MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE
National Technical University
«Kharkiv Polytechnic Institute»

BASIC STRENGTH CALCULATIONS

Methodological instructions for the implementation of practical tasks, independent work and individual tasks in the disciplines «Technical Mechanics» for foreign applicants of the specialty 141 – Electric Power, Electrical Engineering and Electromechanics of the first (bachelor's) level of education of full-time education

Kharkov
NTU «KHPI»
2024

Methodological instructions for the implementation of practical tasks, independent work and individual tasks in the disciplines «Technical Mechanics» for foreign applicants of the specialty 141 – Electric Power, Electrical Engineering and Electromechanics of the first (bachelor's) level of education of full-time education / Compiled by V. V. Klitnoi. Kharkiv: NTU «KhPI», 2024. 58 p.

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CONTENTS

Introduction

The course "Technical Mechanics" is a synthetic course, which includes the main sections of the following disciplines:

1. Theoretical Mechanics.
2. Theory of mechanisms and machines.
3. Strength of materials.
4. Machine parts.

The development of the course is based on the knowledge of the basic laws of physics, higher mathematics, descriptive geometry, engineering, and computer graphics. It is assumed that the trainees must have logical thinking, as well as knowledge of dimensional physical characteristics.

The course "Applied Mechanics" is one of the oldest general engineering training courses in technical colleges. It completes the cycle of engineering disciplines and is a link between the general technical and specialized disciplines.

Practical training 1. The main provisions of the strength of materials science. Internal forces and efforts, voltage, displacement and deformation.

1.1. Objectives, the purpose and object of the science - strength of materials.

Strength of materials is referred to as the science of engineering methods for calculating the strength, stiffness and stability of machine elements (components) and structures.

In contrast to the theoretical mechanics, where all the bodies are considered to be absolutely rigid and non-deformable, in strength of materials they consider real bodies, i.e. such that change their geometry and dimensions (deformed).

Deformation that completely disappears after removal of the load is called elastic deformation.

Deformation that completely or partially remains after removal of the external load is called residual or plastic one.

In strength of materials the emergence of plastic deformation is prohibited and considered to be the beginning of destruction (only elastic deformation is considered).

During the operation of machines and structures their elements (rods, beams, plates, screws, etc.) to some extent participate in structural behavior and are exposed to different forces - loads. For proper operation, the design should satisfy the necessary conditions of strength, stiffness and stability.

The concept of strength, rigidity and stability.

Strength - the ability of the structure, its parts and components under the influence of external forces not to break down and acquire permanent deformation.

Flex - the ability of the structure and its elements under the action of external forces to obtain an elastic deformation not exceeding the allowed values.

Resilience - the ability of a structure or its elements under the influence of external forces not to change its geometry.

In order for the construction to meet on the whole the requirements of strength, stiffness and stability and therefore to be reliable in operation, it is necessary to give the most rational form to its elements, and knowing the properties of the materials from which they will be manufactured, determine the appropriate geometric dimensions, depending on the size and the nature of operating forces.

The discipline strength of materials solves the specified problems based on both theoretical and experimental data being equally important in this field of science. In the theoretical part, this science is based on theoretical mechanics and mathematics, and in the experimental one on physics and materials science.

1.2. Design scheme. Typical forms of the elements of engineering structures.

The design scheme is a simplified representation of a real machine part, where they deliberately do not taken into account a number of less important from the point of view of the calculations performed factors.

The need of schematization can be explained by the fact that the calculation of a relatively simple machine part taking into account all the design factors, even in those cases when it is fundamentally possible, is not practically always acceptable due to its bulkiness and complexity.

Sometimes at calculation of the same object they use design schemes of varying complexity, which allows a refinement of the calculation made on the basis of a rougher design scheme. At the same time, different according to their design and actual purpose real parts may correspond to the same design scheme.

When choosing a design scheme the geometry of the body is simplified, as a result all the bodies are reduced to the 4 forms: squared beam, plate, shell and dimensional body.

Square beam is called a body, in which one dimension (length) substantially exceeds two other (transverse) dimensions (Fig. 1.1). The beam with a straight axis is called the rod.

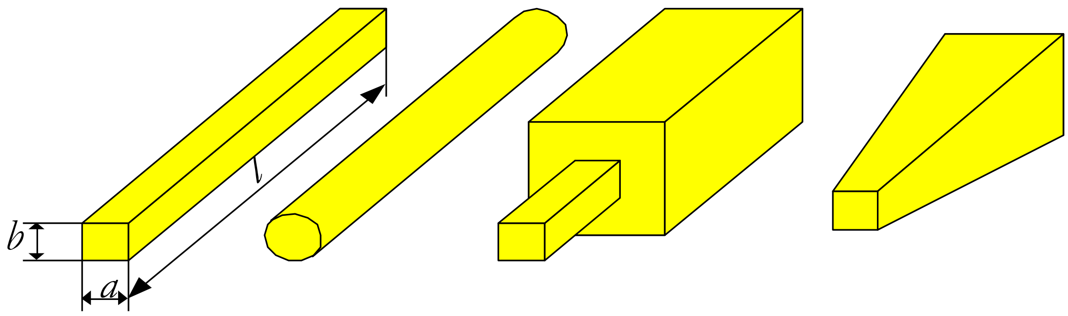
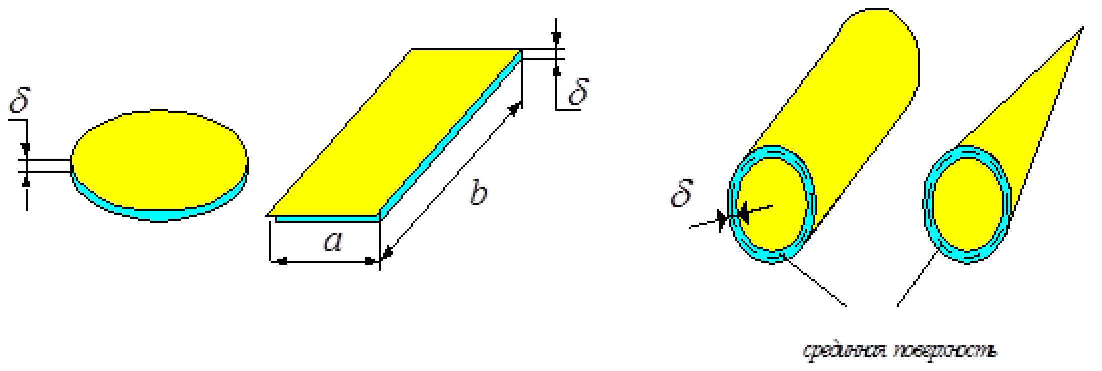


Figure 1.1.

In machines and structures there are found straight, curved, prismatic and variable cross-section rods. Rods whose wall thickness is much less than the cross-sectional dimensions are referred to as thin-walled.



median surface

Figure 1.2.

The plate and shell are bodies whose thickness is much smaller than its other dimensions (Fig. 1.2).

The surface which divides the thickness of both the plate and the shell into equal parts is called the median surface. If the median surface is a plane, then the design object is called - plate. The shells are distinguished according to the shape of the median surface: cylindrical, conical, spherical, etc.

The objects, whose all three dimensions are of the same order, are called volumetric bodies.

1.3. Classification of forces studied in the SM.

1. External and internal forces

External - the forces of bodies interaction with each other.

Internal - forces caused by external forces that seek to restore the body to the undeformed state.

2. Concentrated and distributed forces

Concentrated - forces that act on the body through a small part of the surface, conventionally accepted as a geometric point (abstractedly introduced forces to simplify calculations), these forces are measured in N, kN, the notion of concentrated pair or torque is introduced similarly (Nm, kNm) (Fig. 1.3).

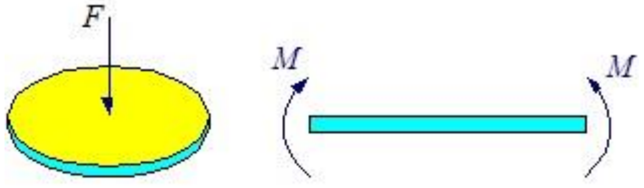


Figure 1.3.

Distributed - forces that act along the line (N/m), the surface (N/m2) or by volume (N/m3, weight force, inertia) (Figure 1.4).

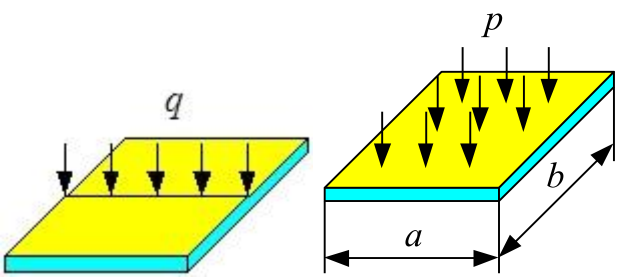


Figure 1.4.

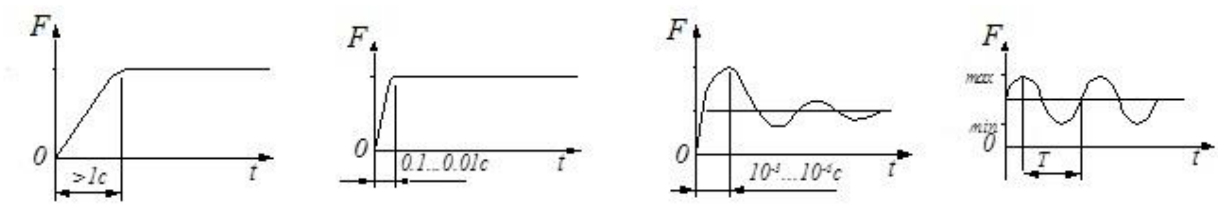
3. Active and reactive forces

Active - external forces acting on the body.

Reacting - forces that arise in constraints retaining the body (constraints).
4. *Static and dynamic forces:*

Static - forces that within a few seconds increase from zero to its maximum value, and retain their value and direction for a long time (Fig. 1.5, a).

Dynamic - forces that are accompanied by significant accelerations of both a deformed body and interacting with them bodies, as this takes place there arise the forces of inertia, which cannot be neglected; the dynamic loads are divided into instantly applied (Figure 1.5 b), percussion (Fig. 1.5, c) and re-variables (Figure 1.5, d).



a b c d

Fig. 1.5.

1.4. The main hypotheses and principles of strength of materials

To construct a theory of strength of materials science there can be accepted several hypotheses:

1. The hypothesis of plane sections.

Every plane section taken before deformation remains plane even after deformation occurs.

2. The hypothesis of deformations smallness.

Elastic deformations experienced by the body are small compared to the linear dimensions of the body (the points of forces application before and after the deformation do not change).

3. The hypothesis of homogeneity and continuity.

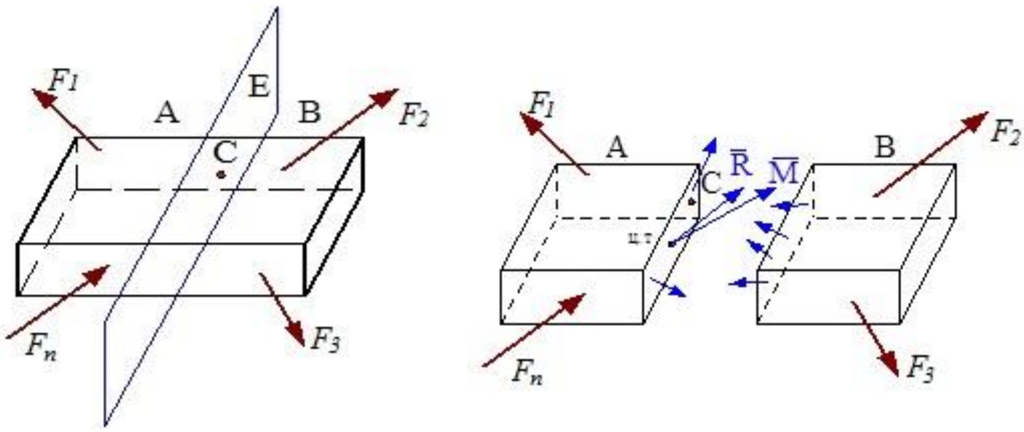
The entire volume of the body is filled with the materials in question and the properties of this material are identical in all points and in all directions (such materials are called isotropic (steel, copper)). If the material properties vary depending on the direction or point of examination, the material is anisotropic (wood, plastic, cast iron).

4. The hypothesis of superposition.

If the body is acted on by several forces, the gross deformation of the body can be represented as the sum of deformations taken by the body separately from each force (superposition principle).

The above-stated hypotheses and principles, as well as certain other assumptions, which will be discussed below, allow solving a wide range of tasks on the strength, stiffness and stability. The results are in good agreement with the experimental data.

1.5. The method of sections. Internal force factors.



a ***b***

Fig. 1.6

The body is acted on by the system of external forces (($\vec{F_1},...,\vec{F_n}$). It is necessary to determine the internal forces arising in a given section (Fig. 1.6, a). To determine the internal forces in the structural element in the resistance of materials they use the method of *cross-sections*. To do this, we mentally cut the body in the cross-section, one part of the body is discarded and the remaining one will be examined in the state of equilibrium under the influence of internal and external forces (Fig. 1.6, b).

The internal forces acting on the left side of the body are equal in magnitude but opposite in direction of the internal forces acting on the right side of the body (they must be such as to satisfy the condition of continuity of deformations - the right and left sides must be blended).

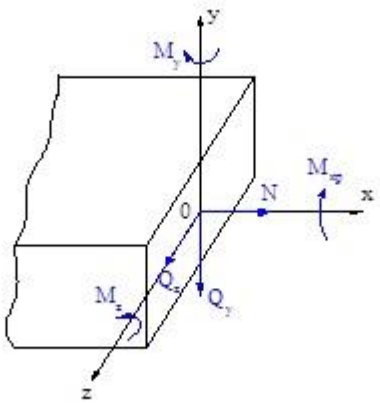


Fig. 1.7.

Bringing them to the center of gravity of the cross-section, we get the main vector of forces $\vec{R}$ and $\vec{M}$ - the main torque. If in the center of gravity of the cross-section one poses the start of the coordinates system x, y, z (axis x - perpendicular to the cross-section; z, y - lie in the cross-sectional plane), then $\vec{R}$ and $\vec{M}$ can be each decomposed into three components (projections): three forces N, Q_y, Q_z and three torques M_y, M_z, $M_{кр}$ (Fig. 1.7).

- N - longitudinal (normal) force;
- Q_y (Q_z) – cross-sectional (shear) forces;
- M_y (M_z) - bending moments in the cross-section;
- $M_{кр}$ - torque.

Under the action of forces shown in Fig. 1.7 this part of the rod is in equilibrium, i.e. six equilibrium equations must be hold for it:

$$\sum X = 0; \quad \sum Y = 0; \quad \sum Z = 0;$$
$$\sum M_x = 0; \quad \sum M_y = 0; \quad \sum M_z = 0. \qquad (1)$$

Thus, we obtain 6 unknown and 6 static equations of which there can be found unknown internal force factors.

Each kind of effort results in typical for it deformation:

1. Tension - compression ($N \neq 0$, the rest is equal to 0);
2. Torsion ($M_{кр} \neq 0$, the rest is equal to 0)
3. Pure bending ($M_y \neq 0$, $M_z \neq 0$, $M = \sqrt{M_y^2 + M_z^2} \neq 0$, the rest is equal to 0);
4. Lateral bending ($Q_y \neq 0$, $Q_z \neq 0$, $M_y \neq 0$, $M_z \neq 0$, the rest is equal to 0);
5. Pure shear ($Q_y \neq 0$, $Q_z \neq 0$, $Q = \sqrt{Q_y^2 + Q_z^2} \neq 0$, the rest is equal to 0);

These types of deformations are called simple.

1/6. Stress. Displacement and deformation.

Stresses characterize the intensity of loading and they are determined by the ratio of internal forces to the area on which they act (Figure 5.8): $\overrightarrow{p_{cp}} = \frac{\Delta \vec{R}}{\Delta A}$ - the average value of the stress; $\vec{p} = \lim_{\Delta F \to 0} \frac{\Delta \vec{R}}{\Delta A} = \frac{d\vec{R}}{dA}$ - the true value of the stress at a given point.

If the internal forces are evenly distributed over the cross-section, then:

$$p = \frac{\vec{R}}{A} \qquad (2)$$

In general, the stress vector is directed at an angle to the cross-section.

If the axis x is perpendicular to the section; z, y - lie in the plane section. Then the projection of the stress $\bar{p}$ on the x-axis is called the normal stress, and the projections on the z-axis and y - the shear stress τ_y , τ_z . They are measured in the units of stress - pascals (Pa) and multiples - (psi, MPa).

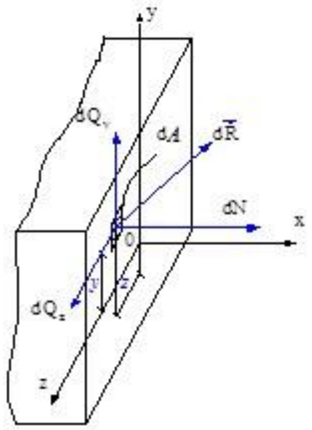


Fig. 1.8.

If the efforts on the element can be considered to be distributed uniformly, then:

$$\sigma = \frac{dN}{dA} \ , \ \tau_y = \frac{dQ_y}{dA}, \ \tau_z = \frac{dQ_z}{dA}. \qquad (3)$$

Full shear stress:

$$\tau = \sqrt{\tau_y^2 + \tau_z^2}. \qquad (4)$$

Full stress at the point of

$$p = \sqrt{\sigma^2 + \tau_y^2 + \tau_z^2}. \qquad (5)$$

The concept of "stress" plays a very important role in the calculations for strength. Therefore, much of the course the strength of materials is given to the study of techniques to calculate the stresses σ and τ.

Under the influence of external forces ($\vec{F_1},...,\vec{F_n}$) of the point they get the displacement Δl (Fig. 1.9).

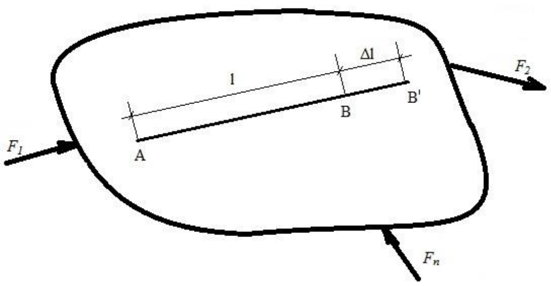


Fig. 1.9.

Deformation is a relative movement: $\varepsilon_{cp} = \frac{\Delta l}{l}$ (average strain of the segment l), $\varepsilon = \lim\limits_{l \to 0} \frac{\Delta l}{l}$ (the true meaning of deformation).

In general, deformation can be seen in projections on the coordinate axes: $\varepsilon_x, \varepsilon_y, \varepsilon_z$.

Total deformation:

$$\varepsilon = \sqrt{\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2} \qquad (6)$$

Practical training ***2***. Tension-compression.

2.1. Tension-compression. The internal forces and stresses.

It is this type of deformation in which under the action of external force factors in each cross-section of the body there is only one internal force factor – the longitudinal elasticity force N . The rest of the force factors are lacking.

The force R acts on the rod and it is necessary to determine the internal force factors acting in section I-I (Fig. 2.1).

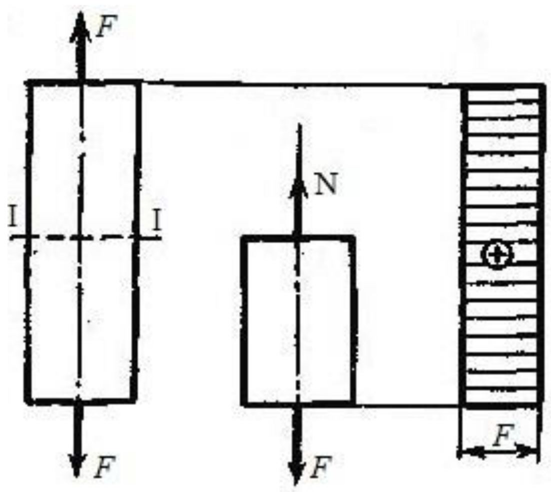


Fig. 2.1.

For this let’s use the method of sections, that is we shall mentally cut the rod in section *I-I*, discard the top of the rod will be discarded, and the bottom will be examined in the state of equilibrium under the influence of external and internal forces.

At each point of the cross-section there arises the internal effort σ , the resultant of which will be the force N .

Since the internal forces in the cross-sections considerably remote from the points of application of concentrated loads are distributed evenly over the cross-section, then

$$N = \int_F \sigma dA; \;\; \sigma \to const \;\; ; \;\; N = \sigma \int_F dA = \sigma A \; . \qquad (1)$$

$\sigma = \dfrac{N}{A}$ - according to stress determination in the cross-section.

From the equilibrium condition of the bottom part $\sum P_x = 0 \quad N = F$ and $\sigma = \dfrac{F}{A}$.

Let's depict graphically the change in the force N along the axis of the rod (Fig. 6.1). This graphic is called orthographic epure (diagram). Each line on the orthographic epure in the scale of construction corresponds to the internal power factor in the section under examination.

If the force N is directed opposite the section it causes tensile strain with a "+" sign if directed to the cross-section - strain compression, taken with the sign "-".

2.2. Hooke's law, Poisson's ratio.

The force F is applied to the rod with the length l and diameter b, under the action of which the rod extends by the amount of the value $\Delta l = l_1 - l$ and contracts by the amount of the value $\Delta b = b - b_1$ (Fig. 2.2).

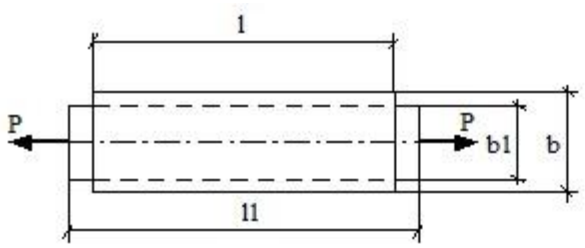


Fig. 2.2.

The ratio of elongation to the original length of the rod is called the longitudinal deformation $\varepsilon = \dfrac{\Delta l}{l}$.

From the experiments it was revealed that between the longitudinal strain and the normal stress there is a directly proportional relationship:

$$\sigma = E\varepsilon . \qquad (1)$$

The above relationship is called the Hooke's law (named after the English scientist, who first discovered it in 1660). It can be stated as follows: the longitudinal strain is directly proportional to the corresponding normal stress.

The value E, which enters the formula that expresses the Hooke's law, is one of the most important physical constants of the material. It characterizes its rigidity, i.e. the ability to resist elastic deformation. This value is called the modulus of longitudinal elasticity. The value E is measured in the same units as the strain, that is, in N/m2 (Pa), N/mm2 (MPa). The values of the elastic moduli for some materials are shown in Table. 6.1.

Table 2.1.

Material	E [MPa]
Steel	2 ... 2.1 * 105
Cast iron	$0,75\ldots1,6*10^5$
Copper	$1,2*10^5$
Aluminium	$0,8*10^5$

Substituting into the formula (6.1) the values of the normal stress $\sigma = \frac{F}{A}$ and the axial strain $\varepsilon = \frac{\Delta l}{l}$, we can determine the change in the length of the rod:

$$\Delta l = \frac{Nl}{EA}. \tag{2}$$

The resulting expression is called the Hooke's law in the components of displacement. It shows that the elongation (shortening) under tension (compression) depends on the magnitude of the longitudinal force N, the cross-sectional area A of the rod, its length l and the modulus of longitudinal elasticity E. The product EA is called the stiffness of the rod under tension (compression).

By analogy with the longitudinal strain $\varepsilon' = \frac{\Delta b}{b}$ is called shear deformation.

Experiments show that the ratio ε'/ε does not depend on N and is determined only by the properties of the material. Absolute value

$$\mu = \left|\frac{\varepsilon'}{\varepsilon}\right|, \tag{3}$$

is called the Poisson's ratio.

The Poisson's ratio is the dimensionless value that characterizes the ability of a material to deform in the transverse direction at its stretching or compressing in the longitudinal direction (for plug μ=0; for rubber μ= 0.48; for steels μ= 0.25-0.3).

The Poisson's ratio for various materials is determined empirically by testing samples.

6.2. Deformation under the joint influence of power and thermal action.

In the process of operation, many machine parts experience the combined effect of power and temperature action. In this case, the change in the length of the rod can be represented as follows:

$$\Delta l = \underbrace{\frac{Nl}{EA}}_{1} + \underbrace{l\alpha\Delta T}_{2} \tag{4}$$

α - the coefficient of linear thermal expansion; ΔT - changes in temperature.

The1st term - the change in length under the action of external forces, the second term - the change in length on exposure to temperature.

After conversion

$$\frac{\Delta l}{l} = \underbrace{\frac{N}{EA}}_{1} + \underbrace{\alpha\Delta T}_{2} \longrightarrow \varepsilon = \frac{\sigma}{E} + \alpha\Delta T \tag{5}$$

we obtain an expression for strain under both the force and temperature action.

For metals, this dependence is performed in the temperature range of 300-400 C °, since at higher temperatures the modulus of elasticity E and the thermal expansion coefficient α depend on time.

3.3. Stresses occurring on sloping planes at stretching strain.

Let's consider an arbitrary sloping section $n - m$. The position of this section is determined by the angle α. σ - the normal stress arising on perpendicular planes. R - stress arising on sloping planes (Fig.6.3). A - the cross-sectional area perpendicular to the plane, A_α - the cross-sectional area of the sloping plane. The dependence between the values of cross-sectional areas - $A_\alpha = \dfrac{A}{\cos\alpha}$.

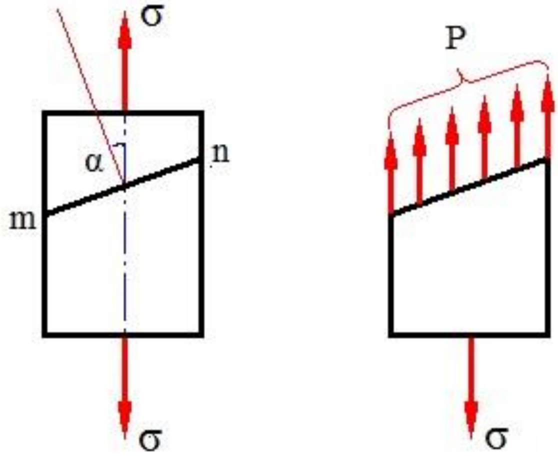


Figure 2.3.

The item is in a state of equilibrium, then we can write the equation of static equilibrium and derive the relationship between the stress σ and P.

$$\sum X = 0 \longrightarrow \sigma A = PA_{\alpha} \longrightarrow \quad P = \sigma \cos\alpha \tag{6}$$

We shall expand the stress P on 2 mutually perpendicular axes and obtain two components of stress σ_{α} and τ_{α} (Fig. 6.4).

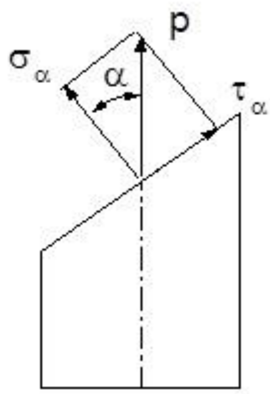


Fig.2.4.

We'll express the components of stress P σ_{α} and τ_{α} through the normal stresses generated on perpendicular planes σ.

$$\begin{aligned} \sigma_{\alpha} &= P \cdot \cos\alpha = \sigma \cdot \cos^2\alpha; \\ \tau_{\alpha} &= P \cdot \sin\alpha = \frac{\sigma}{2} \cdot \sin 2\alpha; \end{aligned} \tag{7}$$

Let’s investigate the obtained expressions (6.7) depending on the angle of the plane slope α.

$$\alpha = 0 \qquad \sigma_\alpha = \sigma \qquad \tau_\alpha = 0$$

$$\alpha = \pm 45 \qquad \sigma_\alpha = \frac{\sigma}{2} \qquad \tau_\alpha = \pm\frac{\sigma}{2}$$

$$\alpha = 90 \qquad \sigma_\alpha = 0 \qquad \tau_\alpha = 0 \tag{8}$$

In longitudinal fibers ($\alpha = 90$), the normal and shear stresses are absent, in transverse fibers ($\alpha = 0$), the normal stresses are maximum; the tangents are equal to zero. At the angle of rotation of $\alpha = \pm 45$ the tangents reach their maximum:

$$\tau_{\max} = \pm\frac{\sigma}{2}. \tag{9}$$

We’ll define the stresses arising on the sloping plane set at an angle of $90 + \alpha$:

$$\sigma_{\alpha+90} = \sigma \cdot \cos^2(\alpha + 90) = \sigma \cdot \sin^2 \alpha \tag{10}$$

$$\tau_{\alpha+90} = \frac{\sigma}{2} \cdot \sin 2(\alpha + 90) = -\frac{\sigma}{2} \cdot \sin 2\alpha \tag{11}$$

From expression (6.11) there follows the law of pairing shear stresses: shear stresses on 2 mutually perpendicular planes are equal in magnitude and opposite in sign (direction) (Fig. 6.5).

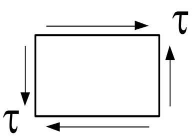


Fig. 2.5.

2.4. Allowable stress. Margin of safety. The condition of the strength and stiffness under tension-compression strain.

Designing begins with the selection of proper material. This problem is solved according to the conditions of the designed structure operation. Additionally, one may take into account the cost considerations and manufacturing techniques. Mechanical testing of the material gives the limiting values of stresses, the achievement of which in machine parts causes either destruction or the occurrence of unacceptable distortions.

The main objective of structural analysis is to ensure its safe operation. For safe operation stresses in a given structure must be below these limiting stresses (there must be satisfied the condition of strength). Therefore, the second issue of designing is the choice of a safe design or allowable stress -$[\sigma]$.

Allowable stress is the greatest stress at which the strength and durability of the projected structural element is provided. Allowable stresses are a certain fraction of the limiting ones - $\sigma_{\lim}$. For static loading the allowable stress values are as follows:

$$[\sigma] = \frac{\sigma_{\lim}}{n} \tag{12}$$

Depending on the type of loading and the material at the choice of an allowable one there is taken a particular limiting stress ($\sigma_{\lim} = \sigma_T$ (tensile stress) for plastic materials; $\sigma_{\lim} = \sigma_B$ (tensile strength) for brittle materials).

The number n, indicating how many times the allowable stress is less than the limiting one is called the margin of safety. The safety margin should be chosen so as to cover the inaccuracy of loads and stresses determination. Moreover n is chosen depending on the longevity and responsibility a given construction should assume.

Establishing of an allowable stress is a very important issue, if it is accepted to be too large, the construction will be unstable, and conversely, when $[\sigma]$ - is low, the structures size will be too large, which will lead to more expensive designs and weighting.

For some areas of mechanical engineering, there are rules of allowable stresses; however, it is impossible to give general rules suitable for all modifications.

For structures, working in tension and compression, the strength condition, composed for a dangerous cross section, can be written as:

$$\sigma_{\max} = \left[\frac{N}{A}\right]_{\max} \leq [\sigma] \ . \tag{13}$$

This equation allows solving the following tasks:

1. For a given external load and allowable stress [σ] to determine the necessary cross-sectional area: $A \geq \frac{N_{\max}}{[\sigma]}$,

2. For a given cross-sectional area and the allowable stress [σ] to determine the permissible load.

3. For a given external load and cross-sectional area to implement the strength test.

In some cases, the working efficiency of the structural element is determined not only by its strength but also hardness, i.e. the ability to take the load without unacceptable elastic deformation. In calculations on stiffness they determine the maximum movement of sections and correlate them with the transferable ones. The condition of rigidity, limiting the change in length of the element has the following general form:

$$\Delta l \leq [\Delta l] \tag{14}$$

where Δl - the change in the machine part dimensions; $[\Delta l]$ - the permissible value of this change.

Considering that at tension (compression) the absolute elongation is generally defined as the algebraic sum of section values, the stringency conditions can be written as follows:

$$\sum_{i=1}^{n} \Delta l_i = \sum_{i=1}^{n} \frac{N_i l_i}{E_i A_i} \leq [\Delta l]. \tag{15}$$

Practical training 3. Geometric characteristics of plane sections

To solve the basic problem of resistance of materials, it is necessary, first of all, to be aware of stresses in the cross-sections of elements. Obviously, it depends on the amount of internal forces and the cross-section area of the element, but with the same characteristics of computational models of beams (length, cross-sectional area, external loads) (Fig. 7.1), deflections (strain), as seen, in the second case are larger; consequently the stress will be larger, too. This is caused by a different orientation of the cross section with respect to the direction of the force F. Thus, at calculations of structures on mechanical reliability one needs to know not only the cross sectional area but also its other geometrical characteristics (static moments, the moments of inertia, cross section area modulus, radii of gyration). These characteristics do not have a physical meaning. They cannot be experimentally determined.

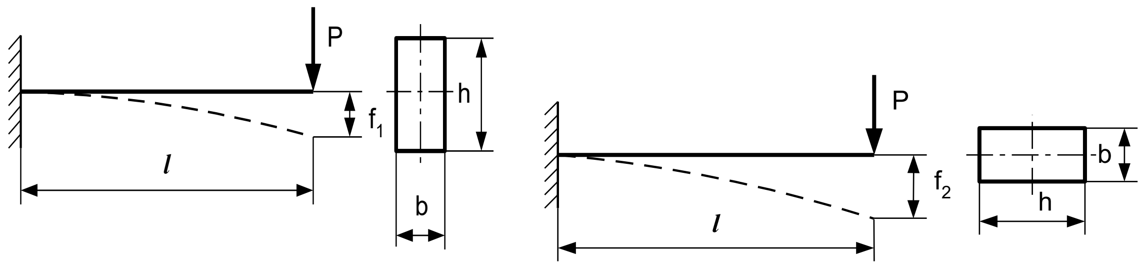


Figure 3.1.

3.1. The static moment of the section. The center of gravity.

Let's consider an arbitrary pattern (cross section of beam/girder), associated with the coordinate axes z and y (fig. 3.2). We shall select the area element dA with coordinates z and y.

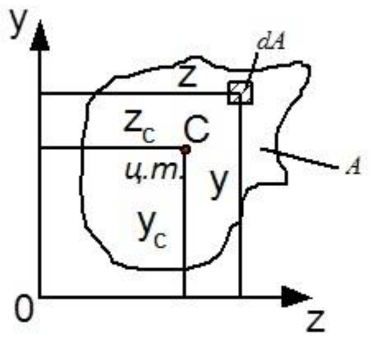


Figure 3.2.

By definition, the static moment is the amount of products of elementary areas on the coordinate of their distance to the corresponding axis.

$$S_z = \int_A y dA \qquad (1)$$

$$S_y = \int_A z dA \qquad (7.2)$$

S_z, S_y - static moments of a planar figure about the axes y and z, respectively.

The dimension of static moments is a unit of length when cubed (mm3, cm3, m3, etc.).

From theoretical mechanics it is known that the center of gravity coordinates of the flat cross section can be determined as follows:

$$z_c = \frac{1}{A}\int_A z dA = \frac{1}{A} S_y, \qquad y_c = \frac{1}{A}\int_A y dA = \frac{1}{A} S_z. \qquad (3)$$

Hence, the static moments of the figure can be found if you know the area of the figure and the coordinates of the center of gravity.

$$S_y = z_c A; \quad S_z = y_c A. \qquad (4)$$

From equation (7.4) it follows that the static moments of the area about the central axes (axes passing through the center of gravity) are equal to zero (since $z_c = 0$, $y_c = 0$).

Thus, the axes, about which the static moment of the cross section is equal to zero, are called the central ones, and the point of intersection of the central axes is the center of gravity of the cross section.

3.2. Changing of the static moment of the cross section when transferring axes.

Let us find the relationship between the static moments of the same section with respect to the parallel axes y, z and y_1, z_1 (Fig. 7.3).

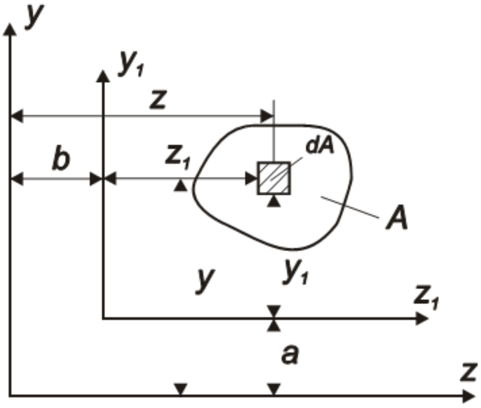


Figure 3.3.

In accordance with the definition:

$$S_z = \int_A y dA \ , \ S_{z1} = \int_A y_1 dA .$$

However, according to the design:

$$y_1 = y - a \ ,$$

$$S_{z1} = \int_A y_1 dA = \int_A (\text{y-a}) dA = S_z - aA \ , \quad (5)$$

by analogy

$$z_1 = z - b\,,$$

$$S_{y1} = \int_A z_1 dA = \int_A (z-b)dA = S_y - bA\,. \tag{6}$$

If the axes z_1, y_1 will pass through the center of gravity of the cross section, from said previously $S_{z1} = 0$, $S_{y1} = 0$, then:

$$z_c = b = \frac{S_y}{A}, \qquad y_c = a = \frac{S_z}{A}\,. \tag{7}$$

Knowing the static moment of the cross section and the cross-sectional area one can determine the coordinate of the center of gravity.

3.3. Determination of the center of gravity of a complex figure.

The static moment of a complex figure about an axis is equal to the sum of the static moments of all parts of this section with respect to the same axis.

$$\begin{aligned} S_z &= S_{z1} + S_{z2} + \ldots + S_{zn} = A_1 y_1 + A_2 y_2 + \ldots + A_n y_n = \sum_{i=1}^{n} A_i y_i; \\ S_y &= S_{y1} + S_{y2} + \ldots + S_{yn} = A_1 z_1 + A_2 z_2 + \ldots + A_n z_n = \sum_{i=1}^{n} A_i z_i. \end{aligned} \tag{8}$$

The coordinates of the center of gravity of a complex figure can be found by using formulas (7.7), then:

$$\begin{aligned} z_c &= \frac{S_y}{A} = \frac{A_1 z_1 + A_2 z_2 + \ldots + A_n z_n}{A_1 + A_2 + \ldots + A_n} = \frac{\sum_{i=1}^{n} A_i z_i}{\sum_{i=1}^{n} A_i}\;; \\ y_c &= \frac{S_z}{A} = \frac{A_1 y_1 + A_2 y_2 + \ldots + A_n y_n}{A_1 + A_2 + \ldots + A_n} = \frac{\sum_{i=1}^{n} A_i y_i}{\sum_{i=1}^{n} A_i}. \end{aligned} \tag{9}$$

The procedure for determining the center of gravity of a complex figure:

• Split the figure into simple parts, for each of which there is known the area A_i and the position of the center of gravity z_i and y_i.

• Determine the coordinates of the center of gravity of the composite cross-section - (9).

3.4. The moments of plane sections inertia.

By definition, the axial moment of inertia about an axis is called, taken over the entire area, the sum of products of elementary sites on the squares of their distances from the axis (Figure 7.4).

$$J_z = \int_A y^2 dA; \quad J_y = \int_A z^2 dA \qquad (10)$$

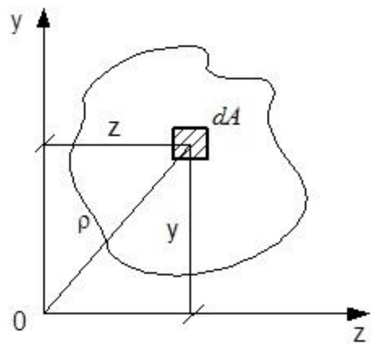


Fig. 3.4.

The polar moment of inertia of the area of the figure relative to a fixed point (O pole) is called the sum of the products of elementary planes on the squares of their distances from the pole.

.
$$J_p = \int_A \rho^2 dA \qquad (11)$$

Since $\rho^2 = y^2 + z^2$, from (7.11) it follows that:

$$J_p = \int_A (y^2 + z^2)dA = \int_F y^2 dA + \int_F z^2 dA = J_z + J_y \,. \tag{12}$$

It should be noted that the values of axial and polar moments of inertia are always positive.

The centrifugal moment of inertia with respect to some frame of reference is called the sum of products of elementary planes areas on their distance from the coordinate axes z and y:

$$J_{zy} = \int_F zy dA \,. \tag{13}$$

Depending on the position of axes the centrifugal moment of inertia can be positive, negative or equal to zero.

The moments of inertia are measured in units of length to the fourth power (mm4, cm4, M4, etc.).

3.5. Changing of the moments of inertia at parallel translation of axes.

Let's assume that the moments of inertia of the figure relative to axes z, y are known:

$$J_z = \int_F y^2 dA; \;\; J_y = \int_F z^2 dA; \;\; J_{zy} = \int_F zy dA.$$

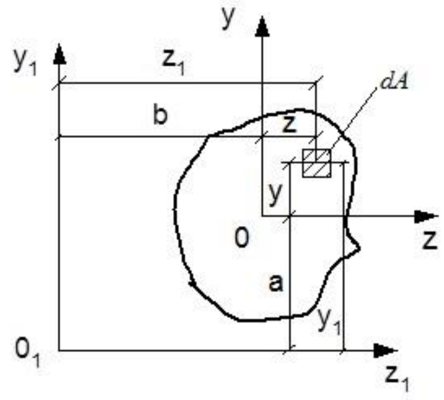


Figure 3.5.

It is required to determine the moment of inertia of the same figure relative to the axes z_1, y_1 (Fig. 3.5):

$$J_{z_1} = \int_A y_1^2 dA;\;\; J_{y_1} = \int_A z_1^2 dA;\;\; J_{z_1 y_1} = \int_A z_1 y_1 dA. \qquad (14)$$

The coordinates of any point in the new system z_1, y_1 can be expressed in terms of the coordinates of the old axes as follows: $z_1 = z + b$; $y_1 = y + a$. We shall substitute these values into formulas (7.14) and integrate:

$$J_{z_1} = \int_A y_1^2 dA = \int_A (y+a)^2 dA = \int_A y^2 dA + a^2 \int_A dA + 2a \int_A y dA = J_z + a^2 A + 2aS_z$$

$$J_{y_1} = \int_A z_1^2 dA = \int_A (z+b)^2 dA = \int_A z^2 dA + b^2 \int_A dA + 2b \int_A z dA = J_y + b^2 A + 2bS_y$$

$$J_{z_1 y_1} = \int_A z_1 y_1 dA = \int_A (z+b)(y+a) dA = \int_A zy dA + ab \int_A dA + a \int_A z dA + b \int_A y dA = J_{zy} + abA + aS_y + bS_z$$

If we assume that the axes z, y pass through the center of gravity of the cross section, the static moments of the section about these axes is equal to zero ($S_z = 0$ and $S_y = 0$). Then:

$$J_{z_1} = J_z + a^2 A;$$
$$J_{y_1} = J_y + b^2 A; \qquad (15)$$
$$J_{z_1 y_1} = J_{zy} + abA.$$

3.6. The main axes. Principal moments of inertia

The principal axes are called the axes around which the centrifugal inertia is equal to zero ($J_{zy} = 0$), and the axial moments take extreme values (the first one - the maximum; the second one - the minimum).

The main axes, passing through the center of gravity of the cross section, are called the principal central axes.

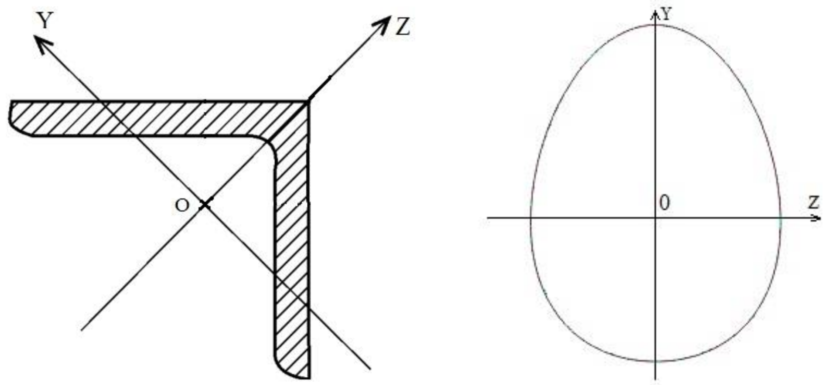


Fig. 3.6.

In many cases it is possible to immediately determine the position of the principal central axes. If the figure has an axis of symmetry, it is one of the main central axes, the second one passes through the center of gravity of the cross section perpendicular to the first one (see examples in Fig. 3.6).

The axial moments of inertia about the principal axes are called the principal moments of inertia.

3.7. Principal moments of inertia of simple figures.

We shall calculate the moments of inertia of the rectangle relative to the principal axes z and y

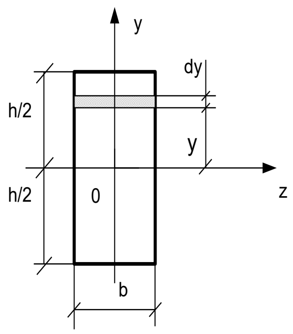


Figure 3.7.

To determine the moment of inertia about the axis z we shall select an elementary area in the form of a narrow rectangle parallel to the axis z. The element width - b, the height - dy.

Consequently, $dA = bdy$

$$J_z = \int_A y^2 dA = b \int_{-h/2}^{h/2} y^2 dy = 2b \int_0^{h/2} y^2 dy = \frac{bh^3}{12}$$

by analogy

$$J_y = \frac{hb^3}{12}$$

We shall calculate the polar moment of the circle inertia relative to the principal axes as well as the moments of inertia.

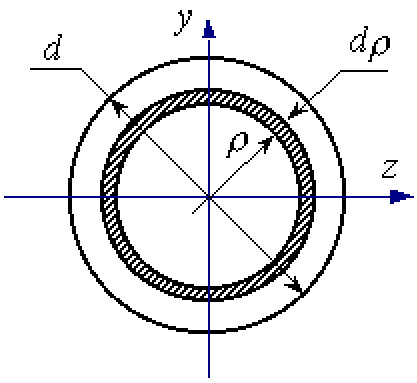


Fig. 3.8.

While calculating the polar moment of inertia, we shall select the elementary strip in the form of a thin ring whose thickness is $d\rho$. The area of this element is $dA = 2\pi\rho d\rho$. The polar moment and the axial moments of inertia are as follows:

$$J_p = \int_A \rho^2 dA = 2\pi \int_0^{\frac{d}{2}} \rho^3 d\rho = \frac{\pi D^4}{32}$$

$$J_x = J_y = \frac{J_p}{2} = \frac{\pi D^4}{64}$$

We shall calculate the polar moment of inertia of the ring relative to the principal axes, and the axial moments of inertia.

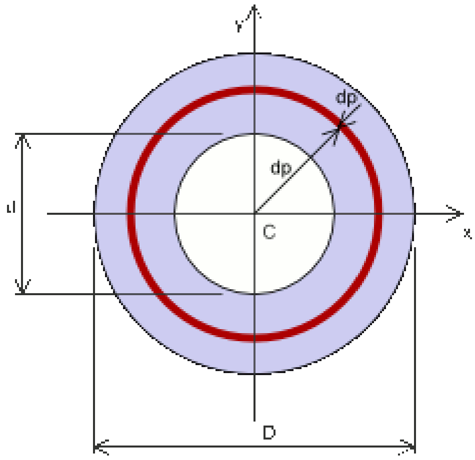


Fig. 3.9.

Let's select the elemental strip in the form of a thin ring whose thickness is $d\rho$. The area of this element is $dA = 2\pi\rho d\rho$. Compared with a circular cross-section the limits of integration will change:

$$J_p = \int_A \rho^2 dA = 2\pi \int_{\frac{d}{2}}^{\frac{D}{2}} \rho^3 d\rho = \frac{\pi\left(D^4 - d^4\right)}{32} =$$

$$= \frac{\pi D^4}{32}(1-\alpha^4) \qquad \alpha = \frac{d}{D}$$

$$J_x = J_y = \frac{J_p}{2} = \frac{\pi\left(D^4 - d^4\right)}{64} = \frac{\pi D^4}{64}(1-\alpha^4)$$

3.8. Principal moments of inertia of complex figures with the axis of symmetry.

The moment of inertia of the complex figure cross section about an axis is equal to the sum of the moments of inertia of its components with respect to the same axis, which follows directly from the properties of the definite integral:

$$J_{zc} = \sum_{i=1}^{n} J_{zci} , \qquad J_{yc} = \sum_{i=1}^{n} J_{yci} . \qquad (16)$$

Thus, to calculate the moment of inertia of the complex figure one should divide it into a series of simple shapes, calculate the moments of inertia of these figures with respect to the principal axes (J_{zci}, J_{yci}). As the figure has an axis of symmetry, it can be divided into simple ones, so that the principal axes of simple figures to be parallel to the principal axes of the complex figure. Then, using the expressions (7.15), the moments of inertia of simple figures in relation to the principal axes may be defined as follows:

$$J_{zci} = J_{zi} + a_i^2 A_i \ , \qquad J_{yci} = J_{yi} + b_i^2 A_i . \qquad (17)$$

where J_{zi}, J_{yi} - the principal moments of inertia of simple figures; A_i - the area of simple figures; a_i, b_i- the distance between the main axes of the simple and complex figures.

By summing J_{zci} and J_{yci} one can determine the required moments of inertia of the complex figure with the axis of symmetry.

$$J_{zc} = \sum_{i=1}^{n}\left(J_{zi} + a_i^2 A_i\right), \qquad J_{yc} = \sum_{i=1}^{n}\left(J_{yi} + b_i^2 A_i\right). \qquad (7.18)$$

3.9. The concept of the radius of gyration and the moment of resistance.

The moment of inertia of the figure relative to any axis can be represented as the product of the figure area of a square called the radius of gyration:

$$J_z = \int_A y^2 dA = A \cdot i_z^2 \qquad (19)$$

where i_z - the radius of inertia about the axis z. From (7.19) it follows that

$$i_z = \sqrt{\frac{J_z}{A}}. \qquad (20)$$

Similarly, the radius of gyration of the cross section relative to the axis y

$$i_y = \sqrt{\frac{J_y}{A}}. \qquad (21)$$

The main radii of gyration correspond to the central axes of inertia:

$$i_u = \sqrt{\frac{J_u}{A}}, i_v = \sqrt{\frac{J_v}{A}}.$$

The axial section modulus is the ratio of the axial moment of inertia to the distance from the outermost point of the section to the respective axis

$$W_z = \frac{J_z}{y_{\max}}; \qquad W_y = \frac{J_y}{z_{\max}}. \qquad (22)$$

Practical training *4.* Bending strain.

4.1. The concept of bending

Bending deformation is called a type of deformation in which under the influence of external forces in each cross-section of the body there is an internal bending moment and shear force.

Flexural assumptions

• Plane sections taken before deformation remain plane after deformation either.
• Plane sections during deformation rotate relative to each other causing tension of particular fibers (lower Fig. 8.1) and compression of the other (upper).
• Neutral fiber during deformation process does not change its length.
• Due to the expansion and contraction of fibers in each section there is an internal bending moment, which aims to restore the body to the undeformed state.

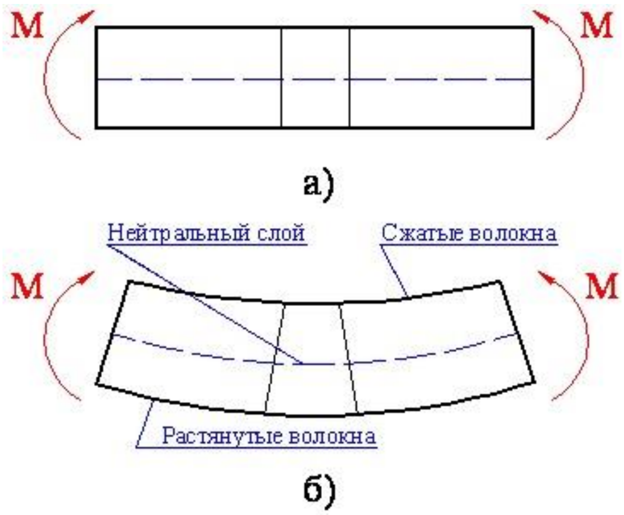


Fig. 4.1.

In case if the internal power factor arises only the bending moment, the deformation is called pure bending. If in addition to the bending moment there is a lateral force, the deformation is called lateral bending.
A rod that works in bending is called - beam.
The bending moment arises in the cross section normal stresses; the sheer force - shearing stresses.

4.2. Differential dependence of bending.

Let's consider a beam loaded with an arbitrary load (Fig. 4.2).

We shall distinguish at the site of action of the positive distributed load the beam element CD with the length dx. Since the element dx is small, in the range of this element the load is considered to be permanent (Fig. 4.3).

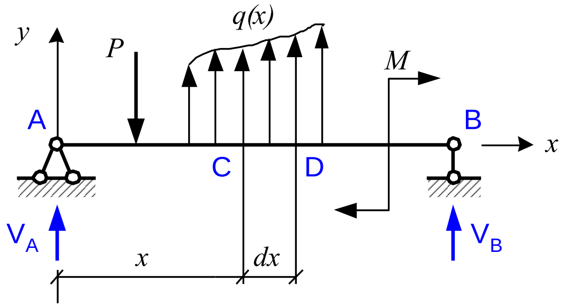


Fig. 4.2.

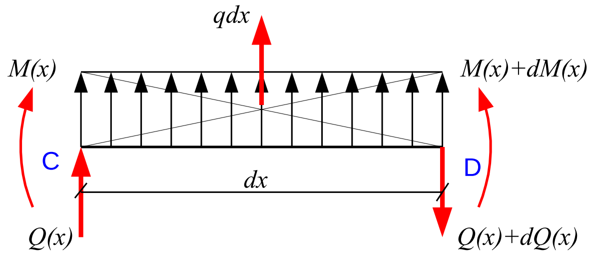


Fig. 4.3.

The item is in a state of equilibrium. We shall write the static equations of equilibrium.

$$\sum Y = Q(x) + qdx - (Q(x) + dQ(x)) = 0\,; \tag{1}$$

$$\sum M_D = M(x) + Q(x)dx + qdx\frac{dx}{2} - (M(x) + dM(x)) = 0\,. \tag{2}$$

From formula (8.1) we have:

$$\frac{dQ(x)}{dx} = q\,. \tag{3}$$

From the second condition of equilibrium (8.2), we shall obtain (the terms of the second order are ignored):

.
$$\frac{dM(x)}{dx} = Q(x) \tag{4}$$

We shall differentiate this expression (8.4):

$$\frac{d^2M(x)}{dx^2} = \frac{dQ(x)}{dx} = q\,. \tag{5}$$

Given that the distributed load is constant $q - const$, we can write:

$$Q = \int q dx = qx + Q_0;$$

$$M = \int Q dx = q\frac{x^2}{2} + Q_0 x + M_0. \qquad (6)$$

The resulting relationship between the effort and intensity of the load dependencies is called differential bending. They are used to control the construction correctness of diagrams $Q(x)$ and $M(x)$.

1. If $q = 0$, $Q(x) = Q_0 = const$, and $M(x) = Q_0 x + M_0$.

Consequently, in those areas where there is no distributed load, the diagram $Q(x)$ is limited to a straight line parallel to the base line, and the diagram $M(x)$ - to an inclined line.

2. If $q(x) = q_0 = const$, then $Q(x) = qx + Q_0$ - the linear function, and $M(x) = q\frac{x^2}{2} + Q_0 x + M_0$ - the parabola. It has a bulge directed toward the load action.

3. Since $\frac{dM(x)}{dx} = Q(x)$, then in those areas where the moment function increases, the sheer force $Q(x)$ is positive, with a decrease - negative. In sections where $Q(x) = 0$ the bending moment reaches extreme values.

4.3. Stresses arising in pure bending.

Let's consider the example of the calculation model of the loaded beam, for which we shall construct the diagrams Q and M, using the rules listed above (Fig. 4.4).

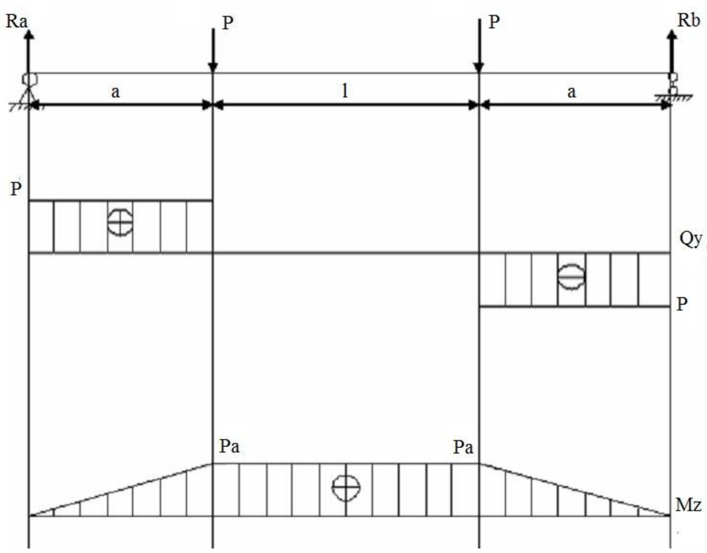
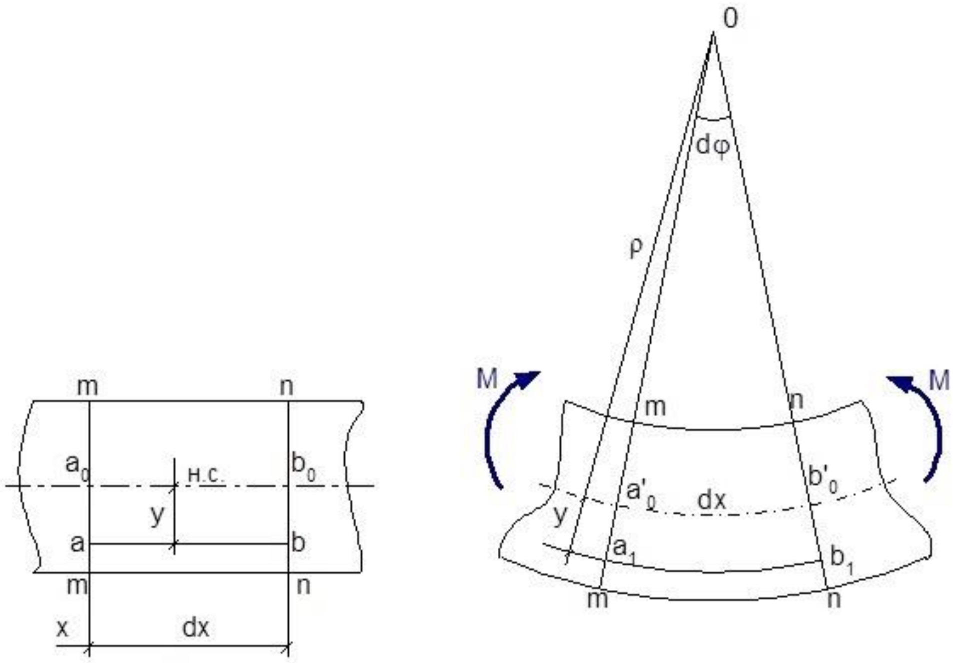


Figure 4.4.

On the area *l*- $Q=0$, $M-const$ (Fig. 4.4), therefore, there is pure bending. The beam in this area takes the form of a circular arc.

We shall consider the element dx, taken in this area (Fig. 4.5, a).



a b

Fig. 4.5.

After deformation of the cross section $m-m$ and $n-n$ remain flat and rotated by a small angle $d\varphi$. The element a_0b_0 of the neutral layer is transformed into the arc $a_0'b_0'$ with a radius of curvature ρ, and the fiber ab, located at the distance y from the neutral sheet, - into a curved fiber a_1b_1 with a radius of curvature $\rho+y$ (Fig. 8.5, b).

We shall define the deformation of the arbitrary fiber *ab*.

$$\varepsilon=\frac{a_1b_1-ab}{ab}, \qquad (7)$$

where, $a_1b_1=(\rho+y)d\varphi$, $ab=dx=\rho d\varphi$, that's why

$$\varepsilon = \frac{(\rho + y)\,d\varphi - \rho d\varphi}{\rho d\varphi}.$$

Hence,

$$\varepsilon = \frac{y}{\rho}. \qquad (8)$$

Since the considered fiber *ab* experiences tensile strain (the internal bending moment arises at each point of the cross section the normal stress), according to Hooke's law:

$$\sigma = \varepsilon E = \frac{y}{\rho} E \qquad (9)$$

(8.9) - Hooke's law for pure bending. In practice, this expression cannot be applied, as the value of curvature radius ρ is not known.

Let's define its value, for this we'll select from the cross-section area the elementary area dA, the position of which is described by the coordinates y and z. The elementary normal force acting in this area will be equal to σdA (Fig. 4.6).

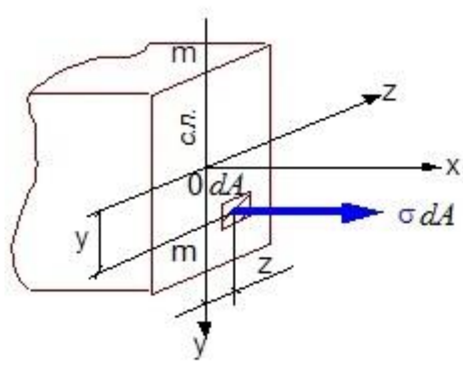


Fig. 4.6.

Let's write the equation of equilibrium for the element in question:

$$\sum F_x = 0 \quad \Rightarrow \quad \int_A \sigma dA = 0, \qquad (10)$$

$$\sum M_y = 0 \quad \Rightarrow \quad \int_A \sigma z dA = 0, \qquad (11)$$

$$\sum M_z = 0 \quad \Rightarrow \quad M - \int_A \sigma y dA = 0. \qquad (12)$$

Then from (8.10):

$$\frac{E}{\rho} \int_A y dA = 0,$$

Hence, the static moment of the cross section is equal to zero $S_z = \int_A y dA = 0$ - the axis z passes through the center of gravity of the section.

From the expression (8.11):

$$\frac{E}{\rho} \int_A yz dA = 0,$$

meaning the vanishing of the product of inertia $I_{yz} = \int_A yz dA = 0$ - the y-axis is the main axis. Consequently axes z and y - the principal central axes of the section.

From the remaining part of the equilibrium equation (8.12) we can define the curvature of the unknown:

$$\frac{1}{\rho} = \frac{M}{EJ_z} \qquad (13)$$

where J_z the principal moment of inertia.

The product EJ_z is called the cross-sectional stiffness section under bending strain.

Substituting the obtained expression (8.13) into Hooke's law for pure bending (8.9) we obtain the following relationship:

$$\sigma = \frac{My}{J_z}. \qquad (14)$$

This formula allows us to calculate the normal stresses in pure bending of the beam at any point in its cross-section.

Analyzing the resulting formula for determining the stress it can be noted: whichever shape and dimensions the cross section has, the stress at the neutral line points is equal to zero; the value σ increases linearly according to the cross section height from the neutral line, and the stress is constant across the width of the cross section. The highest stress values (σ_{max}) are reached in fibers, which are the most remote from the neutral line (Fig. 4.7).

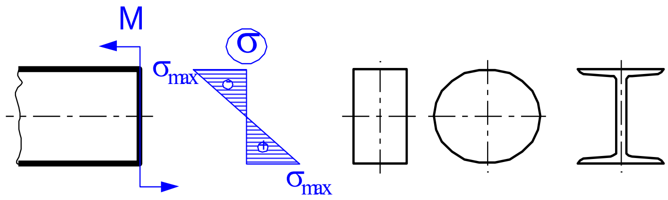


Figure 4.7.

4.4. The condition of strength under pure bending deformation.

The bending moment, which the cross section can withstand safely, is proportional to the W - section modulus. The value of the maximum stress σ_{max} acting in the cross section shall be limited to the allowed value $[\sigma]$

$$\sigma_{max} = \frac{M_{max} y_{max}}{J_z} = \frac{M_{max}}{W_z} \leq [\sigma]. \qquad (15)$$

(8.15) - the condition of strength in pure bending.

4.5. On rational form of the cross section.

Unlike simple tension-compression under bending the stress in the cross section is distributed unevenly. The material located at the neutral sheet is loaded insufficiently. Therefore, in order to reduce the cost and the weight of construction elements working in bending, there should be chosen such sectional shapes, so that the biggest part of the material were removed from the neutral line (Fig. 8.8).

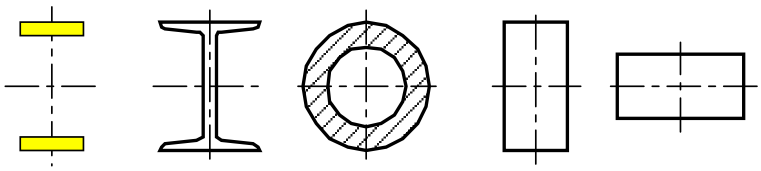


Figure 4.8.

4.6. Stresses arising under lateral bending.

In each cross-section under transverse bending there arise internal bending moments and transverse forces. The bending moment causes the normal stress and the shear force – the shearing stress.

The normal stress with sufficient accuracy for practical purposes can be identified as for pure bending (see (4.14)).

The shearing stresses are defined by Zhuravskiy's formula:

$$\tau = \frac{QS_z(y)}{b_z J_z}, \quad (16)$$

where Q - the lateral force in the section under consideration; J_z - the moment of cross section inertia about the axis z; $S_z(y)$ - the static moment about the neutral axis of the part of the section enclosed between the fiber under consideration and the extreme one; b_z - the width of the cross section in the fiber under consideration. The Zhuravskiy's formula allows drawing some conclusions about the distribution of shear stresses in cross-sections under lateral (transverse) bending (Fig. 4.9):

• the view of the diagram τ depends on the shape of the lateral cross section of the beam;

• in the extreme most remote from the neutral line points τ are always equal to zero;

• shearing stresses reach maximum values for most types of sections on the neutral line of the section.

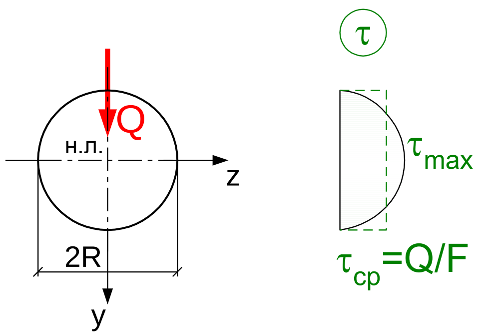


Fig. 4.9.

Practical training ***№5.*** Pure shear. Torsion.

5.1. The concept of pure shear. Deformation under pure shear.

Pure shear is called such a type of deformation in which, under the influence of external forces on the edges of the selected item act only shearing (tangential) stresses.

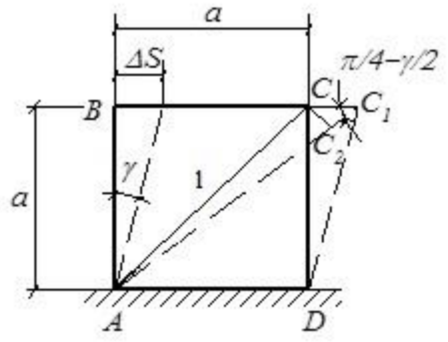


Figure 5.1.

Under pure shear there is observed linear and angular deformity. We shall consider the deformation of the elementary area, cut from the plate exposed to pure shear (Fig. 5.1).

As a result of shear (shift) one cross-section is shifted relative to another by the value ΔS, which is called the absolute magnitude of the shift (the dimension is linear).

The ratio of the absolute value of the shift to the distance between the sections, in which there were applied shear loads, is called the relative deformation of shear or the angle of shear $\frac{\Delta S}{a} = tg\gamma$.

Since, under small strains $tg\gamma \cong \gamma$, then

$$\frac{\Delta S}{a} = \gamma \ . \qquad (1)$$

Let's find the extension of the diagonal AC, the length of which is $\ell = a\sqrt{2}$ ($ABCD$ - a square with the side a).

Considering the geometric pattern of deformation, we get:

$$\Delta\ell = C_1C_2 = CC_1 \cos\left(\frac{\pi}{4} - \frac{\gamma}{2}\right) \cong CC_1 \cos 45^o = \frac{\Delta S}{\sqrt{2}}.$$

Then the relative elongation of the diagonal is equal to

$$\varepsilon = \frac{\Delta\ell}{\ell} = \frac{\Delta S}{\sqrt{2}\cdot a\sqrt{2}} = \frac{1}{2}\frac{\Delta S}{a} = \frac{tg\gamma}{2} \cong \frac{\gamma}{2}. \quad (2)$$

It follows that the linear deformation is equal to half the corner deformation.

5.2. Stresses induced by the arbitrary inclined plane under pure shear strain

Let's cut out from the elementary area subjected to pure shear deformation a three-sided prism with the angle α to the inclined surface. On the inclined surface there arises the normal stress σ_α and the shearing stress τ_α.

Having considered the balance of the triangular prism under the applied loads, we obtain expressions for stresses on the inclined plane:

$$\sigma_\alpha = \tau \sin\alpha \cos\alpha + \tau \cos\alpha \sin\alpha = \tau \sin 2\alpha \quad (3)$$

$$\tau_\alpha = -\tau \sin^2\alpha + \tau \cos^2\alpha = \tau \cos 2\alpha. \quad (4)$$

We shall check the values of stresses on inclined (sloping) planes at different values of the angle α.

$$\alpha = 0 \quad \Rightarrow \sigma_\alpha = 0 \quad \tau_\alpha = \tau$$

$$\alpha = 90 \quad \Rightarrow \sigma_\alpha = 0 \quad \tau_\alpha = -\tau \quad (5)$$

$$\alpha = \pm 45 \quad \Rightarrow \sigma_\alpha = \pm \tau \quad \tau_\alpha = 0$$

It follows that if the item subjected to pure shear strain is turned through the angle of 45 °, then on its edges (faces) only normal stresses will be effective. Thus, on two edges they will be stretching, and on the other two - compressing (Fig. 5.2).

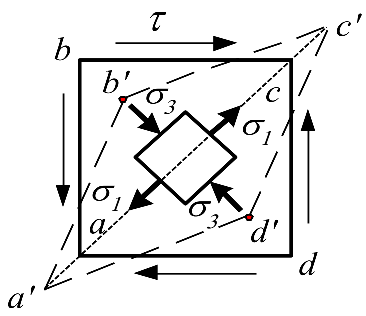


Figure 5.2.

5.3. Hooke's law under pure shear.

Let's define the total strain along the direction of stress action σ_1 (Fig. 5.2).

$$\varepsilon = \varepsilon_1 + \varepsilon_3' \qquad (6)$$

wherein $\varepsilon_1 = \frac{\sigma_1}{E}$, $\varepsilon_3' = -\mu\varepsilon_3 = -\mu\frac{\sigma_3}{E}$, while $\sigma_1 = \tau$; $\sigma_3 = -\tau$ (see equations (5)), then the total deformation taking into account the dependence (relationship) (2):

$$\varepsilon = \frac{\gamma}{2} = \frac{\tau}{E}(1+\mu). \qquad (7)$$

From which we obtain the Hooke's law for pure shear:

$$\tau = \frac{E}{2(1+\mu)}\gamma = G\gamma, \qquad (8)$$

where G - the shear modulus.

5.4. Practical calculations of the compounds working in shear

The connecting structural elements that serve to interconnect the elements and parts of engineering structures basically rely on the shear (rivets, screws, welds, cuttings, etc.)

From a theoretical point of view, these calculations are very imperfect, since they are based on several assumptions simplifying the calculation. For example, that in a section in which shear failure could occur, the shearing stresses are distributed evenly; or that all the rivets in the riveted joints are in the same conditions (force transmitted between the rivets are distributed evenly, which is not true in the elastic zone of material deformation).

5.5. Torsion of the beam of the circular cross section.

Torsion is such a kind of deformation, by which under the action of external forces in each cross section of the body there occurs only internal torque. All other power factors are absent.

When considering the torsional strain they usually identify two tasks:

- determination of stresses in the cross section.
- determination of the angle twist of the section under consideration.

Rods of any cross-sectional shape, working in torsion are called shafts.

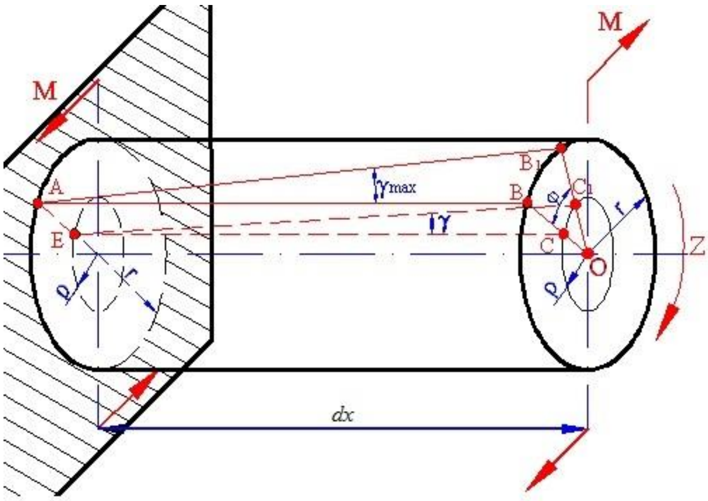


Fig. 5.3.

The task of determining the stresses and twist angles essentially under torsion essentially depends on the shape of the element's cross section. The simplest such problems are solved for the rods of the round and annular cross-section.

Let's consider the element of the shaft with the length *dx*, the far left section which will be considered conditionally fixed (Fig. 5.3). It is easy to show that this element undergoes shear deformation. Indeed, any external generator *AB* or internal generator *EU* shifts under torsion (twisting/rotation), and there arise distortions defined by the shift angles γ max for *AB* or γ - for *EC*. The radius *OC* rotates to the position OC_1 through the angle dφ, called the angle of twist. Since deformations are small, then expressing CC1 as a circular arc, one can obtain the ratio between the angle of shift γ and the twist angle dφ.

On the one hand $CC_1 = \gamma dx$, considering another plane $CC_1 = \rho d\varphi$. Finally, we can write:

$$\gamma = \rho \frac{d\varphi}{dx} = \rho\Theta , \qquad (9)$$

Θ - the relative twist angle.

As the element undergoes pure shear, then using (9.9) and the Hooke's law in shear (9.8) we'll obtain the Hooke's law for torsion:

$$\tau = G\gamma = G\rho\Theta . \qquad (10)$$

To use this formula for determining shearing stresses in torsion is not possible, because the value Θ is not known.

In order to express the value of the relative twist angle Θ we shall consider the equation that relates the torque to stresses

$$M_{кр} = \int_F \tau\rho dF , \qquad (11)$$

where τ - the shearing stress acting on the elementary area dF, located at the distance ρ from the center of the section (Figure 5.4).

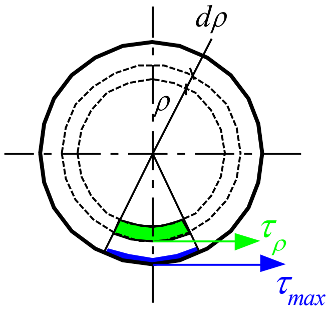


Fig. 5.4.

Let's transform the expression (9.11) further

$$M_{кр} = \int_F \tau\rho dF = \int_F G\Theta\rho^2 dF = G\Theta\int_F \rho^2 dF = G\Theta J_p ,$$

where J_p - the polar moment of inertia.
Hence, we obtain the formula for the relative angle of round rod twist

$$\theta = \frac{d\varphi}{dx} = \frac{M_{кр}}{GJ_\rho} , \qquad (12)$$

where GJ_ρ - the torsion stiffness of the cross section.
Knowing the expression (12), we can write the formula for calculating the angles of twist:

$$\varphi = \int_0^\ell \frac{M_{кр}}{GJ_\rho} dx. \qquad (13)$$

If the torque is constant within the cylindrical portion of the rod with the length ℓ, then

$$\varphi = \theta\ell = \frac{M_{кр}\ell}{GJ_\rho} \qquad (14)$$

To determine the shear stress at any point of the rod, we substitute into (10), the expression for θ (9.12). Then

$$\tau = \frac{M_{кр} \cdot \rho}{J_\rho}. \qquad (15)$$

Shear stresses are distributed over the cross section depending on the triangular relationship with the maximum in the extreme fibers.

$$\tau_{\max} = \frac{M_{кр} \cdot \rho_{\max}}{J_\rho} = \frac{M_{кр}}{W_\rho}, \qquad (16)$$

where $W_\rho = J_\rho / \rho_{\max}$ - the polar moment of resistance.

5.6. The condition of torsional strength.

The condition of torsion strength, taking into account the notation adopted is stated as follows: the maximum shearing stresses arising in a dangerous section of the shaft shall not exceed the allowable stresses and can be written as:

$$\tau_{\max} = \frac{M_{кр}}{W_\rho} \le [\tau], \qquad (17)$$

where $[\tau]$- the allowable torsion stress (pure shear).

Practical training №6 Fundamentals of the theory of complex stress state.

6.1. The state of stress at a point.

Let's consider a body in equilibrium under the influence of the spatial system of forces. To study the stress state of the body, we'll choose the arbitrary point A and define the stresses emerging at this point (Figure 10.1).

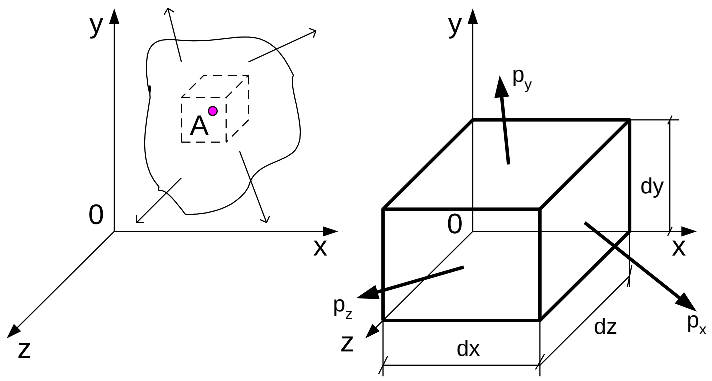


Fig. 6.1.

Since the material of the body under consideration is uniform and continuous, it is possible to move from the point to an infinitely small volume, comprising the point A. On the faces of the parallelepiped there act internal forces that replace the action of the discarded body parts. These forces $p_x,\ p_y,\ p_z$ (Fig. 6.1) are called the full stress. Here, the indices correspond to the normal to the elementary areas (planes), which are acted on by stresses. Since the selected element is small, it can be assumed that the stresses on each face are distributed evenly.

Let's decompose (distribute) the stress vector in three mutually perpendicular directions, which coincides with the coordinate axes.

Stresses, perpendicular to the plane, are indicated σ with the index corresponding to the normal to the plane on which they act are called normal. Stresses acting in the plane of the parallelepiped are called tangents - τ with two indices: the first one corresponds to the normal to the area, the second - the direction of the stress (Fig. 6.2).

Thus, on each side of the selected element there act three components of the total stress. The set of stresses acting on all the faces may be represented in the matrix form (the stress tensor):

$$\begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{vmatrix}. \qquad (1)$$

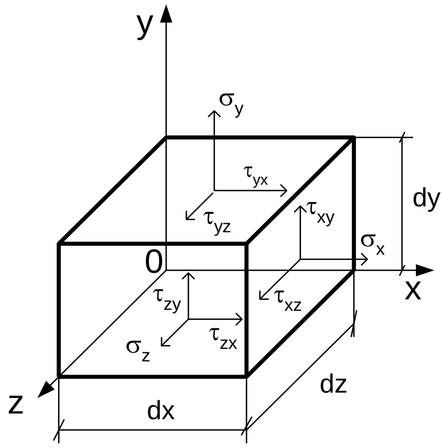


Figure 6.2.

The value of each of the stresses referred above depends on the platform orientation in space, but the magnitude of the total stress at point A depends only on the external forces.

On the basis of the law of pairing of shearing stresses, we can write:

$$\tau_{xy} = \tau_{yx}, \qquad \tau_{yz} = \tau_{zy}, \qquad \tau_{xz} = \tau_{zx}. \tag{2}$$

Thus, we have six unknowns (three normal and three tangential stresses). Formulating 6 static equations we get a statically determinate system of which it is possible to find the unknown values of stresses.

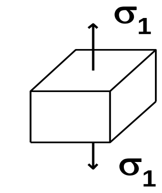
6.2. The main areas (planes) and the principal stresses

Rotating the element in question (Fig. 6.2) in the space one can find a position at which the shearing stresses on the faces of the element will be equal to zero. These faces are called major platforms, the axes perpendicular to them - the principal axes, and the normal stresses acting on these areas - the principal stresses.

The principal stresses indicate σ_1, σ_2, σ_3. However, they are ordered as follows - $\sigma_1 > \sigma_2 > \sigma_3$.

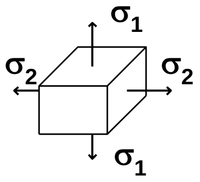
If one of the principal stresses, or two, or all three at the same time are different from zero, we get different kinds of stress states, namely:

-



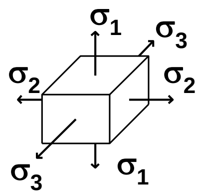
– uniaxial, linear

$\sigma_1 \neq 0\,,\quad \sigma_2 = \sigma_3 = 0$



– biaxial flat

$\sigma_1 > \sigma_2 \neq 0\,,\ \sigma_3 = 0$



- three-axis, dimensional

$\sigma_1 > \sigma_2 > \sigma_3 \neq 0$

6.3. Mohr's circle of stress.

Convenient two-dimensional geometric representation of three-dimensional stress state was proposed by the German scientist Mohr.

Let’s define the stresses generated on the sloping area, if you know the principal stresses and the area is perpendicular to one of the main areas.

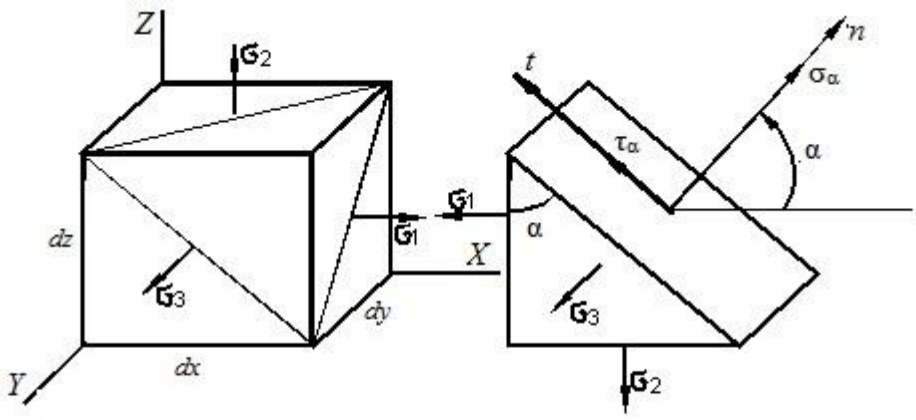


Fig. 6.3.

Out of the parallelepiped there was cut a triangular prism with the angle α, the inclined surface is perpendicular to the main board (area) of stresses σ_2 (Fig. 6.3). We shall consider the equilibrium of a three-sided prism.

$$\sigma_\alpha dx \frac{dy}{\cos\alpha}\cos\alpha + \tau_\alpha dx \frac{dy}{\cos\alpha}\sin\alpha = \sigma_1 dxdy,$$
$$\sigma_\alpha dx \frac{dy}{\cos\alpha}\sin\alpha + \tau_\alpha dx \frac{dy}{\cos\alpha}\cos\alpha = \sigma_3 dxdytg\alpha. \qquad (1)$$

After the transformation:

$$\sigma_\alpha = \sigma_1 \cos^2\alpha + \sigma_3 \sin^2\alpha = \frac{\sigma_1+\sigma_3}{2} + \frac{\sigma_1-\sigma_3}{2}\cos 2\alpha \qquad (2)$$

$$\tau_\alpha = \frac{\sigma_1-\sigma_3}{2}\sin 2\alpha\ . \qquad (3)$$

σ_α and τ_α - the normal and shearing stresses arising in the inclined plane parallel to one of the principal axes.

We raise each of the resulting equations to the square and then add. As a result, we obtain the equation of the circle – the Mohr's circle:

$$\left(\sigma_\alpha - \frac{\sigma_1+\sigma_3}{2}\right)^2 + \tau_\alpha{}^2 = \left(\frac{\sigma_1-\sigma_3}{2}\right)^2. \qquad (4)$$

We have obtained the equation of the circle, where the center of the circle lies on the axis σ and is offset from *0* by the distance of $\frac{\sigma_1+\sigma_3}{2}$. This dependence makes it possible according to the slope of the area α to determine the normal and shearing stresses induced on it.

6.4. Building the circle of Mora. Direct and inverse problems of the Mohr’s circle.

In the coordinates σ and τ the circle of Mohr is a parametric equation of the circle where the angle α acts as a parameter.

Let’s analyze the values of normal and shearing stresses on the inclined platforms, depending on the angle α:

$$\alpha = 0 \Rightarrow \quad \sigma_\alpha = \sigma_1 \quad \tau_\alpha = 0$$
$$\alpha = 90 \Rightarrow \quad \sigma_\alpha = \sigma_3 \quad \tau_\alpha = 0$$
$$\alpha = 45 \Rightarrow \quad \sigma_\alpha = \frac{\sigma_1+\sigma_3}{2} \quad \tau_\alpha = \frac{\sigma_1-\sigma_3}{2}$$

One may note the maximum values of stresses:

$$\sigma_{\max} = \sigma_1, \qquad \tau_{\max} = \frac{\sigma_1 - \sigma_3}{2}. \tag{5}$$

In theory of the stress state they identify two main problems: direct and inverse problems of the Mohr's circle.

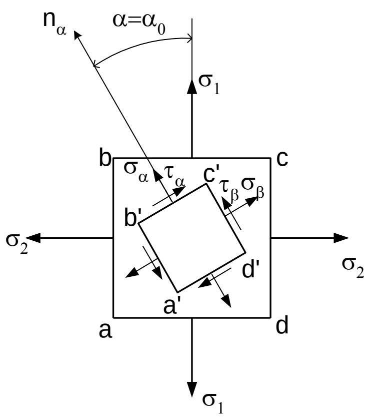


Fig. 6.4.

The direct problem. At a given point there is known the positions of the main areas and the corresponding principal stresses; it is required to find the normal and shearing stresses on the areas, inclined at the predetermined angle α to the main ones (It is known: σ_1, $\sigma_2, \sigma_3, \alpha$; to be determined: $\sigma_\alpha, \tau_\alpha$). (Fig. 6.4).

The analytical solution of the direct problem is given by formulas (2) - (3).

We shall analyze the state of stress, using a graphical construction. To do this, we'll introduce a geometric plane and relate it to rectangular coordinate axes σ, τ. Choosing for stresses a certain scale, we lay off as abscissa (Fig. 6.5) the segments $OA = \sigma_1$; $OB = \sigma_2$. On both AB and the diameter we draw a circle with the center at point C. The constructed circle will be the stress circle of Mohr.

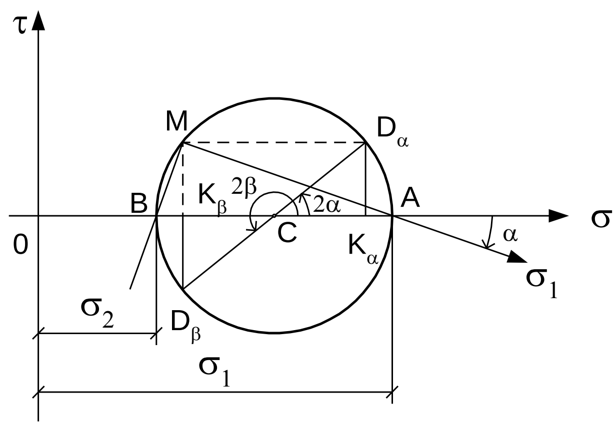


Fig. 6.5.

The coordinates of the circle points correspond to the normal and shearing stresses on various areas. So, for determination of stresses on the area projected at the angle α from the center of the circle C we'll draw a ray at the angle 2α to intersect the circle at the point D_α $(\alpha > 0)$. The abscissa of the resulting point D_α is equal to the normal stress σ_α, and the ordinate – to shearing stress τ_α.

$$OK_\alpha = \sigma_\alpha, \quad K_\alpha D_\alpha = \tau_\alpha.$$

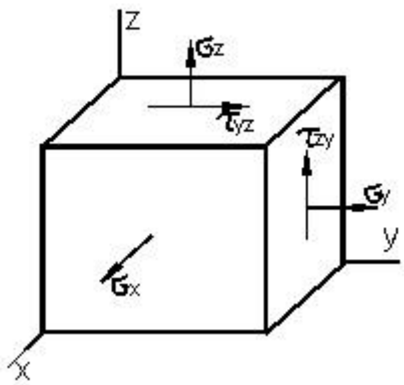


Figure 6.6.

Inverse problem. At a given point there are known the normal and shearing stresses acting in two orthogonal areas, passing through a given point; it is required to find the principal directions and the principal stresses (it is known: $\sigma_z, \sigma_y, \sigma_x$, $\tau_{zy} = \tau_{yz} = \tau$; to be determined: $\sigma_1, \sigma_2, \sigma_3$),) (Fig. 6.6).

One of the given normal stresses is the main one (σ_x - Fig. 6.6), as tangential stresses do not act on this area. To determine the other two principal stresses, first they determine the position of points D_α and D_β (Fig. 6.5), characterizing the stresses on the respective faces of the element. To do this, they mark off in the appropriate scale the following segments: $OK_\alpha = \sigma_y$, $OK_\beta = \sigma_z$, $K_\alpha D_\alpha = K_\beta D_\beta = \tau$.. Further, on $D_\alpha D_\beta$ with the center in point C we draw a circle – the Mohr's circle. The points of intersection of the Mohr's circle with the axis σ will determine the remaining two main stresses $OA = \sigma'$; $OB = \sigma''$. Ranking the principal stresses σ_x, σ', σ'', we assign them to the appropriate indexes.

The values of the principal stresses can also be found analytically:

$$\sigma' = \frac{\sigma_z + \sigma_x}{2} + \sqrt{\left(\frac{\sigma_z - \sigma_x}{2}\right)^2 + \tau^2};$$
$$\sigma'' = \frac{\sigma_z + \sigma_x}{2} - \sqrt{\left(\frac{\sigma_z - \sigma_x}{2}\right)^2 + \tau^2}. \qquad (6)$$

6.5. The generalized Hooke's law.

This law establishes a relationship between the amount of strain and stresses for the complex stress state.

Let's consider the deformation of the body element by selecting the element in the form of a cuboid with sides a, b, c. Along the edges of the parallelepiped there act the principal stresses σ_1, σ_2 , σ_3 (Fig. 6.7).

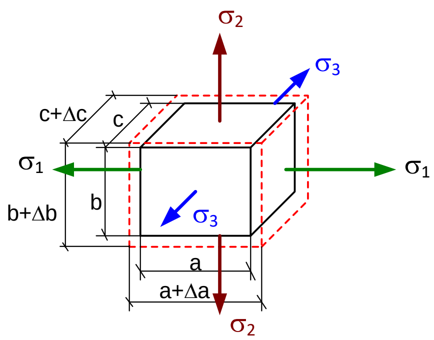


Fig. 10.7. Applying the principle of superposition, we can write:

$$\varepsilon_1 = \varepsilon_1' + \varepsilon_1'' + \varepsilon_1''', \qquad (7)$$

wherein ε_1' - relative elongation in the direction of σ_1, caused by the action of σ_1($\sigma_2 = \sigma_3 = 0$); ε_1'' - elongation in the direction of σ_1, caused by the action of σ_2($\sigma_1 = \sigma_3 = 0$); ε_1''' - extension in the direction of σ_1, caused by the force σ_3 ($\sigma_1 = \sigma_2 = 0$).

Suppose, in accordance with the basic hypotheses and assumptions that the material obeys the Hooke's law, and the deformations are small, while taking into account the fact that the direction σ_1 for tension σ_1 is longitudinal and for stresses σ_2 and σ_3- cross-sectional one:

$$\varepsilon_1' = \frac{\sigma_1}{E}, \qquad \varepsilon_1'' = -\mu\frac{\sigma_2}{E}, \qquad \varepsilon_1''' = -\mu\frac{\sigma_3}{E}. \qquad (8)$$

Based on expressions (8):

$$\varepsilon_1 = \frac{1}{E}\left[\sigma_1 - \mu(\sigma_2 + \sigma_3)\right]. \tag{9}$$

Similarly, we obtain expressions for other strains:

$$\begin{aligned} \varepsilon_2 &= \frac{1}{E}\left[\sigma_2 - \mu(\sigma_1 + \sigma_3)\right], \\ \varepsilon_3 &= \frac{1}{E}\left[\sigma_3 - \mu(\sigma_1 + \sigma_2)\right]. \end{aligned} \tag{10}$$

The formulas (10.9-10.10) express the generalized Hooke's law for an isotropic body.

We shall define the total strain for the considered case:

$$\varepsilon = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \frac{1-2\mu}{E}(\sigma_x + \sigma_y + \sigma_z)\,. \tag{11}$$

From this expression it is possible to define the limits of variation of the Poisson's ratio. Assuming that all of the stresses are tensile $\sigma_1 + \sigma_2 + \sigma_3 \geq 0$, it is logical that the total strain must be greater than zero $\varepsilon \geq 0$, and it is satisfied if $1 - 2\mu \geq 0 \Rightarrow \quad 0 \leq \mu \leq 0.5$, as confirmed experimentally.

6.6. The theory of strength.

Evaluation of strength reliability is a common engineering problem in which the state of stress in the dangerous point is compared with the limit state. Such an estimation is accurate enough in case of a uniaxial stress state (tension - compression). However, many structural elements operate in a complex stress state. For this the hard-stress state is replaced by the simple stress state which is considered as a uniaxial tension and the condition is tested:

$$\sigma_{экв} \leq [\sigma]_p\,. \tag{12}$$

Equivalent stress $\sigma_{экв}$ is called the stress that should be set up in an extended sample, so that his stress state become equally dangerous to the given stress state.

To determine the equivalent stresses, scientists have proposed a number of hypotheses (theories) of strength for assessing the risk of transition to the ultimate state of the material of construction elements that are in a complex stress state.

In each theory of strength there is used a particular hypothesis of strength, which is an assumption concerning advantageous effect on the strength of the material of any given factor. The most important factors associated with the occurrence of the dangerous state of the material are the normal and shear stresses, the linear deformation and the potential strain energy.

The hypothesis of strength out of many factors that influence the strength of material chooses one and ignores the rest. The reliability of the hypothesis of strength is tested empirically.

1. Maximum normal stress.

The hypothesis of primary influence of the largest according to the absolute value normal stresses is basic to the theory of the greatest stress.

According to this theory of strength, the dangerous condition of the material under complex stress state occurs when the greatest in modulus (in absolute value) the main stress reaches the limit for a given material in simple tension. The strength condition is as follows:

$$\sigma_{эквI} = \sigma_1 \le [\sigma]_p \qquad (13)$$

where σ_1 - the largest of the maximum stresses.

This theory of strength provides positive results only for some brittle materials.

2. Maximum linear deformations.

The given theory is based on the assumption that all materials regardless of the state of stress are destroyed when the relatively greatest elongation in any direction reaches a value at which rupture at stretching occurs.

According to this theory of strength the dangerous condition of the material under complex stress state occurs when the largest in absolute value the relative linear deformation reaches the limit in simple tension.

$$|\varepsilon_{\max}| \le [\varepsilon]_P . \qquad (14)$$

Expressing the maximum relative deformations according to the generalized Hooke's law, and analyzing the magnitude of the maximum value of the relative deformation in tension:

$$\varepsilon_{\max} = \varepsilon_1 = \frac{1}{E}\left[\sigma_1 - \mu\left(\sigma_2 + \sigma_3\right)\right] \le \frac{[\sigma]}{E} .$$

Then, under the condition of equality of volumetric and linear elastic modulus the condition of strength will be:

$$\sigma_{эквII} = \sigma_1 - \mu\left(\sigma_2 + \sigma_3\right) \le \left[\sigma\right]_p . \quad (15)$$

Experimental verification of this hypothesis has identified a number of significant drawbacks. Best results are obtained for brittle materials (alloy cast iron, high-strength steel after low tempering, etc.).

3. Maximum shearing stresses.

According to this theory of strength the dangerous condition of the material under complex stress occurs when the shear stress reaches a maximum value limit for this material.

$$\tau_{\max} \le \left[\tau\right]_P . \quad (16)$$

With three dimensional stress

$$\tau_{\max} = \frac{1}{2}\left(\sigma_1 - \sigma_3\right) \le \left[\tau\right]_P = \frac{\left[\sigma\right]_P}{2}$$

Then the condition of strength according to the third theory will be:

$$\sigma_{эквIII} = \sigma_1 - \sigma_3 \le \left[\sigma\right]_p . \quad (17)$$

This theory gives a good agreement with the experimental results for ductile materials.

4. The energy theory of strength.

This theory is based on the assumption that a dangerous condition, regardless of the type of stress state, occurs when the specific strain (stress) energy, associated with the change of form, reaches the limit for the given material value.

$$u_{Фсл} \le \left[u_Ф\right]_p . \quad (18)$$

With three dimensional stress state the condition of strength in this case takes the following form:

$$\sigma_{эквIV} = \sqrt{\frac{1}{2}\left[\left(\sigma_1-\sigma_2\right)^2+\left(\sigma_2-\sigma_3\right)^2+\left(\sigma_3-\sigma_1\right)^2\right]} \le \left[\sigma\right]_p . \qquad (19)$$

The experiments confirm well enough the fourth theory for ductile materials, working equally in tension and compression. The emergence of small plastic deformations in the material are determined more accurately by the fourth theory than by the third one.

5. Mohr's theory of strength.

The hypothesis of strength of Mohr allows to take into account the difference in the properties of the material. It can be obtained by modifying the greatest shear stress hypothesis:

$$\sigma_{эквV} = \sigma_1 - \frac{\left[\sigma\right]_P}{\left[\sigma\right]_{сж}}\sigma_3 \le \left[\sigma\right]_P . \qquad (20)$$

If the properties of the material in tension and compression are the same, the fifth theory is converted into the third theory of strength.

In practice, the first and second theory is not used (are historical in nature). For plastic materials they use the third and fourth strength theory, for fragile - the fifth strength theory.

Educational edition

Methodological instructions for the implementation of practical tasks, independent work and individual tasks in the disciplines «Technical Mechanics» for foreign applicants of the specialty 141 – Electric Power, Electrical Engineering and Electromechanics of the first (bachelor's) level of education of full-time education

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The work was recommended for publication by A.V. Gaydamaka

In the author's edition