 4 \cdot (-2) + 6 = 18,$$
$$f(2) = 2^2 - 4 \cdot 2 + 6 = 2,$$
$$f(4) = 4^2 - 4 \cdot 4 + 6 = 6.$$

Thus, the global maximum value is 18 and the global minimum value is 2, so the difference *d* is equal to

$$d = f_{max} - f_{min} = 18 - 2 = 16.$$

Example 3. Find the global extrema of the function $f(x) = 3x^4 - 6x^2 + 2$ in the interval $[-2,2]$.
This function is defined and differentiable on the whole real axis. In this case, all local extrema can be found from the equation $f'(x) = 0$:

$$f'(x) = (3x^4 - 6x^2 + 2)' = 12x^3 - 12x = 12x(x^2 - 1) = 12x(x-1)(x+1);$$
$$f'(x) = 0, \Rightarrow 12x(x-1)(x+1) = 0, \Rightarrow x_1 = 0, x_2 = -1, x_3 = 1.$$

As it can be seen, the function has three local extrema and all these points fall in the given interval $[-2,2]$. Calculate the values of the function at the points of extremum and at the endpoints of the interval:

$$f(0) = 3 \cdot 0^4 - 6 \cdot 0^2 + 2 = 2; f(-1) = 3 \cdot (-1)^4 - 6 \cdot (-1)^2 + 2 = -1$$

$$f(-2) = 3 \cdot (-2)^4 - 6 \cdot (-2)^2 + 2 = 26.$$

Since the function is even, we can write:

$$f(1) = f(-1) = -1, f(2) = f(-2) = 26.$$

Thus, the function has the minimum value -1 at two points: at $x = -1$ and $x = 1$ The maximum value 26 is also attained at two points: at $x = -2$ and $x = 2$. A schematic graph of the function is given in Figure