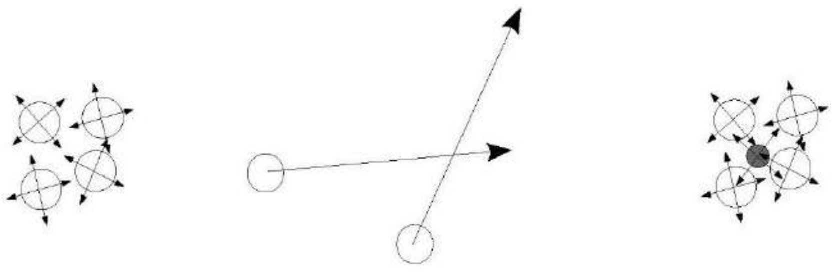
HYDRAULICS

Methodological instructions for the implementation of practical tasks, independent work and individual tasks in the disciplines «Hydraulics» for foreign applicants of the specialty 131 – Applied Mechanics of the first (bachelor's) level of education of full-time education

1. The nature and properties of fluids, forces, and flows

1.1 Properties of fluids

1.1.1 Definition of a fluid



(a) Liquid (b) Gas (c) Liquid with other molecule

Figure 1-1. Behaviour of typical molecules (a) of a liquid, (b) a gas, and (c) a different molecule in a liquid

A fluid is matter which, if subject to an unbalanced external force, suffers a continuous deformation. The forces which may be sustained by a fluid follow from the structure, whereby the molecules are able to move freely. Gas molecules have much larger paths, while liquid molecules tend to be closer to each other and hence are heavier.

Fluids can withstand large compressive forces (pressure), but only negligibly small tensile forces. Fluids at rest cannot sustain shear forces, however fluids in relative motion do give rise to shear forces, resulting from a momentum exchange between the more slowly moving particles and those which are moving faster. The momentum exchange is made possible because the molecules move relatively freely.

Especially in the case of a liquid, we can use the analogy of smooth spheres (ball bearings, billiard balls) to model the motion of molecules. Clearly they withstand compression, but not tension.

Molecular characteristic	Solids	Fluids	
		Liquids	Gases
Spacing	Small (material is heavy)		Large (light)
Activity	Very little	Vibratory	Great, molecules moving at large velocities and colliding
Structure	Rigid, molecules do not move relative to each other. Stress is proportional to strain	If confined, elastic in compression, not in tension or shear.	
Response to force	Resisted continuously, static or dynamic	Molecules free to move and slip past one another. If a force is applied it continues to change the alignment of particles. Liquid resistance is dynamic (inertial and viscous).	

Table 1-1. Characteristics of solids and fluids

1.1.2 The continuum hypothesis

In dealing with fluid flows we cannot consider individual molecules. We replace the actual molecular structure by a hypothetical continuous medium, which at a point has the mean properties of the molecules surrounding the point, over a sphere of radius large compared with the mean molecular spacing. The term "fluid particle" is taken to mean a small element of fluid which contains many molecules and which possesses the mean fluid properties at its position in space.

1.1.3 Density

The density ρ is the mass per unit volume of the fluid. It may be considered as a point property of the fluid, and is the limit of the ratio of the mass δm contained in a small volume δV to that volume:

$$\rho = \lim_{\delta V \to 0} \frac{\delta m}{\delta V}.$$

In fact the limit introduced above must be used with caution. As we take the limit $\delta V \to 0$, the density behaves as shown in Figure 1-2. The real limit is $\delta V \to \delta V^*$, which for gases is the cube of the mean free path, and for liquids is the volume of the molecule. For smaller volumes considered the fact that the fluid is actually an assemblage of particles becomes important. In this course we will assume that the fluid is continuous, the *continuum hypothesis*.

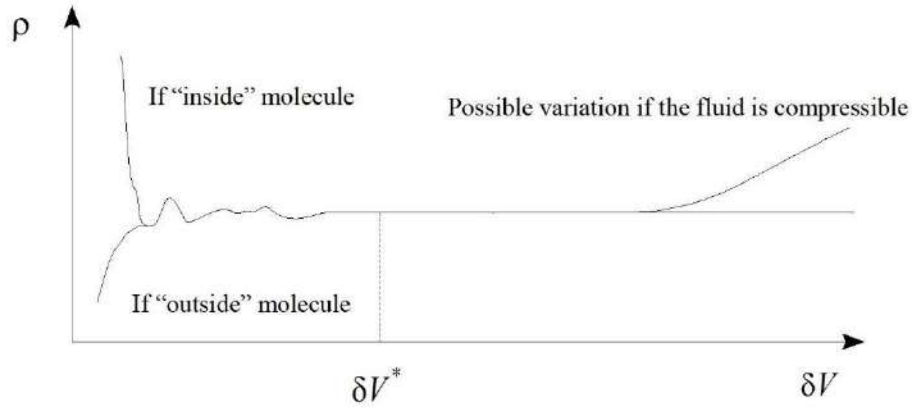


Figure 1-2. Density of a fluid as obtained by considering successively smaller volumes δV, showing the apparent limit of a constant finite value at a point, but beyond which the continuum hypothesis breaks down.

As the temperature of a fluid increases, the energy of the molecules as shown in Figure 1-1 increases, each molecule requires more space, and the density decreases. This will be quantified below in §1.1.8, where it will be seen that the effect for water is small.

1.1.4 Surface tension

This is an important determinant of the exchange processes between the air and water, such as, for example, the purification of water in a reservoir, or the nature of violent flow down a spillway. For the purposes of this course it is not important and will not be considered further.

1.1.5 Bulk modulus and compressibility of fluids

The effect of a pressure change δp is to bring about a compression or expansion of the fluid by an amount δV. The two are related by the bulk modulus K, constant for a constant temperature, defined:

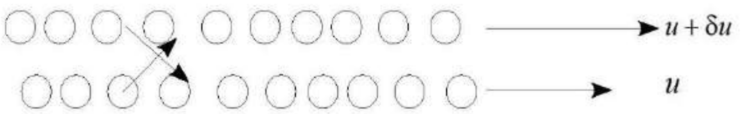
$$\delta p = -K\frac{\delta V}{V}.$$

The latter term is the "volumetric strain". K is large for liquids, so that density changes due to pressure changes may be neglected in many cases. For water it is $2.070\ \mathrm{G\,N\,m^{-2}}$. If the pressure of water is reduced from 1 atmosphere (101,000 $\mathrm{N\,m^{-2}}$) to zero, the density is reduced by 0.005%. Thus, for many practical purposes water is incompressible with change of pressure.

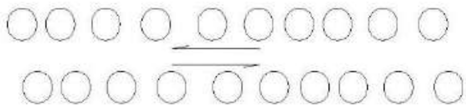
1.1.7 Viscosity

If, instead of the fluid being stationary in a mean sense, if parts of it are moving relative to other parts, then if a fluid particle moves randomly, as shown above, it will in general move into a region of different velocity, there will be some momentum transferred, and an apparent force or stress appears. This gives rise to the phenomenon of *viscosity*, which is a diffusivity associated with the momentum or velocity of the moving fluid, and has the effect of smoothing out velocity gradients throughout the fluid in the same way that we saw that diffusivity of a substance in water leads to smoothing out of irregularities.

A model of a viscous fluid: Consider two parallel moving rows of particles, each of mass m, with N in each row. The bottom row is initially moving at velocity u, the upper at $u + \delta u$, as shown in the figure. At a certain instant, one particle moves, in the random fashion associated with the movement of molecules even in a still fluid, as shown earlier, from the lower to the upper row and another moves in the other direction.



The velocities of the two rows after the exchange are written as v and $v + \delta v$. Now considering the momenta of the two layers, the effect of the exchange of particles has been to increase the momentum of the slow row and decrease that of the fast row. Hence there has been an apparent force across the interface, a *shear stress*.



This can be quantified:

Initial x-momentum of bottom row	=	$(Nm)u$
Initial x-momentum of top row	=	$(Nm)(u + \delta u)$
Change of x-momentum of bottom row	=	$+m\delta u$
Change of x-momentum of top row	=	$-m\delta u$
Final x-momentum of bottom row	=	$(Nm)v$
Final x-momentum of top row	=	$(Nm)(v + \delta v)$

In the absence of other forces, equating the momenta gives

$$\begin{aligned} Nmv &= Nmu + m\,\delta u \\ Nm(v + \delta v) &= Nm(u + \delta u) - m\,\delta u \end{aligned}$$

Thus,

$$\begin{aligned} v &= u + \frac{\delta u}{N}, \quad \text{and} \\ v + \delta v &= u + \delta u - \frac{\delta u}{N}. \end{aligned}$$

Hence the lower slower fluid is moving slightly faster and the faster upper fluid is moving slightly slower than before. Subtracting, the velocity difference between them now gives

$$\delta v = \delta u - 2\frac{\delta u}{N},$$

thus $\delta v < \delta u$, and we can see the effect of viscosity in reducing differences throughout the fluid.

In fact, if we consider the whole shear flow as shown in Figure 1-4, we can obtain an expression for shear stress in the fluid in terms of the velocity gradient, as follows:

$$\text{Impulse of top row on bottom} = m\,\delta u.$$

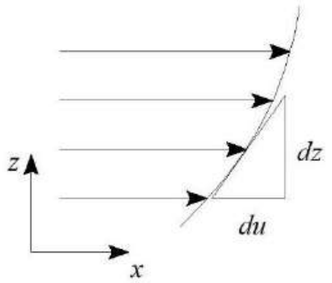


Figure 1-4. Shear flow, where the velocity in one direction u is a function of the transverse velocity co-ordinate z

If T is the average time between such momentum exchanges, then

$$\text{mean shear force} = \frac{m\,\delta u}{T}.$$

As the horizontal velocity u is a function of z, then if the two rows are δz apart, the differential of u is

$$\delta u = \frac{du}{dz}\delta z,$$

and the mean shear stress per unit distance normal to the flow is

$$\text{mean shear stress} = \frac{\text{mean shear force}}{\text{length} \times 1} = \frac{m\,du/dz\,\delta z}{T\,N\,\delta x},$$

where δx is the mean particle spacing in x. Now, m, N T, δx and δz are characteristic of the fluid and not of the flow, and so we have

$$\text{mean shear stress} \propto \frac{du}{dz},$$

the transverse velocity shear gradient. Fluids for which this holds are known as *Newtonian Fluids*, as are most common fluids. The law is written

$$\tau = \mu\frac{du}{dz}, \tag{1.2}$$

where μ is the *coefficient of dynamic viscosity*, which is defined here. Although we have only considered parallel flows here, the result has been found to apply throughout Newtonian fluid flow.

Effects of viscosity: The law (equation 1.2) shows how a transverse shear flow gives rise to shear stresses, which acts, as suggested more immediately by the molecular argument above, so as to redistribute momentum throughout a fluid. (Consider how stirring a bucket of water with a thin rod can bring all the water into rotation - imagine trying to stir a fluid without viscosity!) If the fluid at the boundary of a flow has a certain specified velocity (*i.e.* momentum per unit mass), then viscosity acts so as to distribute that momentum throughout the fluid. This can be shown by considering the entry of flow into a pipe as shown in Figure 1-5. On the pipe wall the fluid velocity must be zero, as the molecules adhere to the wall. Immediately inside the pipe entrance the velocity differences are large, as viscosity has not yet had time to act, but as flow passes down the pipe viscosity acts so as to smooth out the differences, until a balance is reached between the shear stresses set up by the viscosity, the driving pressure gradient, and the momentum of the fluid.

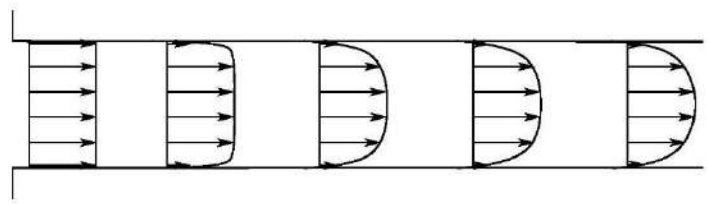


Figure 1-5. Entry of viscous flow into a pipe, showing the effect of viscosity smoothing the initially discontinuous velocity distribution

1.1.8 Typical values of physical properties of water

Temperature (°C)	0	4	10	20	30	...	50	...	100
Density ($kg\,m^{-3}$)	999.8	1000	999.7	998.3	995.7	...	988	...	958.1
Kinematic viscosity (units of $10^{-6}\ m^2 s^{-1}$)	1.780	1.584	1.300	1.006	0.805	...	0.556	...	0.294

Table 1-2. Variation with temperature of the properties of pure water with no dissolved substances, from Jirka (2005) for the companion subject to this

The density of water depends on temperature (see Table 1-2), dissolved solid concentration (especially salt), but little with pressure, as we have seen above. Changes in temperature and salinity are responsible for a number of phenomena associated with convection currents and gravity currents, particularly in the ocean and lakes. However in many fluid flows in rivers and pipes, these differences are small and can be ignored.

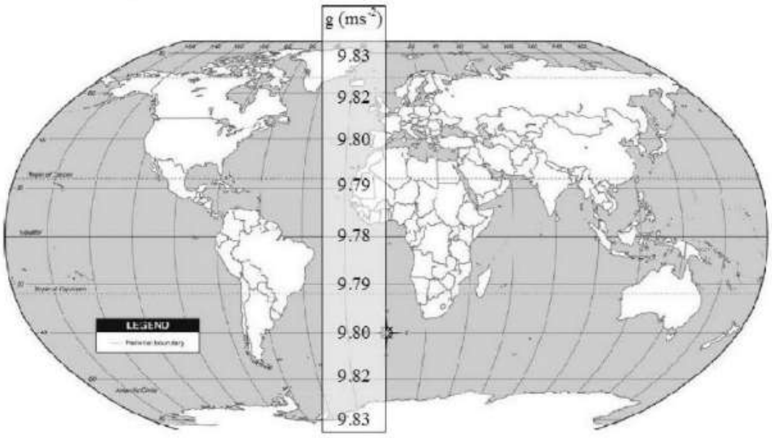


Figure 1-7. Variation of gravitational acceleration g ($m\,s^{-2}$) with latitude.

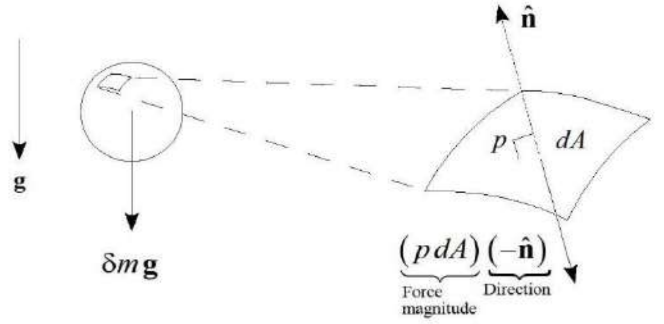
Gravitational acceleration varies between $9.78\,m\,s^{-2}$ at the equator and $9.83\,m\,s^{-2}$ at the poles, so that students could use an appropriate value, as portrayed in Figure 1-7, however it is clear that a value of $9.8\,m\,s^{-2}$ is accurate enough, and even a value of $10\ m\,s^{-2}$ (correct to 2%) could be used with an accuracy commensurate with almost all of the theories and methods presented in this course.

Density of fresh water ρ	1000	$kg\,m^{-3}$
Density of sea water ρ	1025	$kg\,m^{-3}$
Gravitational acceleration g	$9.8 \approx 10$	$m\,s^{-2}$
Kinematic viscosity ν	10^{-6}	$m^2\,s^{-1}$

Table 1-3. Typical values of physical properties associated with water problems

In view of all the above remarks, in many engineering problems and in this course typical constant values can be assumed, which are set out in Table 1-3. It is a strange fact, and with the exception of the density of fresh water, an accidental one, that all of these quantities are close to an integer power of 10!

1.2 Forces acting on a fluid



(a) Body force vector acting through centre of mass of fluid particle

(b) Pressure force vector acting normal to local surface of fluid

Figure 1-8. The two types of forces acting on a fluid particle – body force such as gravity acting through the centre of mass, and surface force, in this case pressure acting normally to the local surface.

Consider figure 1-8 showing the two dominant types of forces acting on fluid particles.

1. **Body forces** – this type of force acts at a distance, penetrating deep inside the fluid. - the most common is that due to gravity. It is usually expressed as an acceleration, or, force per unit mass:

$$\delta\mathbf{F}_{\text{body}} = \delta m\,\mathbf{g} = \rho\mathbf{g}\,\delta V,$$

   where $\mathbf{g}$ is the acceleration due to gravity, $\mathbf{g} = (0, 0, -g)$, where we have assumed that the z co-ordinate is vertically upwards.

2. **Surface forces**
   a. **Pressure forces** – These are due to molecular motions of particles. Let $\hat{\mathbf{n}}$ be a unit vector normal to the surface, directed into the fluid, then

$$\delta\mathbf{F}_{\text{pressure}} = -p\,dA\,\hat{\mathbf{n}}.$$

   Pressure is a scalar quantity. In a static fluid (one where all particles have the same velocity) there are no other stresses acting, and equilibrium of a finite volume leads to Pascal's Law[3]: *pressure exerted anywhere in a static fluid is transmitted equally in all directions*.

   b. **Shear forces** – the relative motion of real fluids is accompanied by tangential stresses. Those due to viscous effects, whereby momentum exchange due to random movement of molecules occurs, are relatively small in hydraulics problems. Rather more important than that of individual molecules is the momentum exchange by turbulence and large masses of fluid.

1.3 Units

Throughout we will use the *Système Internationale*, in terms of metres, kilograms and seconds, the fundamental units of mass (M), length (L) and time (T) respectively. Other quantities are derived from these. All are set out in Table 1-4. Some of the derived quantities will be described further below.

Quantity	Dimensions	Units
	Fundamental quantities	
mass	M	kg
length	L	m
time	T	s
temperature	θ	$^\circ$C or $^\circ$K $= ^\circ$C $+ 273.15$
	Derived quantities	
linear velocity	LT^{-1}	$\mathrm{m\,s}^{-1}$
angular velocity	T^{-1}	s^{-1}
linear acceleration	LT^{-2}	$\mathrm{m\,s}^{-2}$
volume flow rate	$\mathrm{L}^3\mathrm{T}^{-1}$	$\mathrm{m}^3\,\mathrm{s}^{-1}$
mass flow rate	MT^{-1}	$\mathrm{kg\,s}^{-1}$
linear momentum	$\mathrm{M\,L\,T}^{-1}$	$\mathrm{kg\,m\,s}^{-1}$
force	MLT^{-2}	$1\ \mathrm{kg\,m\,s}^{-2} = 1\ \mathrm{N}$ (Newton)
work, energy	$\mathrm{ML}^2\mathrm{T}^{-2}$	$1\,\mathrm{N\,m} = 1\,\mathrm{J}$ (Joule)
power	$\mathrm{ML}^2\mathrm{T}^{-3}$	$1\,\mathrm{J\,s}^{-1} = 1\,\mathrm{W}$ (Watt)
pressure, stress	$\mathrm{ML}^{-1}\mathrm{T}^{-2}$	$1\,\mathrm{N\,m}^{-2} = 1\,\mathrm{Pa}$ (Pascal) $= 10^{-5}$ bar
surface tension	MT^{-2}	$\mathrm{N\,m}^{-1}$
dynamic viscosity	$\mathrm{ML}^{-1}\mathrm{T}^{-1}$	$1\,\mathrm{kg\,m}^{-1}\,\mathrm{s}^{-1} = 10$ Poise
kinematic viscosity	$\mathrm{L}^2\mathrm{T}^{-1}$	$1\,\mathrm{m}^2\,\mathrm{s}^{-1} = 10^4$ Stokes

Table 1-4. Quantities, dimensions, and units

1.4 Turbulent flow and the nature of most flows in hydraulics

1.4.1 Reynolds' experiments (1883)

Reynolds[4] non-dimensionalised the differential equations which govern the flow of viscous fluid, and found that

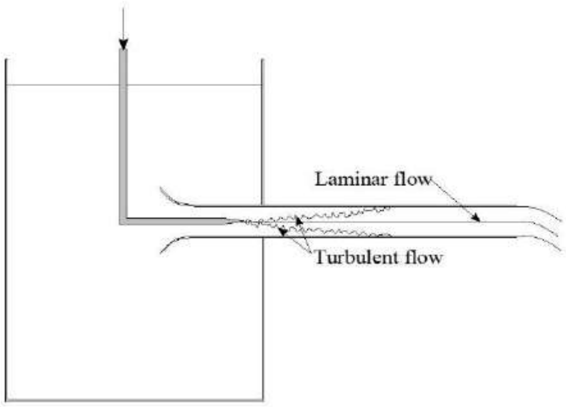


Figure 1-9. Reynolds' apparatus – the diagram showing the dye trace in the pipe for two different flow cases.

for dynamic similarity between two geometrically similar flow situations the dimensionless group Velocity scale $\times$ Length scale $/\nu$ must be the same in the two cases, the quantity which is now called the Reynolds number (for these pipe experiments the mean velocity U and pipe diameter were used). To determine the significance of the dimensionless group Reynolds conducted his experiments on flow of water through glass tubes, as shown in Figure 1-9, with a smooth bellmouth entrance and dye injected into the tube. For small flows the dye stream moved as a straight line through the tube showing that the flow was laminar. With increasing velocity, and hence Reynolds number, the dye stream began to waver and then suddenly broke up and was mixed through the tube. The flow had changed to turbulent flow with its violent interchange of momentum. Starting with turbulent flow in the glass tube, Reynolds found that it always becomes laminar when the velocity is reduced to make UD/ν less than 2000. Usually flow will change from laminar to turbulent in the range of Reynolds numbers from 2000 to 4000. In laminar flow the losses are proportional to the average velocity, while in turbulent flow, proportional to a power of velocity from 1.7 to 2.

The Reynolds number: It is an historical accident that in his 1883 paper, Osborne Reynolds introduced the quantity that has since been called the *Reynolds Number* and given the symbol R, which is the *inverse* of the dimensionless viscosity, defined as

$$R = \frac{UL}{\nu} = \frac{1}{\nu_*},$$

which has been almost universally used as the measure of the relative importance of viscosity. It would have been more satisfying if that number had been defined upside down as dimensionless viscosity! With the traditional definition, high Reynolds number flows are those which are large and/or fast such that the effects of viscosity are small. In environmental hydraulics problems, with, say, a typical length scale of $1\,\mathrm{m}$, a velocity scale of $1\,\mathrm{m\,s^{-1}}$, and a typical value for water at 20°C of $\nu = 1 \times 10^{-6}\,\mathrm{m^2\,s^{-1}}$, a value of $R = 10^6$ ($\nu_* = 10^{-6}$) is obtained, showing how viscosity is unimportant in many outdoor problems. Flows in pipes, however, because they can be smaller and have slower flow, may have Reynolds numbers of the order of 10^3, when viscous effects may be present.

The term “droplet liquid” “fluid” is a low-compressible medium, and the term "compressible liquid" is a gas. Consequently, a liquid is understood as any medium with the property of fluidity. Fluid has a certain volume, which practically does not change under the influence of forces. Gases, occupying a certain space, can significantly change the volume, contracting and expanding under the influence of forces.
This means that fluid easily change shape, in contrast to solids, but they hardly change the volume, and gases easily change both shape and volume.

Conventional Derivation of the Van der Waals Equation

The state of a given amount of any substance can be described by three parameters: pressure p, volume V, and temperature T. These parameters are related to each other. Their relationship is described by the equation of state, which in the general case has the form:

$$F(p, V, T) = 0.$$

The specific form of the equation depends on the substance. For example, a rarefied gas at a sufficiently high temperature is well described by the ideal gas model. Its equation of state is the well-known ideal gas law stated by Emile Clapeyron $(1799-1864)$ in 1834 :

$$pV = \frac{m}{M} RT.$$

Here m is the mass of the gas, M is the molar mass (i.e. the mass of one mole of the gas), R is the universal gas constant. For one mole of gas, this equation takes the following form:

$$pV = RT.$$

Subsequent experiments revealed deviations in the behavior of real gases from the ideal gas law. These results were summarized by the Dutch physicist Johannes Diderik van der Waals $(1837-1923)$ who in 1873 proposed a more accurate equation of state for a real gas.

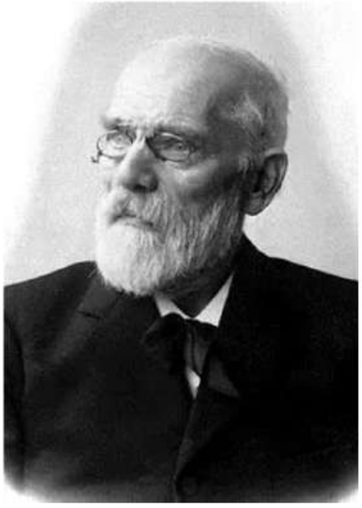


Fig.1 Johannes Diderik van der Waals (1837-1923)

It is called the Van der Waals equation. For one mole of a gas, it can be written as

$$\left(p+\frac{a}{V^2}\right)(V-b)=RT.$$

This equation takes into account the attractive and repulsive forces between molecules. The attractive forces are taken into account through the near-wall effect.

Van der Waals Isotherms

At a fixed temperature, the Van der Waals equation describes the dependence $p(V)$. In the pV-plane, this dependence is represented as a family of isotherms, each of which corresponds to a certain temperature.

For a fixed value of p, the resulting equation is a third degree equation with respect to the variable V. It is known that a cubic equation can have 1 or 3 real roots. The first case occurs at high temperatures T (the green isotherm AB in Figure 3).

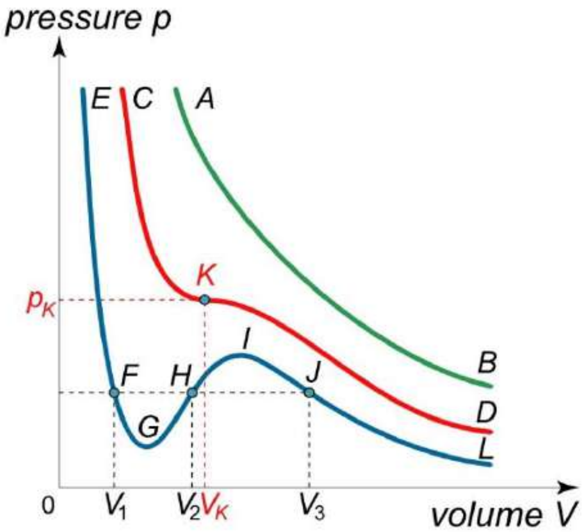


Figure 3.

With lowering the temperature, an undulating region appears on the isotherm. In this case, there are three roots (the blue isotherm $EFGHIJL$). The transition between the two types of isotherms occurs at a certain temperature T_K, which is called the critical temperature.

To quantify just how compressible substances are, it is necessary to define the property. The **isothermal compressibility** is defined by the fractional differential change in volume due to a change in pressure.

$$\kappa_T \equiv -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T$$

The isothermal bulk modulus of an ideal gas is the ratio of change in pressure (Δp) to the fractional change in volume ($\Delta V/V$) at constant gas temperature T. Differentiate the ideal gas equation $pV = nRT$ to get $p\Delta V + V\Delta p = 0$ i.e., $\Delta V/V = -\Delta p/p$. Note that $\Delta T = 0$ because the temperature is constant. Substitute $\Delta V/V$ in the defining equation to get the isothermal bulk modulus

$$B_{\text{isothermal}} = -\frac{\Delta p}{(\Delta V/V)} = p.$$

2. Hydrostatics

An understanding of fluid statics is essential for the design of hydraulic structures, tanks, ships, pressure measurement and meteorology. We will consider the equilibrium of a mass of fluid which is at rest, or in uniform motion, when no element of fluid moves relative to any other element. As there are no velocity gradients, there are no shear stresses. The pressure forces balance applied body forces, in accordance with Newton's second law.

2.1 Fundamentals

2.1.1 Pressure at a point

The pressure in a fluid in static equilibrium is the same in all directions. Pressure is a scalar quantity that arises from the non-directional nature of the oscillations of fluid particles, nominally at rest here. This was considered in §1.2.

2.1.2 The pressure in a fluid under static equilibrium

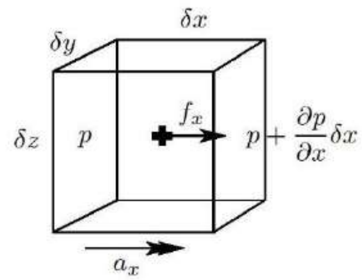


Figure 2-1.

Consider an element of fluid in static equilibrium with components of body forces f_x, f_y and f_z per unit mass. The fluid has accelerations a_x, a_y and a_z. Considering motion in the x direction only at this stage, as shown in Figure 2-1, the acceleration in the x direction is caused by the net force in the x direction on the faces which are normal to that direction. The net force in the x direction due to pressure forces is

$$-\left(p + \frac{\partial p}{\partial x}\,\delta x\right)\,\delta y\,\delta z + p\,\delta y\,\delta z = -\frac{\partial p}{\partial x}\,\delta x\,\delta y\,\delta z. \tag{2.1}$$

Considering Newton's second law in the x direction: mass$\times$acceleration in x direction=net force in x, this gives

$$\begin{aligned}(\rho\,\delta x\,\delta y\,\delta z)\,a_x &= \text{body force} + \text{net pressure force}\\ &= (\rho\,\delta x\,\delta y\,\delta z)\,f_x - \frac{\partial p}{\partial x}\,\delta x\,\delta y\,\delta z,\end{aligned}$$

and cancelling the common factors of the volume of the body $\delta x\,\delta y\,\delta z$, gives

$$\frac{\partial p}{\partial x} = \rho\,(f_x - a_x) \tag{2.2}$$

and we obtain expressions for the y and z components which are the same, with x replaced throughout by y and z respectively. Thus we have:

The pressure gradient at a point = fluid density $\times$ (body force per unit mass - fluid acceleration),

or in vector terms

$$\nabla p = \rho\,(\mathbf{f} - \mathbf{a}), \tag{2.3}$$

where $\mathbf{f} = (f_x, f_y, f_z)$ is the body force per unit mass, and $\mathbf{a} = (a_x, a_y, a_z)$ is the fluid acceleration.

2.1.3 Fluid at rest in a gravity field

In the usual case of a body of fluid at rest in a field of constant gravity, where x and y are in a horizontal plane and z is vertically upwards,

$$a_x = a_y = a_z = 0 \quad \text{and} \quad f_x = f_y = 0,\ f_z = -g \approx -9.8\,\mathrm{m\,s^{-2}},$$

and substituting into equation (2.2) with similar equations for the other co-ordinates, we obtain

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0, \tag{2.4a}$$

$$\frac{\partial p}{\partial z} = -\rho g. \tag{2.4b}$$

That is, the pressure does not vary with x or y, such that on any horizontal plane the pressure is constant, such as anywhere on the plane XX or the plane YY *within the fluid* in Figure 2-2. This is also known as Pascal's law.

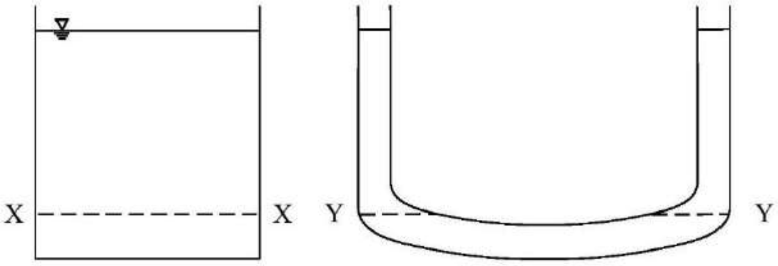


Figure 2-2. Two vessels and two typical horizontal planes, such that within the fluid the pressure on each plane is constant

Equation (2.4b) can be integrated in common cases where the density ρ is a known function of pressure p. For most problems in hydraulics the fluid can be considered incompressible, such that ρ is constant, and the integration is simple:

$$\begin{aligned} \int \frac{\partial p}{\partial z} dz &= \int (-\rho g) dz, \quad \text{giving} \\ \int \mathrm{d}p &= -\rho g \int dz, \quad \text{and integrating,} \\ p &= -\rho g z + C(x, y), \end{aligned}$$

where because we integrated a partial derivative the quantity C is in general a function of the other two variables x and y. However, the other differential equations (2.4a) show that C cannot be a function of x or y so that it is a constant throughout the fluid. Thus we have the equation governing the pressure in an incompressible fluid in a constant gravity field, the

Hydrostatic Pressure Equation:

$$p + \rho g z = \text{Constant throughout the fluid} \tag{2.5}$$

It is simpler to solve many problems in the form of equation (2.5), as we will see below while considering some pressure measurement devices. However, many other problems are most easily solved by expressing the pressure as a function of depth below the surface rather than elevation above a point. To do this the constant is evaluated by considering a special point in the fluid, usually on the surface.

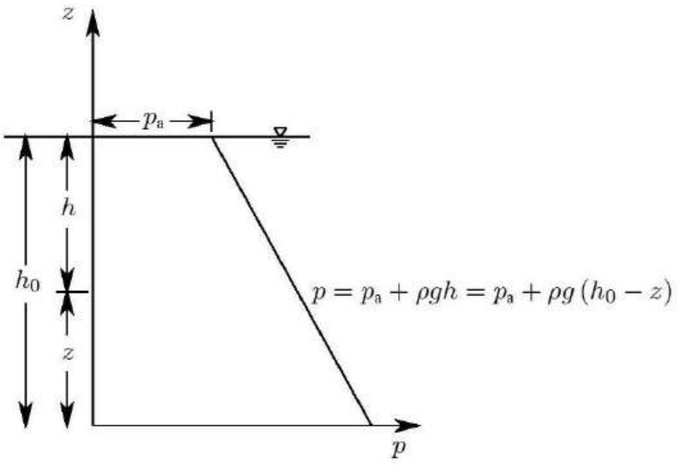


Figure 2-3. Hydrostatic pressure variation with depth

Consider the general situation shown in Figure 2-3 that shows a graph of pressure plotted horizontally against elevation using a datum (reference level) that is a vertical distance h_0 below the surface. When $z = h_0$ the pressure is atmospheric, denoted by p_a, and so

$$p + \rho g z = \text{Constant} = p_a + \rho g h_0,$$

thereby evaluating the constant and enabling us to write the equation for pressure p at an arbitrary point with elevation z as

$$p - p_a = \rho g(h_0 - z) \quad \text{or} \quad p = p_a + \rho g(h_0 - z).$$

Since $h_0 - z = h$, the height of the free surface above the point, this becomes

$$p - p_a = \rho g h$$

and almost always we measure pressure as *gauge pressure*, relative to the atmosphere, as shown in Figure 2-4, so that we usually just write

$$p = \rho g h, \tag{2.6}$$

so that the pressure at a point is given by the density ρ multiplied by gravitational acceleration g multiplied by the depth of water above the point.

Gauge, absolute, and vacuum pressures Figure 2-4 shows, on an absolute pressure scale, how the gauge pressure, widely used for engineering purposes, is measured relative to the (variable) atmospheric pressure; and vacuum pressure, used in engineering applications such as brakes for vehicles, is measured from the same datum but in the other direction. An absolute vacuum corresponds to a vacuum pressure of one atmosphere.

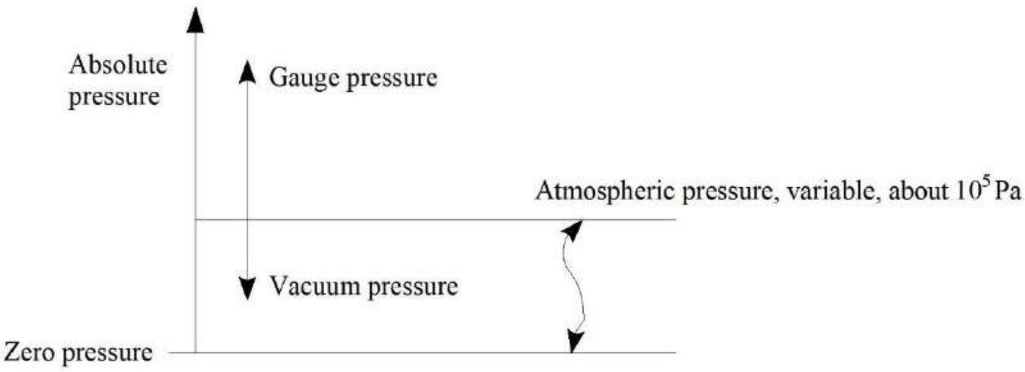


Figure 2-4. Definitions of gauge, absolute, and vacuum pressures.

Equivalent static head: Throughout engineering, pressures are often expressed as the equivalent height of liquid which can be supported by that pressure, called the *head*, here denoted by h:

$$h = \frac{p}{\rho g}.$$

2.1.4 Units

Pressure is a force per unit area, so in fundamental *SI* units it is $\mathrm{N\,m^{-2}}$, for which a special name is used, a Pascal, or Pa. We will see that this is an appropriate name, and that it has a value of about 10^5 Pa. Atmospheric pressure used to be specified in terms of 1/1000 of that, or millibar, however they are not *SI* units. Instead of a millibar, the exactly numerical equivalent value of "hectoPascal" or "hPa" is used, which is a 1/100 of a Pascal. Atmospheric pressure varies usually in the range $980 - 1030$ hPa. Hurricane Wilma was the most intense hurricane ever recorded in the Atlantic basin, in the 2005 season; the eye pressure was 882 hPa.

List of Pressure Measuring Devices

The mercury barometer and atmospheric pressure

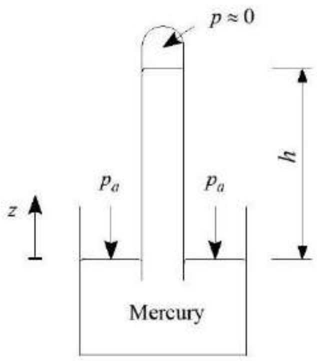


Figure 2-5. Mercury barometer.

Consider the hydrostatic pressure equation (2.5) for all points within the mercury of the barometer shown in Figure 2-5:

$$p + \rho g z = \text{Constant.}$$

We apply this at two points: at the surface open to the air and secondly at the surface in the tube, at which the pressure is the vapour pressure of mercury, close to zero

$$\begin{aligned} p_a + 0 &= 0 + \rho g h, \quad \text{thus} \\ p_a &= \rho g h. \end{aligned}$$

The density of mercury is about $13\,600\ \mathrm{kg\,m^{-3}}$, and typically h is about $0.76\ \mathrm{m}$, hence

$$p_a \approx 13\,600 \times 9.8 \times 0.76 \approx 1 \times 10^5\ \mathrm{N\,m^{-2}}.$$

In terms of the equivalent height of water, which it is sometimes convenient to use, $h = p/\rho g \approx 10^5/1000\ \ 10 \approx 10\,\mathrm{m}$.

Mercury is ideally suited for use in a barometer due to its high density (needing therefore only a short tube) and its very low vapour pressure.

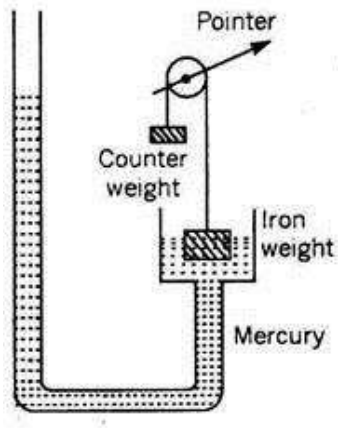
The altitude of a place and weather conditions influence the reading of the barometer. A reading of a barometer recorded at a spot indicates only the local atmospheric pressure.

The International standard atmospheric pressure is 101.325 kPa corresponding to 10.325 m of water or 760 mm of mercury.

The Siphon Barometer

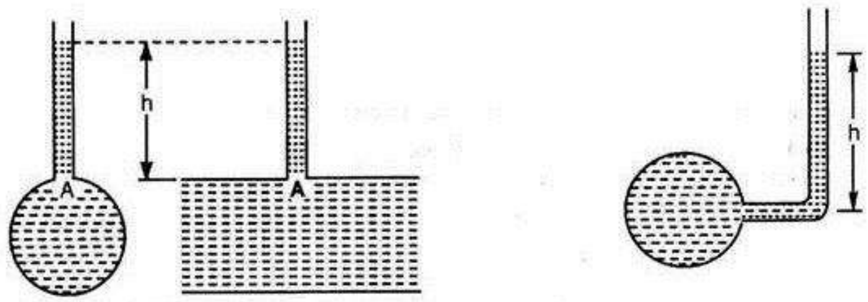
This instrument is conveniently used as a household barometer. This device consists of a glass tube bent at the lower part to form a U-tube. The open end of the U-tube is enlarged. This enlarged part takes the place of the bowl or reservoir of the ordinary barometer. An iron block of small weight is supported on the mercury surface partly by up thrust of mercury on it and partly by a counterweight.

The iron block and the counterweight are connected by a string taken over a pulley. Variation of atmospheric pressure brings about rise and fall of the mercury surface in the open end of the U-tube which in turn causes the pulley to rotate by some angle. A pointer attached to the pulley will move over a circular scale from which the atmospheric pressure may be read.



Piezometer or Pressure Tube

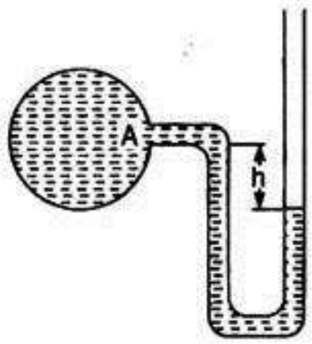
The piezometer is used to measure the static pressure head of a liquid flowing at any section of a pipe. It consists of a tube whose open lower end is mounted flush with the inside wall of the pipe. The other end of the tube is exposed to the atmosphere. In the arrangement shown in Fig. the height h to which the liquid rises in the tube represents the pressure head at the level A where the tube is connected to the pipe.



The piezometer has limitations for its use due to the following reasons:

(I) It is very difficult or impracticable to measure high pressures. Particularly for liquids of low specific gravity, the height of the liquid column in the piezometer will be inconveniently high requiring a very long piezometer tube.

(II) The piezometer cannot work for negative gauge pressure since air would flow into the container through the tube.



The piezometer tube may take the form shown in Fig. for measurement of small negative gauge pressures. In this arrangement, the free surface of the liquid in the tube will be at a level lower than the level A inside the container where the pressure is to be gauged. If the free liquid surface in the tube is h units below A, then the pressure head at A

(III) Rapid changes of pressure which may take place continuously cannot be effectively measured. This is because change in the piezometer level will lag behind corresponding rapid change of pressure.

Manometers:

Manometers are pressure gauging devices using columns of different liquids. The fluid whose pressure is to be determined is called the metered fluid while the other fluid is called the manometer fluid. The manometer fluid may be of higher density or lower density than that of the metered fluid. These devices can be used to gauge pressures of liquids as well as gases. Manometers have connecting U-shaped tubes containing different fluids.

In a manometer when one limb of the device is open to the atmosphere it records the pressure of the source connected to the other limb. When both the limbs are connected to pressure sources, the manometer records the difference of pressure between the two pressure sources. Accordingly, these manometers are called simple manometer and differential manometer.

The simple manometer

Consider the U-tube in Figure 2-6 filled with manometric fluid of density ρ_m with one end attached to a point X where the pressure of the fluid, of density ρ_f, is to be measured. The other end is open to the atmosphere.

In the fluid, using the hydrostatic pressure equation (2.5) at both X and B:

$$p_X + \rho_f g h_1 = p_B + 0.$$

In the manometric fluid, also using the hydrostatic pressure equation at B and at the surface open to the air:

$$p_B + 0 = p_a + \rho_m g h_2.$$

Eliminating the unknown intermediate pressure p_B:

$$p_X - p_a = \rho_m g h_2 - \rho_f g h_1,$$

thus by measuring the height difference of the fluid in the manometer we can calculate the gauge pressure at X. In

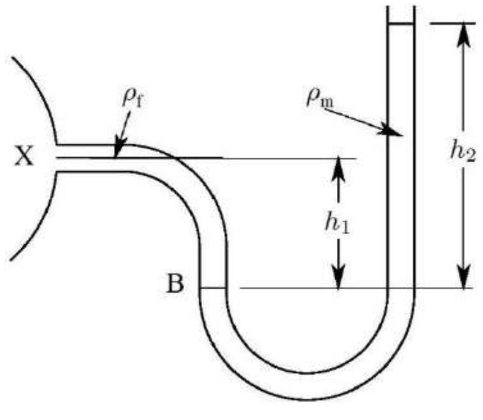
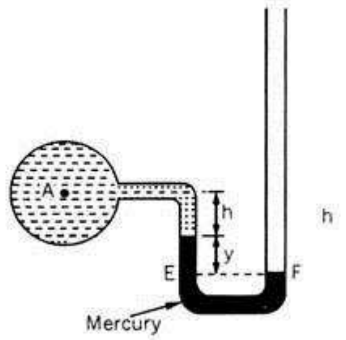


Figure 2-6. Simple manometer, measuring the pressure in a vessel relative to the atmosphere

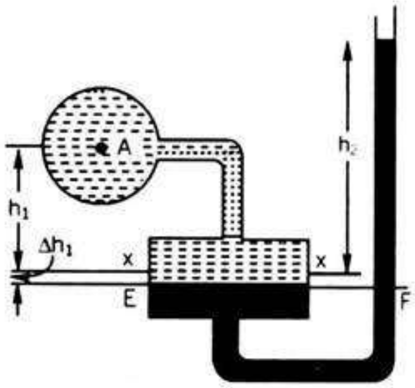
the special case where the same fluid is used throughout,

$$p_X - p_a = \rho_f g(h_2 - h_1).$$



Sensitive Manometers

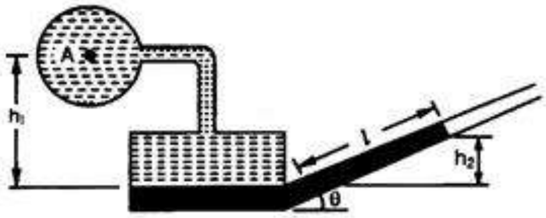
The single column manometer shown in Fig. is a modified form of the ordinary U-tube manometer. This manometer also has two limbs, one of which is made very large in area compared with the other. The area of the larger limb (also called the basin) may be made 100 times the area of the other limb. The manometer contains a heavy liquid like mercury. The pipe in which the pressure is to be determined is connected to the larger limb.



Any pressure change in the pipe may only produce a very small change in level of the manometer liquid surface in the basin. This change in level may be neglected. Hence the reading in the narrow tube only is to be taken. Since there is no need to take any reading corresponding to the liquid surface in the basin, it need not be made transparent. Usually it is made of iron. The other limb i.e., the narrow tube may be vertical or inclined, to make it more sensitive.

The Inclined Tube Manometer

This is an improvement over the single column manometer. In this case the manometer tube is made inclined in order to make it more sensitive. Fig. shows this type of manometer. In this case the displacement of the heavy liquid in the narrow tube is relatively greater and hence readings can be taken more accurately.

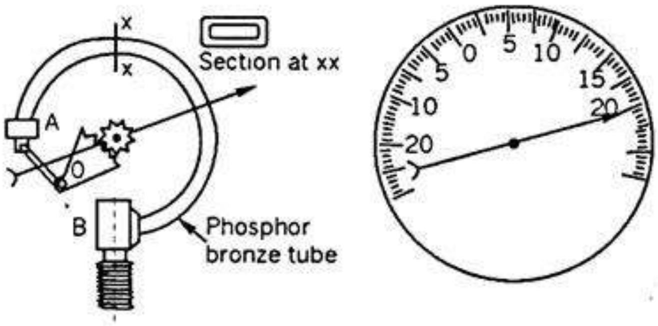


The Bourdon Gauge

This device consists of a metallic tube of elliptical section closed at one end A, the other end B being fitted to the gauge point where the pressure is to be measured. As the fluid enters the tube, the tube tends to straighten.

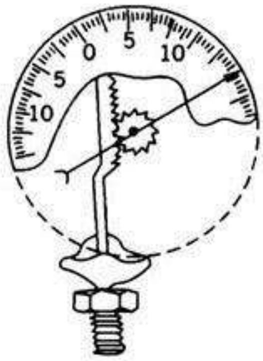
By using a pinion-sector arrangement the small elastic deformation of the tube is communicated to a pointer in an amplified manner. The pointer moves over a graduated dial. The device is calibrated by subjecting it to various known pressures.

The Bourdon gauge is suitable for measuring not only high pressures such as those in a steam boiler or a water main but also negative or vacuum pressures. A gauge which is so devised to measure positive as well as negative pressures is called a compound gauge.



The Diaphragm Pressure Gauge

This device is based on the same principle as that of the Bourdon gauge. In this case a corrugated diaphragm is provided instead of the Bourdon tube. When the device is fitted to any gauge point, the diaphragm will undergo an elastic deformation.



This device is found suitable for measuring relatively low pressures.

What a siphon is and how it works in practice.

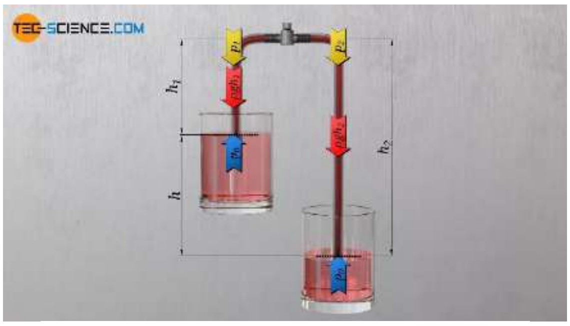
If, for example, you want to empty a pool by a garden hose, you only have to place one end of the hose over the edge into the pool and the other end outside. The outside end of the hose only has to be lower than the water level.

As long as the lower end is always held lower than the water level, the water is also carried over larger heights, such as over the edge of the pool (to the maximum height to be overcome later more). Such an arrangement is also called a *siphon or siphon spillway*.

A flexible tube is mounted on the screw cap of the bottle. If the bottle is now turned upside down, the water will begin to flow out through the tube. The outflow of the water leads to an increase in the volume of air inside the bottle. Since no air can inflow through the relatively small hose, a negative pressure is created inside the bottle.

The resulting negative pressure in the bottle can then be used to suck water from another vessel.

With an atmospheric air pressure of 1 bar, this ambient pressure can thus push a maximum water column of 10 meters upwards. The ambient pressure is therefore not sufficient for higher heights.

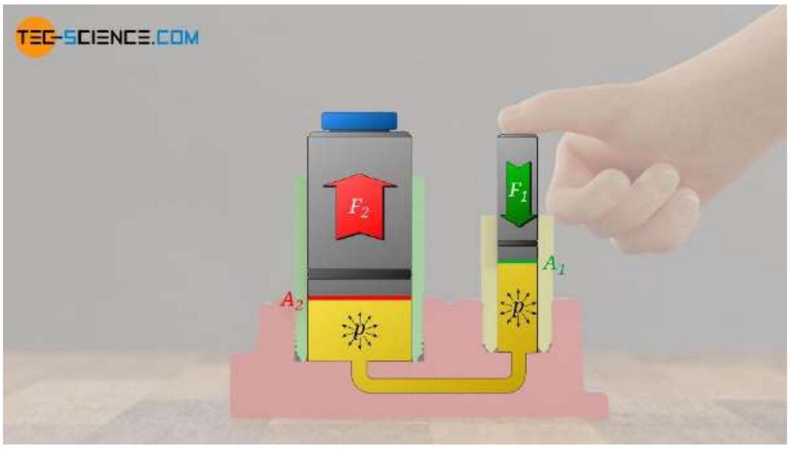


$$p_1 = p_0 - \rho \cdot g \cdot h_1$$

$$p_2 = p_0 - \rho \cdot g \cdot h_2$$

$$\begin{aligned}\Delta p &= p_1 - p_2 \\ &= p_0 - \rho \cdot g \cdot h_1 - (p_0 - \rho \cdot g \cdot h_2) \\ &= \cancel{p_0} - \rho \cdot g \cdot h_1 - \cancel{p_0} + \rho \cdot g \cdot h_2 \\ &= \rho \cdot g \cdot h_2 - \rho \cdot g \cdot h_1 \\ &= \rho \cdot g \cdot (h_2 - h_1) \\ &= \rho \cdot g \cdot h\end{aligned}$$

Hydraulic jack



An application – industrial hydraulics

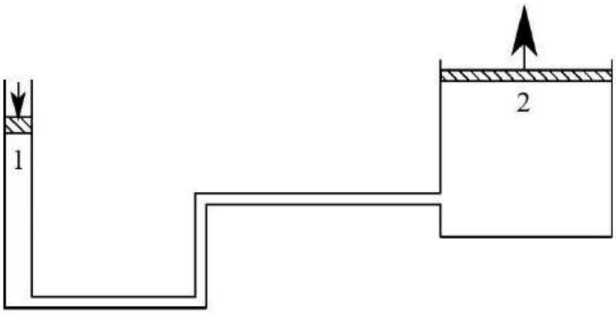


Figure 2-8. Operation of an hydraulic lift or jack

Consider the problem shown in Figure 2-8 in which an actuator (a small cylinder with a piston in it, but probably a pump in more modern applications) can apply a force to a fluid, and the pressure so generated is then transmitted throughout the fluid to the other end of the fluid line where a larger cylinder and piston are placed. Use of the hydrostatic pressure equation in the fluid between points 1 and 2 gives

$$p_1 = p_2 + \rho g \left(z_2 - z_1\right).$$

Now, $p_1 = F_1/A_1$, with a similar result for piston 2, giving

$$\frac{F_1}{A_1} = \frac{F_2}{A_2} + \rho g \left(z_2 - z_1\right).$$

In practical cases the term involving the difference in elevation is negligible, so that the force which can be lifted by the hydraulic jack is

$$F_2 \approx F_1 \left(\frac{A_2}{A_1}\right).$$

For large values of A_2/A_1 large forces can be exerted, such as the control surfaces on a large aircraft.

Pressures in accelerating fluids

Consider equation (2.2) for the pressure gradient:

$$\frac{\partial p}{\partial x} = \rho(f_x - a_x), \tag{2.2}$$

with similar equations for y and z.

1. **Fluid accelerating vertically**

Let the vertical acceleration be a_z, upwards, the other components being zero. Equation (2.3) gives

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0, \frac{\partial p}{\partial z} = \rho(-g - a_z) = -\rho(a_z + g),$$

which are the same equations as obtained before, but with g replaced by $a + g$, so that the vertical acceleration and gravity simply combine algebraically.

2. **Fluid accelerating horizontally**

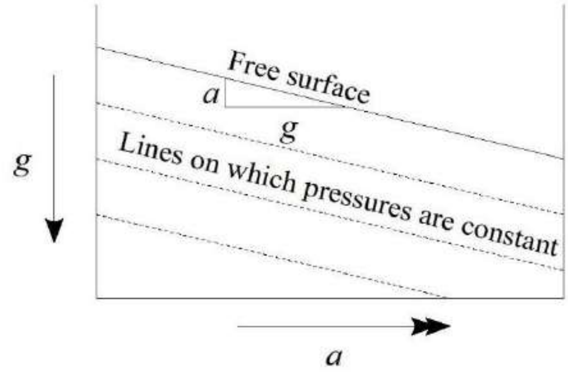


Figure 2-9. Vessel accelerating to the right, showing tilted water surface and pressure contours.

Consider the container in Figure 2-9 which is accelerating in the positive x direction with acceleration a. Equation (2.2) gives

$$\frac{\partial p}{\partial x} = \rho(0 - a), \quad \frac{\partial p}{\partial y} = 0, \quad \text{and} \quad \frac{\partial p}{\partial z} = \rho(-g - 0).$$

These equations may be integrated to give

$$\begin{aligned} p &= -\rho a x - \rho g z + C \\ &= -\rho(ax + gz) + C, \end{aligned}$$

where C is a constant of integration. From this we can find the equation of surfaces on which the pressure is constant, p_1 say:

$$\frac{p_1 - C}{-\rho} = ax + gz,$$

hence

$$z = D - \frac{a}{g}x,$$

where D is a constant. Clearly, surfaces on which pressure is constant are planes, with a gradient in the x-direction of $-a/g$. The free surface, on which $p = p_a$ is a special case.

2.2 Forces on submerged planar objects

Consider the submerged plate of arbitrary shape shown in Figure 2-10, inclined at an angle θ to the horizontal. We introduce the co-ordinate s with origin at the axis OO where the plane of the plate intersects the plane of the free surface. The shape of the plate is defined by a specified breadth as a function of s, $b(s)$. At any horizontal line on the plate which is h below the surface the hydrostatic pressure equation gives,

$$p = p_a + \rho g h$$

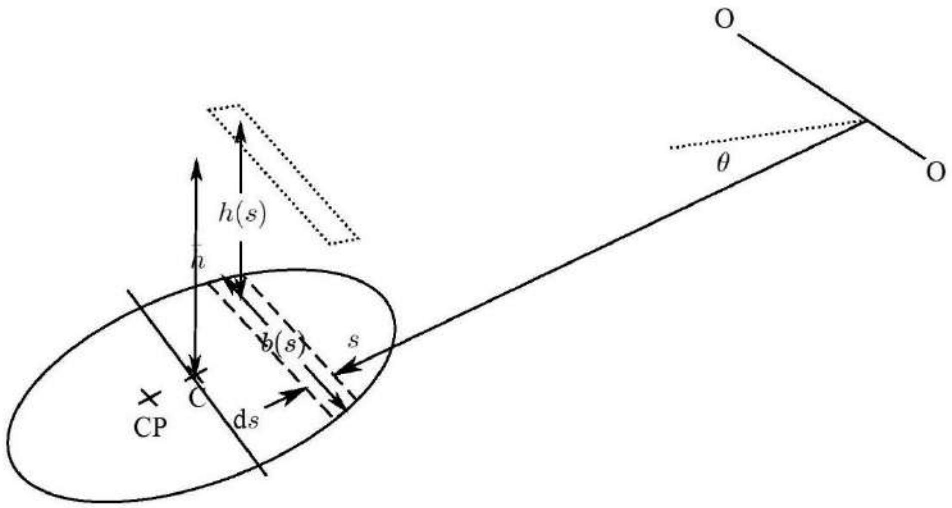


Figure 2-10. Inclined underwater plane surface and axis OO where it intersects the plane of the surface

and the force on an element which is ds wide and $b(s)$ long is

$$\begin{aligned} dF &= p\,dA, \text{ where } dA \text{ is the area of the element,} \\ &= p\,b(s)\,ds \\ &= (p_a + \rho g h)\,b(s)\,ds. \end{aligned}$$

Now, $h = s\,\sin\theta$, therefore

$$dF = p_a b(s)\,ds + \rho g \sin\theta\, s\, b(s)\,ds. \tag{2.10}$$

Integrating over the whole plate to find the total force on one side gives

$$F = \int dF = p_a \int b(s)\,ds + \rho g \sin\theta \int b(s)\,s\,ds,$$

where the first integral is simply the *area* of the plate, A, while the second is its *first moment of area* M_{OO} *about the intersection line* OO *of the plane of the plate and the free surface*, which has be written as $M_{OO} = A\bar{s}$, where $\bar{s}$ is the co-ordinate of the centroid C, giving

$$F = p_a A + \rho g \sin\theta\, A\,\bar{s}. \tag{2.11}$$

However the depth of the centroid $\bar{h} = \bar{s}\sin\theta$, so force can be written quite simply as

$$F = p_a A + \rho g A \bar{h}, \tag{2.12}$$

which is often able to be simply calculated without taking moments, if the plate is a simple shape. As the pressure at the centroid is $p_C = p_a + \rho g \bar{h}$, the result can be simply stated

Force on plate = Area × Pressure at centroid.

In design calculations it is often necessary to know the position of *Centre of Pressure* (CP), the point at which the total force can be considered to act. Taking moments about OO, substituting equation (2.10), and integrating:

$$\begin{aligned} \text{Moment of force about O} &= \int s\,dF = p_a \int b(s)\,s\,ds + \rho g \sin\theta \int b(s)\,s^2\,ds \\ &= p_a A\,\bar{s} + \rho g \sin\theta\, I_{OO}, \end{aligned} \tag{2.13}$$

where I_{OO} is the *Second Moment of Area* of the plate about the axis OO. If s_{CP} is , By definition, s_{CP}, the s co-ordinate of the centre of pressure is such that the moment of the force is equal to $F\,s_{CP}$, and so, substituting equation (2.11) into that and setting equal to the result of equation (2.13), we have

$$s_{CP} = \frac{p_a A\,\bar{s} + \rho g \sin\theta\, I_{OO}}{p_a A + \rho g A \bar{h}}. \tag{2.14}$$

Almost always the atmospheric pressure is ignored, as calculating the force on one side of a plate usually means that the other side of the plate also is subject to atmospheric pressure, *e.g.* the side of a ship shown in Figure 2-11, or the interior of a submarine.

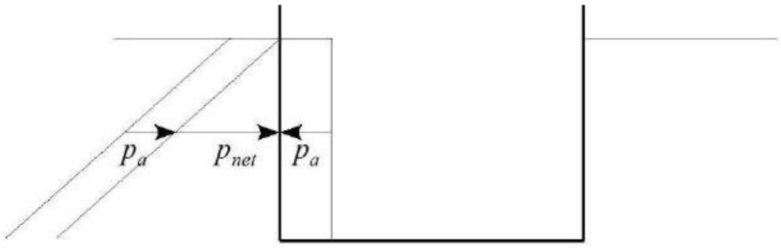


Figure 2-11. Situation, such as the side of a ship, where the atmospheric pressure contributes equally on both sides such that it can be ignored.

Substituting $p_a = 0$ into equation (2.14) and using $\bar{h} = \bar{s}\sin\theta$ gives the expressions

$$s_{CP} = \frac{\sin\theta\, I_{OO}}{A\bar{h}} = \frac{I_{OO}}{A\bar{s}}. \tag{2.15}$$

Note: although the force can be calculated by using the depth of the centroid of the plate, *it does not act through the centroid, i.e.* $s_{CP} \neq \bar{s}$.

In some cases, for simple shapes such as rectangles and circles, the second moment of area I_{CC} of the plane about an axis through its centroid is already known and does not have to be determined by integration. It is convenient to use the *Parallel Axis Theorem*, which in this case is

$$I_{OO} = I_{CC} + A\,\bar{s}^2, \tag{2.16}$$

to calculate the distance of the centre of pressure from the centroid. Substituting this into equation (2.15)

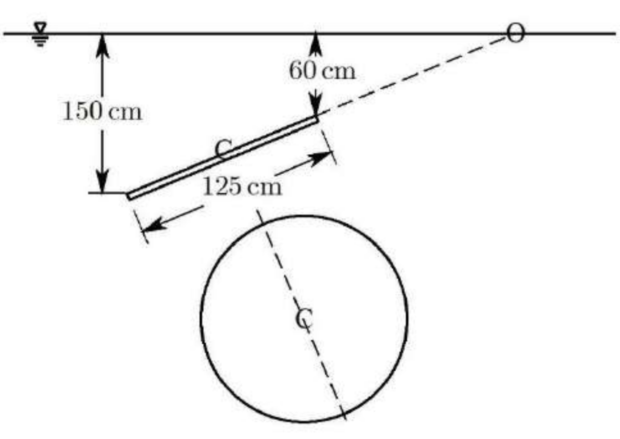
$$s_{CP} = \frac{I_{CC} + A\,\bar{s}^2}{A\bar{s}} = \frac{I_{CC}}{A\bar{s}} + \bar{s}. \tag{2.17}$$

This formula can be re-written to give the expression for the *distance of the centre of pressure from the centroid in the plane of the plate*:

$$s_{CP} - \bar{s} = \frac{I_{CC}}{A\bar{s}} = \frac{\text{2nd Moment of area about axis at the centroid}}{\text{1st Moment of area about axis at the surface}}. \tag{2.18}$$

In the limit where the plate is much smaller than the distance from the axis, this will tend to zero, where the relative variation of pressure over the plate is small.

Example 2.2 A circular cover 125 cm in diameter is immersed in water so that the deepest part is 150 cm below the surface, and the shallowest part 60 cm below the surface. Find the total force due to water acting on one side of the cover, and the distance of the centre of pressure from the centroid. You may assume that the second moment of area of a circle of diameter D about a diameter is $I_{CC} = \pi D^4/64$.



Circle:	Area $A = \frac{\pi}{4} \times 1.25^2 = 1.228\,\mathrm{m}^2$,
Depth to centroid $\bar{h}$:	By symmetry: $\bar{h} = 0.5(0.6 + 1.5) = 1.05\,\mathrm{m}$,
Force: equation (2.12):	$F = \rho g A\bar{h} = 1000 \times 9.8 \times 1.228 \times 1.05 = 12.6\,\mathrm{kN}$ (*Ans.*)
Simple trigonometry:	$\sin\theta = (150 - 60)/125 = 0.72$,
Simple trigonometry:	$\bar{s} = \bar{h}/\sin\theta = 1.05/0.72 = 1.458\ \mathrm{m}$,
Formula given:	$I_{\mathrm{CC}} = \pi D^4/64 = 0.120\,\mathrm{m}^4$,
Position of Centre of Pressure (equation 2.18):	$s_{\mathrm{CP}} - \bar{s} = I_{\mathrm{CC}}/A\bar{s} = 0.120/\left(1.228 \times 1.458\right) = 0.067\ \mathrm{m}$,
	Centre of pressure is $6.7\,\mathrm{cm}$ from centroid

2.3 Forces on submerged boundaries of general shape

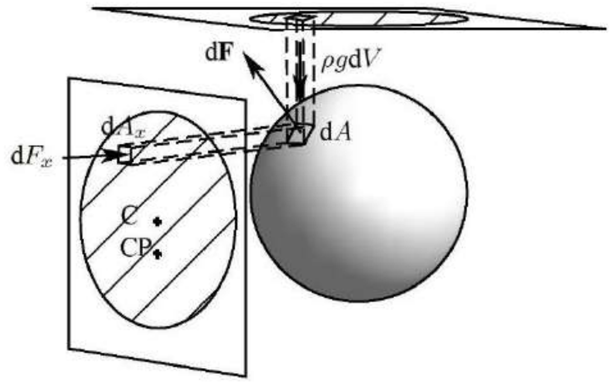


Figure 2-12. Projection of boundary element onto two planes: (a) the free surface, to calculate the vertical force, and (b) a vertical plane perpendicular to the desired horizontal force direction

Forces on submerged boundaries of general shape can be calculated surprisingly simply. Consider the underwater boundary in figure 2-12, with an element of area $\mathrm{d}A$. Consider also the two elemental prisms constructed from $\mathrm{d}A$, one to the free surface, which has a volume $\mathrm{d}V$, and one in a horizontal direction, denoted by x, in which direction we require the force, with elemental cross-sectional area $\mathrm{d}A_x$ when projected onto a plane perpendicular to that direction. The results we obtain for this prism will be valid for all horizontal directions, so it will not be necessary to consider separately here another direction perpendicular to this.

2.3.1 Horizontal

The horizontal element has a vertical force on it due to the weight force of the fluid in it, but for it we are only concerned with horizontal forces in the direction along the element, and so, as there are no other horizontal forces on the prism, this means that the x-component of the force of the fluid on the curved surface is equal to $\mathrm{d}F_x$, the force of fluid on the projection $\mathrm{d}A_x$ onto a vertical plane. This holds for all elements of the surface, and so we can write for the whole surface:

Any horizontal component of force on a submerged surface is equal to the force on a projection of that surface onto a vertical plane perpendicular to the component.

For such a calculation, we can use all the results of §2.2 for the forces on a planar surface, for the special case of a vertical plane $\theta = \pi/2$. Equations (2.12) and (2.18) give,

$$F_x = \rho g A_x \bar{h}_x \quad \text{and} \quad h_{\mathrm{CP}} - \bar{h}_x = \frac{I_{\mathrm{CC}}}{A_x \bar{h}_x}, \tag{2.19}$$

where we have used the notation $\bar{h}_x$ for the depth of the centroid of the projected area A_x.

Example 2.3 A tank attached to the side wall of an underwater structure is of the shape of half a sphere, a hemisphere. It has a diameter D and its centre is d below the surface. Calculate the horizontal force and its position.

By symmetry, the net force will be normal to the side wall, and there is no force to left or right in the plane of the wall. To calculate the force we have simply to use equation (2.19), where the projected area of the hemisphere is a circle, with centroid at its centre, d below the surface:

$$F_x = \rho g A_x \bar{h}_x = \rho g \frac{\pi D^2}{4} d.$$

To obtain the position of its resultant point of application, we use the fact that for a circle, $I_{CC} = \pi D^4/64$,

$$h_{CP} - d = \frac{I_{CC}}{Ad} = \frac{\pi D^4}{64} \frac{4}{\pi D^2 d} = \frac{D^2}{16d}.$$

Note that if we re-express this to calculate the distance between centre of pressure and centroid relative to the *radius* (half the diameter) we find

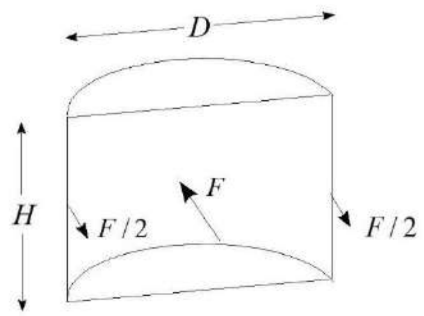
$$\frac{h_{CP} - d}{D/2} = \frac{D}{8d},$$

showing that for large submergence, $D/d \to 0$, and the centre of pressure will approach the centroid, which holds for all submerged surfaces. in the deep water limit

Example 2.4 A mixing tank is a vertical cylinder of height H and diameter D filled with fluid of density ρ.

(a) Calculate the total horizontal force F on one half of the tank? (this would be used to design the tank walls, which must carry forces of $F/2$ as shown).

(b) How far above the base of the tank does it act?



Answer: (a) The force on the semi-circular cylinder is equal to that on its projected area, which is the rectangle across its centre, with dimensions $H \times D$. The centroid of the area is at its geometric centre, such that $\bar{h} = H/2$.

$$\begin{aligned} F &= \rho g A_x \bar{h}_x \\ &= \rho g H D \times H/2 \\ &= \tfrac{1}{2} \rho g H^2 D. \end{aligned}$$

(b) The distance between centre of pressure and centroid:

$$h_{CP} - \bar{h}_x = \frac{I_{CC}}{A\bar{h}_x}.$$

For the rectangle, the second moment of area I_{CC} about a horizontal axis through the centroid is, $I_{CC} = DH^3/12$, hence

$$h_{CP} - \bar{h}_x = \frac{DH^3}{12} \frac{1}{HD \times H/2} = \frac{H}{6},$$

below the centroid, so the distance of the centre of pressure above the base is $H/2 - H/6 = H/3$.

2.3.2 Vertical

Considering the vertical element of fluid, there are only two vertical forces acting on it: one is the weight force of the fluid $\rho g\, dV$, and the other is the vertical component of the element on the fluid. For equilibrium of the element, the two must be equal. This holds for all elements of the boundary, and so we can write that the vertical force on the whole boundary is equal to $\rho g V$, the weight force on the fluid between the boundary and the plane of the free surface.

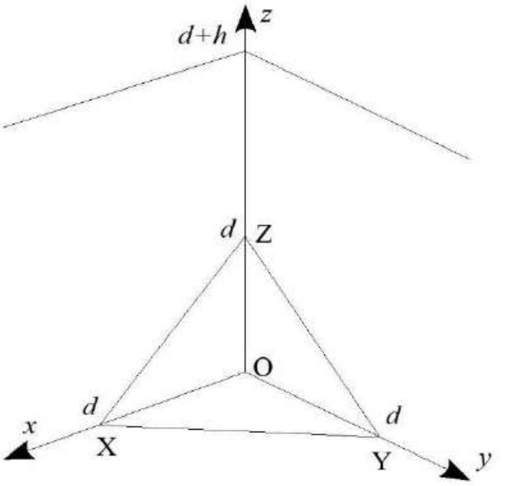
Now, however, there is an important generalisation, and that is if any or all of the region between the boundary and the surface is not occupied by fluid, for example if there were another body in the fluid, breaking up the prismatic volume. In fact, this has no effect on the results, for the pressure on the boundary is the same, whether or not part of the volume is unoccupied by fluid, and the force on the boundary is the same as if all the region between boundary and surface were occupied by fluid. Hence we can write the relatively simple and powerful result:

> *The force component in the direction of gravity on a submerged boundary is equal to the weight force on the fluid in the volume bounded by:*
> *1. the boundary itself, and*
> *2. its projection onto a plane at the level of the free surface.*
> *Contributions are (downwards/upwards) if the surface is (below/above) the local fluid.*

To locate the position of the resultant vertical force, the body force of the vertical prismatic element is at the centre of the prism, and integrating over all such prisms to form the body, the resultant acts through the centre of volume of the body, giving

> *The vertical force component acts through the centre of gravity of the volume between the submerged boundary and the water surface.*

Example 2.5 The corner of a tank is bevelled by equal dimensions d as shown in the figure. It is filled to h above the top of the bevel. What is the force on the triangular corner?



$$
\begin{aligned}
\text{Force in } x \text{ direction} &= \text{Force on projection OYZ} \\
&= \rho g A_{OYZ}\, \bar{h}_{OYZ} \\
&= \rho g \times \tfrac{1}{2} d^2 \times \left(h + \tfrac{2}{3} d\right) = \frac{\rho g d^2}{6}\,(3h + 2d) \\
&= \text{Force in } y \text{ direction by symmetry}
\end{aligned}
$$

$$
\begin{aligned}
\text{Force in } z \text{ direction} &= \text{Weight force on prism above XYZ} \\
&= \rho g \times \text{ Volume} \\
&= \rho g \times \frac{d^2}{2} \times \tfrac{1}{3}(h + d + h + d + h) \\
&= \frac{\rho g d^2}{6}\,(3h + 2d)\,.
\end{aligned}
$$

Thus, each of the force components is the same. This is what we would expect for such a surface whose direction cosines are the same for all 3 directions.

2.3.3 Horizontal force on a vertical wall with the water level at the top

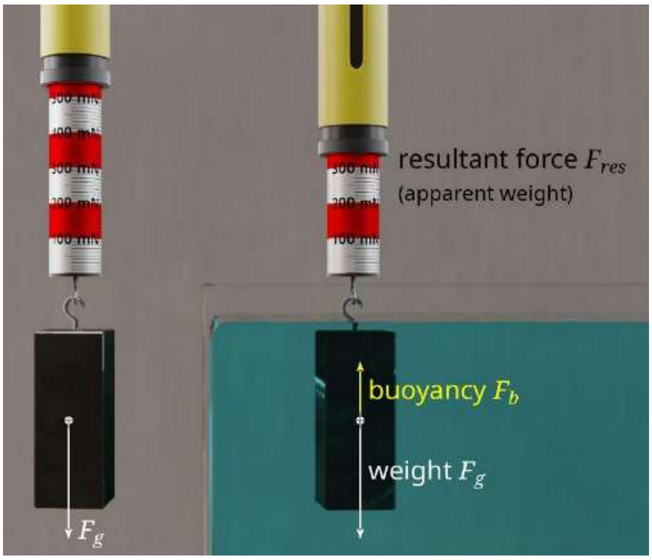
The force on a straight vertical wall with water level at the top is a very common problem, such as sea walls, tanks, and retaining walls. The answer has been provided in Example 2.4 above, originally for the force on a curved wall, which we saw was the same as for a rectangular wall. We replace the diameter of the tank by the length L of the wall and the results immediately follow:

$$F = \tfrac{1}{2}\rho g H^2 L, \quad \text{or,force per unit length } \frac{F}{L} = \tfrac{1}{2}\rho g H^2.$$

$$\text{Height of resultant force above the base} = \frac{H}{3} \text{ or, } \frac{2H}{3} \text{ from the surface.}$$

Buoyancy is the force directed against gravity that an object experiences when submerged in a fluid (liquid or gas).

Everyone may have tried to lift another person and found that this requires a lot of strength. However, if you try to lift this person in water, it is much easier. The reason for this is due to the so-called buoyancy, which an object experiences as soon as it is submerged in a liquid. This buoyant force is also responsible for the fact that even steel ships weighing tons do not sink but float on the water. The cause of the buoyancy will be discussed in more detail in this part.



$$F_b = \Delta m \cdot g$$

$$\boxed{F_b = \Delta V \cdot \rho_l \cdot g}$$

2.4 The buoyancy and stability of submerged and floating bodies

The methods used above to calculate the force on a surface can be used to calculate the buoyancy forces on totally or partially submerged bodies, that is, surfaces which are closed within the fluid or closed by a plane surface at the level of the free surface.

2.4.1 Totally-submerged body

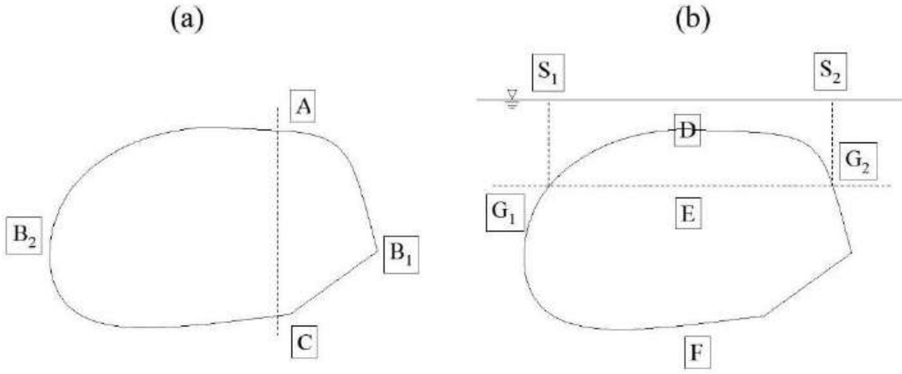


Figure 2-15. Submerged body showing arbitrary points for force calculations

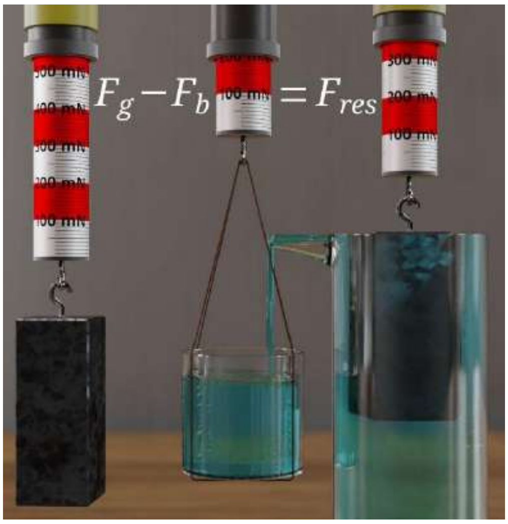
Consider the submerged body in Figure 2-15(a) intersected by an arbitrary vertical plane AC, which is the projected area of each of the two surfaces closed by the plane, AB_1C and AB_2C, so that the forces on each of the two surfaces are equal (and opposite), hence there is no resultant force on the body. Consider how human development might have progressed had it been possible for an irregularly-shaped object to propel itself!

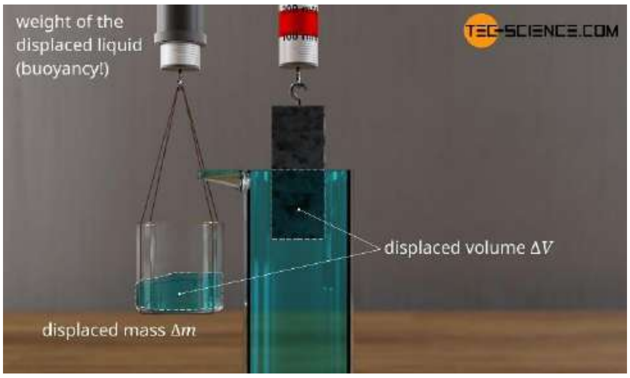
Now consider the body intersected by an arbitrary horizontal plane G_1G_2 as in Figure 2-15(b), where S_1 and S_2 are points in the free surface where the intersection of the plane and the body are projected to the surface. The vertical (downwards as sketched) force on G_1DG_2 is equal to the weight of fluid in $G_1DG_2S_2S_1G_1$ while the vertical force (almost all of it upwards here) on G_1FG_2 is equal to the weight of fluid which would occupy the volume $G_1FG_2S_2S_1G_1$. Hence, the *net buoyancy force*, the net upwards force, is the weight of fluid which would occupy *the difference* between the two volumes, namely the volume of the body.

That is, the buoyancy force is upwards and is equal to the gravity force on the volume of fluid displaced. This is *Archimedes' Principle*, and is easily extended to the case where the body is floating. In this case the body floats at such a level in the water that the weight of the whole body equals that of the weight of fluid displaced by that part of the body beneath the waterline.

The buoyancy: The Archimedes' principle

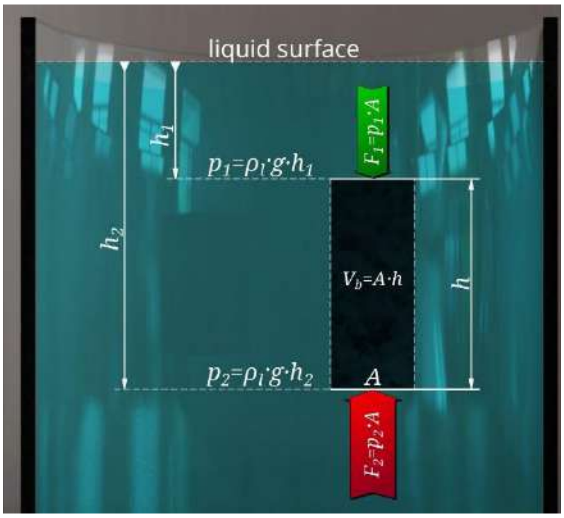
The scientist Archimedes experimented with the phenomenon buoyancy already 250 years B.C. He was able to show that the buoyant force by which a submerged body appears to become lighter corresponds to the weight of the displaced liquid.





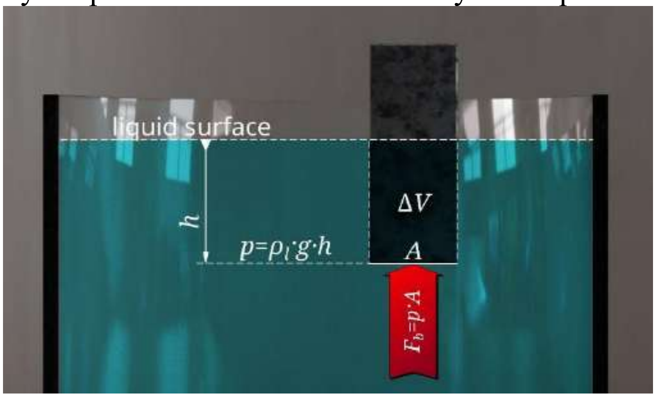
Derivation of the buoyant force

The buoyancy is due to the different hydrostatic pressures at the top and bottom of a submerged body.



$$F_b = F_2 - F_1$$
$$F_b = \rho_l \cdot g \cdot h_2 \cdot A - \rho_l \cdot g \cdot h_1 \cdot A$$
$$F_b = \rho_l \cdot g \cdot A \cdot (h_2 - h_1)$$
$$\boxed{F_b = \Delta V \cdot \rho_l \cdot g}$$

Derivation of the buoyancy for partial immersion of the body in a liquid



2.4.2 Centre of buoyancy

The *centre of buoyancy* is the position in space where the buoyant force may be considered to act for the purposes of taking moments. It is at the centre of mass of the fluid displaced by the body, whether submerged or floating.

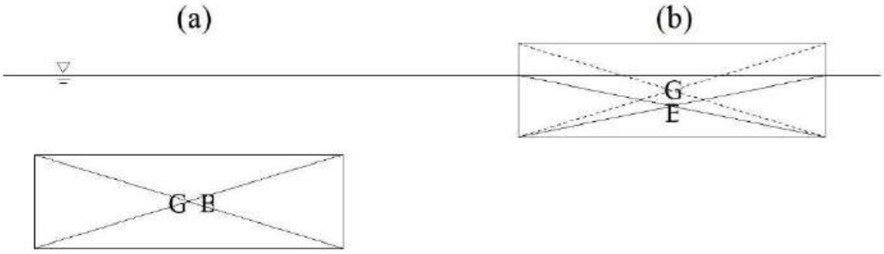


Figure 2-16. Illustration of the centre of buoyancy of a regular body which is (a) submerged, or (b) floating.

Consider two solid rectangular blocks of the same size, but different densities. The one in (a) is heavier than the liquid, it does not float, and if it is homogeneous the centres of buoyancy B and gravity G will coincide. In case (b) the block is lighter than the fluid, it floats, and the displaced volume is less than its total volume, and G is above B as shown.

2.4.3 Stability of submerged bodies

Consider the balloon in the illustration. Its centre of buoyancy is close to the centre of the envelope, while its centre of gravity is near the gondola. It has been given a small positive (anti-clockwise) displacement by a gust of wind. Consider the lines of action of the equal and opposite weight and buoyancy forces – it is clear that the moment set up by the displacement is negative and acts so as to reduce the displacement. In this case, and for any submerged body, as long as B is above G, the configuration is stable.

Now consider Figure 2-16(b) for the floating block, where G is *above* B. Do we expect that configuration to be unstable? If not, why not? The explanation and limits for stability will be given in the following section.

2.4.4 Stability of floating bodies

For most floating bodies, the centre of gravity is above the centre of buoyancy, and from the above we might expect

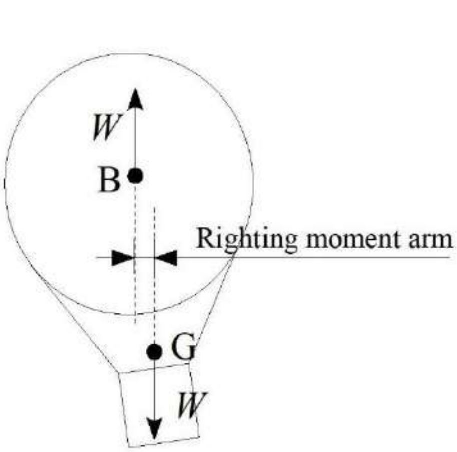


Figure 2-17. Forces acting on a submerged body

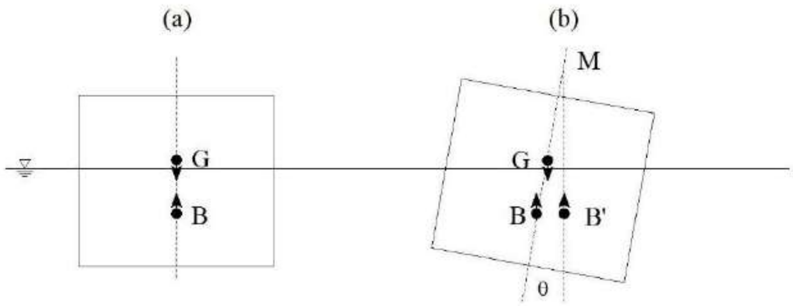


Figure 2-18. (a) Floating body with the common potentially-unstable situation where the centre of gravity is above the centre of buoyancy, and (b) showing how for an angle of rotation the displaced fluid is now a trapezoidal shape and the centre of buoyancy has moved sideways, enough in this case for there to be a restoring force on the body.

the system to be unstable. However, if the body rotates, the shape of the displaced volume changes, such that the centre of buoyancy moves laterally. Hence we have to calculate how much movement there is for a particular body or ship to determine whether it is stable or otherwise. Figure 2-18 shows a stable situation where the centre of buoyancy for an angle of roll θ has moved from B to B$'$, enough that a restoring force has been set up. The amount of movement depends on θ, so that a more fundamental quantity is the distance BM, where M is as shown in the figure, such that for a small angle of rotation θ the distance that the centre of buoyancy moves is

$$\mathrm{BB}' = \mathrm{BM}\,\theta. \tag{2.20}$$

The condition that the body be stable is that M be above G, such that the *Metacentric Height* GM is positive.

We can calculate the distance BB$'$ by taking moments of volume about the axis of rotation of the body at the waterline, as the centre of buoyancy is at the centre of the mass of fluid displaced. After a rotation θ, the change of first moment of volume is due to the change of first moment of volume just of the wedge-shaped regions shown in Figure 2-19, as the displaced volume below them is unchanged. Hence,

$$\text{Displaced volume} \times \mathrm{BB}' = \begin{array}{l}\text{Change of first moment of volume in the region}\\ \text{between the unperturbed and rotated states.}\end{array}$$

3. Fluid kinematics

3.1 Kinematic definitions

Fluids in motion are rather more complicated: the motion varies from place to place and from time to time. In this section we are concerned with common terminology and descriptions of the flow, and the specification of fluid motion.

Steady/unsteady flow: where the flow at each place (does not change / does change) with *time*.

Uniform/nonuniform flow: where the flow (does not vary / does vary) with *position*.

Laminar flow: where fluid particles move along smooth paths in laminas or layers. This occurs where velocities are small or viscosity is large or if the size of the flow is small, *e.g.* the flow of honey, the motion around a dust particle in air. In civil and environmental engineering flows flow is almost never laminar, but is *turbulent*.

Turbulent flow: where the fluid flow fluctuates in time, apparently randomly, about some mean condition, *e.g.* the flow of wind, water in pipes, water in a river. In practice we tend to work with mean flow properties, however in this course we will adopt empirical means of incorporating some of the effects of turbulence. Consider the x component of velocity at u a point written as a sum of steady ($\bar{u}$) and fluctuating (u') components:

$$u = \bar{u} + u'.$$

Let us compute the time mean value of u at a point by integrating over a long period of time T:

$$\bar{u} = \frac{1}{T}\int_0^T u\,dt = \frac{1}{T}\int_0^T (\bar{u} + u')\,dt = \bar{u} + \frac{1}{T}\int_0^T u'\,dt,$$

and we see that by definition, the mean of the fluctuations, which we write as $\overline{u'}$, is

$$\overline{u'} = \frac{1}{T}\int_0^T u'\,dt = 0. \tag{3.1}$$

Now let us compute the mean value of the square of the velocity, such as we might find in computing the mean pressure on an object in the flow:

$$\begin{aligned}
\overline{u^2} &= \overline{(\bar{u} + u')^2} = \overline{\bar{u}^2 + 2\bar{u}u' + u'^2}, \text{ expanding,} \\
&= \overline{\bar{u}^2} + \overline{2\bar{u}\,u'} + \overline{u'^2}, \text{ considering each term in turn,} \\
&= \bar{u}^2 + 2\bar{u}\,\overline{u'} + \overline{u'^2}, \text{ but, as } \overline{u'} = 0 \text{ from(3.1),} \\
&= \bar{u}^2 + \overline{u'^2}.
\end{aligned} \tag{3.2}$$

hence we see that the mean of the square of the fluctuating velocity is not equal to the square of the mean of the fluctuating velocity, but that there is also a component $\overline{u'^2}$, the mean of the fluctuating components.

Eulerian and Lagrangian descriptions: Lagrangian descriptions use the motion of fluid particles. Eulerian descriptions study the motion at points in space, each point being occupied by different fluid particles at different times.

The fluid properties we usually need to specify a flow at all points and times are:

1. Fluid velocity $\mathbf{u} = (u, v, w)$ a vector quantity, in terms of components in (x, y, z) co-ordinates,
2. Pressure p, a scalar, which in compressible flow will determine the density at a point, and
3. For flow with a free surface the height of that surface is also important.

Flow lines

Streamlines: a streamline is a line in space such that everywhere the local velocity vector is tangential to it, whether unsteady or not, whether in three dimensions or not.

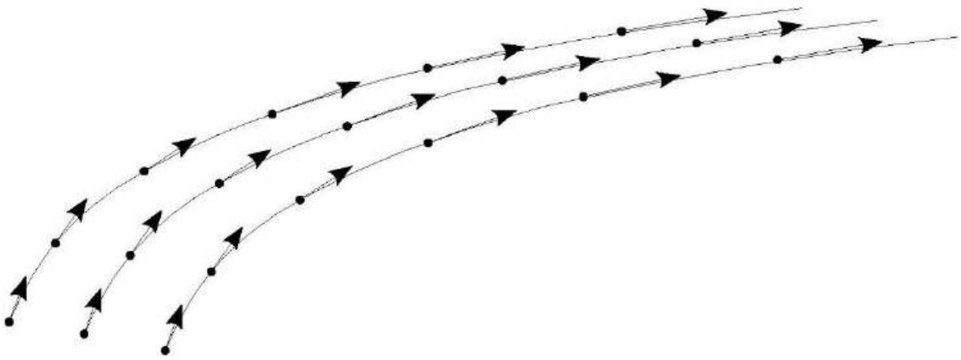


Figure 3-1. Typical streamlines showing how the velocity vectors are tangential

Figure 3-1 shows typical streamlines and velocity vectors. As the velocity vector is tangential at all points of a streamline, there is no flow across a streamline, and in steady 2-D flow, the mass rate of flow between any two streamlines is constant. Where streamlines converge, as shown in the figure, the velocity must increase as shown to maintain that flow. Hence, a plot of streamlines implicitly shows us the direction and magnitude of the velocities.

Streamtubes: In 3-D flows the equivalent is a streamtube, which is a closed surface made up of streamlines, as shown in Figure 3-2. There is no flow through a streamtube surface, hence the mass rate of flow through the streamtube is constant at all sections.

Pathlines: a pathline is the path followed by an individual particle, which is a more Lagrangian concept. For steady flow streamlines and pathlines coincide.

Streaklines: a streakline is a line joining the positions of all particles which have passed through a certain point. They also coincide with streamlines and pathlines for steady flow.

3.2 Flux of volume, mass, momentum and energy across a surface

It is necessary for us to be able to calculate the total quantity of fluid and integral quantities such as mass, momentum, and energy flowing across an arbitrary surface in space, which we will then apply to the rather more simple case of control surfaces. Consider an element of an arbitrary surface shown in Figure 3-3 through which fluid flows at velocity $\mathbf{u}$. The velocity component perpendicular to the surface is $|\mathbf{u}|\cos\theta = \mathbf{u}\cdot\hat{\mathbf{n}}$. In a time dt the volume of fluid which passes across the surface is $\mathbf{u}\cdot\hat{\mathbf{n}}\, dt\, dA$, or, the *rate* of volume transport is $\mathbf{u}\cdot\hat{\mathbf{n}}\, dA$. Other quantities easily follow from this: multiplying by density ρ gives the rate of *mass* transport, multiplying by velocity $\mathbf{u}$ gives the rate of transport of *momentum* due to fluid inertia (there is another contribution due to pressure

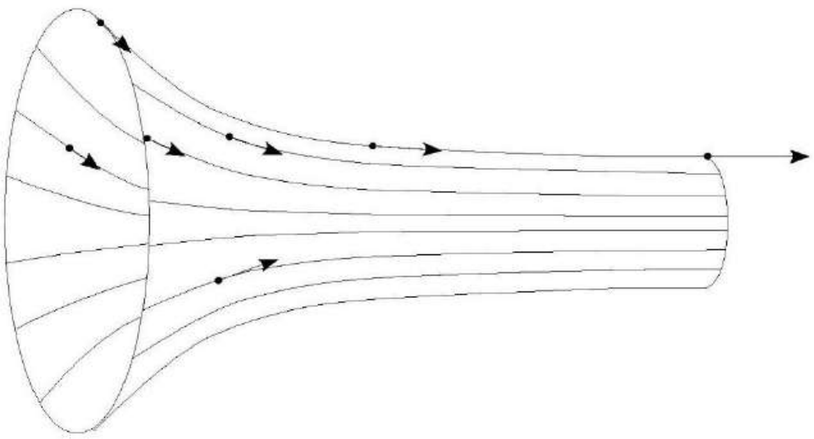


Figure 3-2. Streamtube with velocity vectors tangential to the component streamlines and greater where velocity is greatest.

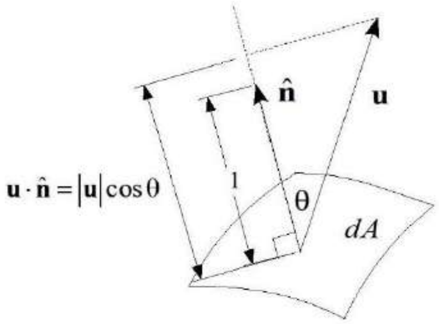


Figure 3-3. Element of surface dA with local velocity vector $\mathbf{u}$ showing how the velocity component normal to the surface is $\mathbf{u}\cdot\hat{\mathbf{n}}$.

for total momentum), and if e is the energy per unit mass, multiplying by e gives the rate of energy transport across the element. By integrating over the whole surface A, not necessarily closed, gives the transport of each of the quantities, so that we can write

$$\text{Rate of}\left\{\begin{array}{c}\text{volume}\\ \text{mass}\\ \text{inertial momentum}\\ \text{energy}\end{array}\right. \text{transport across surface } A = \int_A \left[\begin{array}{c}1\\ \rho\\ \rho\mathbf{u}\\ \rho e\end{array}\right] \mathbf{u}\cdot\hat{\mathbf{n}}\,dA. \tag{3.3}$$

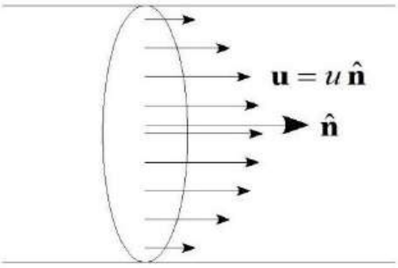


Figure 3-4. Pipe flow, showing a transverse section such that $\mathbf{u}$ and $\hat{\mathbf{n}}$ are parallel. Note that the magnitude of the velocity varies over the section.

Note that as $\mathbf{u}\cdot\hat{\mathbf{n}}$ is a scalar there is no problem in multiplying this simply by either a vector or a scalar. In hydraulic practice such integrals are usually evaluated more easily. For example, across a pipe or channel which is locally straight, to calculate the rates of transport we choose a surface perpendicular to the flow, as in Figure 3-4, as is described below.

Flux across solid boundaries: There can be no velocity component normal to a solid boundary, such that every solid boundary satisfies the boundary condition

$$\mathbf{u} \cdot \hat{\mathbf{n}} = 0 \tag{3.4}$$

and so from equation (3.3) there can be no volume, mass, momentum, or energy transfer across solid boundaries.

3.3 Control volume, control surface

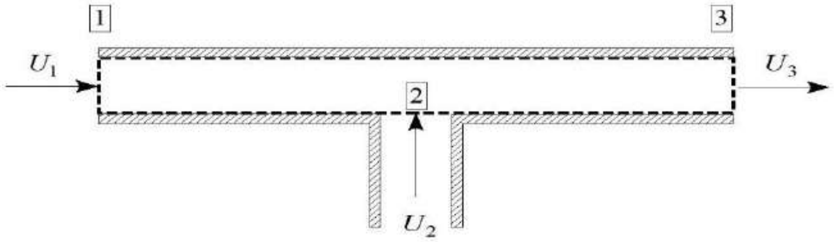


Figure 3-5. Typical control volume for a problem where one pipe joins another.

A *control volume* refers to a fixed region in space through which fluid flows, forming an open system. The boundary of this system is its *control surface*. A typical control surface is shown in Figure 3-5. The control volume for a particular problem is chosen for reasons of convenience. The control surface will usually follow solid boundaries where these are present and will not cut through solid boundaries, as sometimes we have to calculate all the forces on a control surface, which is difficult to do with solid boundaries. Where the control surface cuts the flow direction it will usually be chosen so as to do so at right angles. Problems where control surfaces are most useful are those where the fluid enters or leaves via pipes or channels whose cross-sectional area is relatively small, which fortunately, is usually the case. The reasons for this will be made obvious below.

4. Conservation of mass – the continuity equation

Now we consider successively conservation of mass, momentum, and energy, for *steady* flows, although the effects of turbulence will be incorporated. Each conservation principle helps us to solve a variety of problems in hydromechanics.

For *steady* flow, the mass conservation equation for a fluid within a control surface (CS) can be written

$$\int_{\mathrm{CS}} \rho\, \mathbf{u} \cdot \hat{\mathbf{n}}\, dA = 0, \tag{4.1}$$

such that the integral over the control surface of the mass rate of flow is zero, so that matter does not accumulate within the control volume. This is the mass conservation equation. Equation (3.4) states that $\mathbf{u} \cdot \hat{\mathbf{n}} = 0$ on solid boundaries, and so for practical problems we have

$$\int_{\text{Flow boundaries}} \rho\, \mathbf{u} \cdot \hat{\mathbf{n}}\, dA = 0, \tag{4.2}$$

where we only have to consider those parts of the control surface through which fluid flows. In most hydraulics the density of water varies very little and so ρ can be assumed to be a constant, so that it can be taken outside the integral sign, and as it is a common factor, it can be neglected altogether.

If the flow through each flow boundary cuts the boundary at right angles, we can write the velocity as $\mathbf{u} = \pm u\, \hat{\mathbf{n}}$, such that $\mathbf{u} \cdot \hat{\mathbf{n}} = \pm u$, where the plus/minus sign is taken when the flow leaves/enters the control volume. Then across any section of area A we have the contribution $\int_A \mathbf{u} \cdot \hat{\mathbf{n}}\, dA = \pm \int_A u\, dA$, which is $\pm Q$, the *volume flow rate* or *discharge* across the section. Sometimes it is convenient to express this in terms of U, the mean velocity, such that

$$\text{Rate of volume transport across surface } = \int_A u\, dA = Q = UA.$$

The mass conservation equation now becomes, for a control surface through which an incompressible fluid crosses each boundary at right angles,

$$\pm Q_1 \pm Q_2 \pm \ldots = \pm U_1 A_1 \pm U_2 A_2 \pm \ldots = 0, \tag{4.3}$$

which is the *continuity equation*, where the sign in each case is chosen according as to whether the velocity or discharge is assumed to be leaving or entering the control volume. For the velocities as shown in Figure 3-5, this would become

$$Q_3 - Q_1 - Q_2 = U_3 A_3 - U_1 A_1 - U_2 A_2 = 0.$$

Example 4.1 Fluid flows down a circular pipe of diameter D_1 at speed U_1. It passes through a contraction to a smaller diameter D_2, as shown in Figure 4-1. What is the mean velocity in the second pipe?

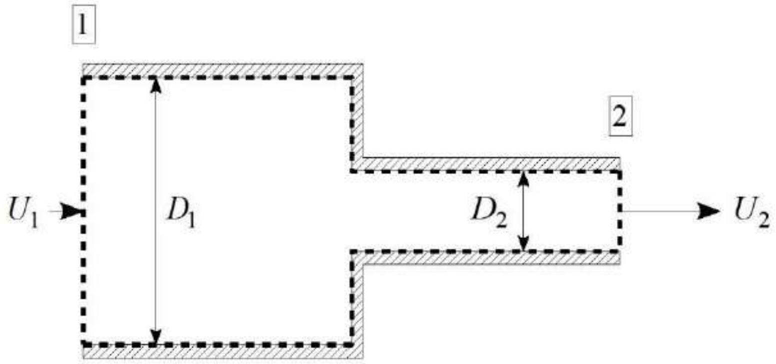


Figure 4-1. Control volume for example

Note that the control surface follows the solid boundaries or crosses the flow direction at right-angles. We have the convention that flows out of the control volume are positive and those in are negative, hence

$$\begin{aligned} -U_1 A_1 + U_2 A_2 &= 0, \\ -U_1 \frac{\pi}{4} D_1^2 + U_2 \frac{\pi}{4} D_2^2 &= 0, \\ \text{Hence } U_2 &= U_1 \left(\frac{D_1}{D_2}\right)^2. \end{aligned}$$

5. Conservation of momentum and forces on bodies

Formulation: Newton's second law states that the net rate of change of momentum is equal to the force applied. In equation (3.3) we obtained

$$\text{Momentum transport across a surface} = \int_A \rho \mathbf{u}\, \mathbf{u} \cdot \hat{\mathbf{n}}\, dA, \tag{5.1}$$

hence for a closed control surface CS we have to add all such contributions, so that

$$\text{Net rate of change of momentum transport across control surface} = \int_{\text{CS}} \rho \mathbf{u}\, \mathbf{u} \cdot \hat{\mathbf{n}}\, dA.$$

There are two main contributions to the force applied. One is due to surface forces, the pressure p over the surface. On an element of the control surface with area dA and outwardly-directed normal $\hat{\mathbf{n}}$ the pressure force on the fluid in the control surface has a magnitude of $p\,dA$ (simply pressure multiplied by area) and a direction $-\hat{\mathbf{n}}$, because the pressure acts normal to the surface and the direction of the force on the fluid is directed inwards to the control volume.

The other contribution is the sum of all the body forces, which will be usually due to gravity. We let these be denoted by $\mathbf{F}_{\text{body}}$. In many problems the body force is relatively unimportant. Equating the rate of change of momentum to the applied forces and taking the pressure force over to the other side we obtain the *integral* *momentum theorem* for steady flow

$$\int_{\text{CS}} \rho \mathbf{u}\, \mathbf{u}\cdot\hat{\mathbf{n}}\, dA + \int_{\text{CS}} p\, \hat{\mathbf{n}}\, dA = \mathbf{F}_{\text{body}}. \tag{5.2}$$

This form enables us to solve a number of problems yielding the force of fluid on objects and structures .Now the integrals in equation (5.2) will be separated into those over surfaces through which fluid flows and solid surfaces:

$$\sum_{\substack{\text{Fluid}\\\text{surfaces}}} \left(\int_A (\rho \mathbf{u}\, \mathbf{u}\cdot\hat{\mathbf{n}} + p\, \hat{\mathbf{n}})\, dA \right) + \sum_{\substack{\text{Solid}\\\text{surfaces}}} \left(\int_A \rho \mathbf{u}\, \mathbf{u}\cdot\hat{\mathbf{n}}\, dA + \int_A p\, \hat{\mathbf{n}}\, dA \right) = \mathbf{F}_{\text{body}}.$$

However, as $\mathbf{u}\cdot\hat{\mathbf{n}} = 0$ on all solid surfaces, there are no contributions. Also on the solid surfaces, unless we know all details of the flow field, we do not know the pressure p. However the sum of all those contributions is the total force $\mathbf{P}$ of the fluid on the surrounding structure. Hence we have the theorem in a more practical form for calculating the force on objects:

$$\text{Total force on solid surfaces} = \mathbf{P} = -\sum_{\substack{\text{Fluid}\\\text{surfaces}}} \left(\underbrace{\int_A (\rho \mathbf{u}\, \mathbf{u}\cdot\hat{\mathbf{n}} + p\, \hat{\mathbf{n}})\, dA}_{\text{Momentum flux}} \right) + \mathbf{F}_{\text{body}}. \tag{5.3}$$

Note the use of the term *momentum flux* for the integral shown – it includes contributions from the inertial momentum flux and from pressure.

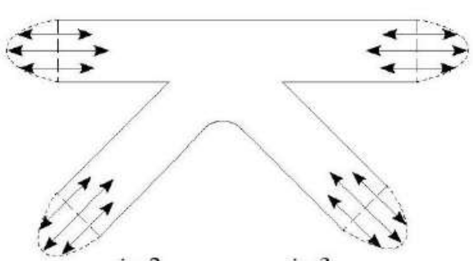


Figure 5-1. Control surface with fluid which is entering or leaving everywhere crossing the control surface (long dashes) at right angles, but where velocity may vary across the element, as shown with short dashes.

Inertial momentum flux: Here we evaluate the integral describing the transport of momentum by fluid velocity. In many situations *we can choose the control surface such that on each part where the fluid crosses it, the local surface element is planar, and the velocity crosses it at right angles*, as shown in Figure 5-1. We write for one control surface element, denoted by j, $\mathbf{u}_j = \pm u_j \hat{\mathbf{n}}_j$,where u_j is the fluid speed, whose magnitude might vary over that part of the control surface through which it passes, but whose direction is perpendicular either in the direction of the unit normal or opposite to it. Hence, for a particular element j,

$$\int_{A_j} \rho \mathbf{u}_j\, \mathbf{u}_j\cdot\hat{\mathbf{n}}_j\, dA = \int_{A_j} \rho\, (\pm u_j \hat{\mathbf{n}}_j)\, (\pm u_j)\, dA = +\hat{\mathbf{n}}_j\, \rho \int_{A_j} u_j^2\, dA, \tag{5.4}$$

where, as $(\pm) \times (\pm)$ is always positive, the surprising result has been obtained that the contribution to momentum flux is always in the direction of the outwardly-directed normal, whether the fluid is entering or leaving the control volume. Also we have assumed that the area A_j is planar such that $\hat{\mathbf{n}}_j$ is constant, so that we have been able to take the $\hat{\mathbf{n}}_j$ outside the integral sign.

Approximation of the integral allowing for turbulence and boundary layers: Although it has not been written explicitly, it is understood that equation (5.4) is evaluated in a time mean sense. In equation (3.2) we saw that if a flow is turbulent, then $\overline{u^2} = \bar{u}^2 + \overline{u'^2}$, such that the time mean of the square of the velocity is greater than the square of the mean velocity. In this way, we should include the effects of turbulence in the inertial momentum flux by writing the integral on the right of equation (5.4) as

$$\int_{A_j} u_j^2 \, dA = \int_{A_j} \left(\bar{u}_j^2 + \overline{u_j'^2} \right) \, dA. \tag{5.5}$$

Usually we do not know the nature of the turbulence structure, or even the actual velocity distribution across the flow, so that we approximate this in a simple sense by recognising that the time mean velocity at any point and the magnitude of the turbulent fluctuations are all of the scale of the mean velocity in the flow in a time and spatial mean sense, $\bar{U}_j = Q_j / A_j$, such that we write for the integral in space of the time mean of the squared velocities:

$$\int_{A_j} u_j^2 \, dA = \int_{A_j} \left(\bar{u}_j^2 + \overline{u_j'^2} \right) \, dA \approx \beta_j \bar{U}_j^2 A_j = \beta_j \left(\frac{Q_j}{A_j} \right)^2 A_j = \beta_j \frac{Q_j^2}{A_j}. \tag{e23}$$

The coefficient β_j is called a Boussinesq[7] coefficient, who introduced it to allow for the spatial variation of velocity. Allowing for the effects of time variation, turbulence, has been noted recently (Fenton 2005).

The coefficients β_j have typical values of 1.2. Almost all, if not all, textbooks introduce this quantity for open channel flow (without turbulence) but then assume it is equal to 1. Surprisingly, for pipe flow it seems not to have been used at all. In this course we consider it important and will include it.

We are left with the result that over the surface A_j the contribution to momentum flux due to fluid velocity is $\rho \beta_j \bar{U}_j^2 A_j \hat{\mathbf{n}}_j$, where β_j is the Boussinesq coefficient.

Momentum flux due to pressure: The contribution due to pressure at each element in equation (5.3) is $\int_{A_j} p \, \hat{\mathbf{n}} \, dA$. In most problems where simple momentum considerations are used, the pressure variation across a section is not complicated. To evaluate the term a hydrostatic pressure distribution could be assumed so that we can replace the integral by $\bar{p}_j \, A_j \, \hat{\mathbf{n}}_j$,where $\bar{p}_j$ is the mean pressure at that section, obtained by assuming a hydrostatic pressure distribution, or as the relative size of the section is usually not large, effects of gravity are ignored altogether and we just assume the pressure is constant.

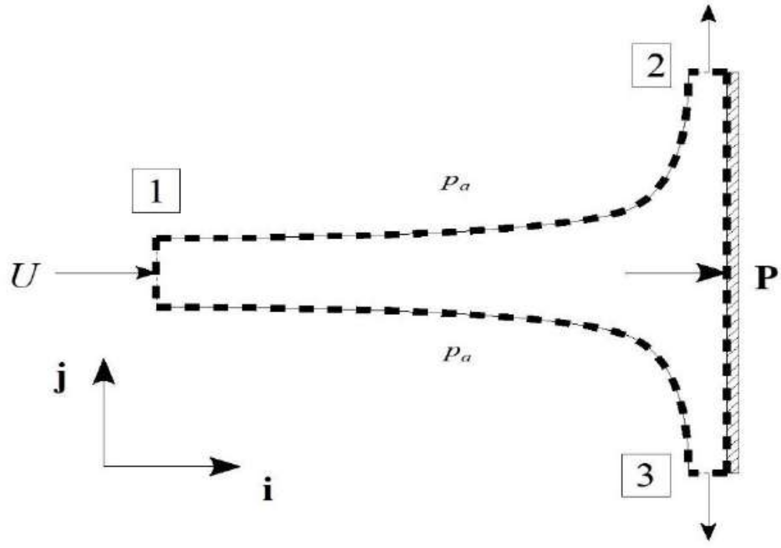
Combining: Collecting terms due to velocity and pressure and summing over all such surfaces we have a simple approximation to equation (5.3) above:

$$\mathbf{P} = \sum_j \left(\rho \beta_j \bar{U}_j^2 + \bar{p}_j \right) A_j \left(-\hat{\mathbf{n}}_j \right) + \mathbf{F}_{\text{body}}, \text{ in terms of velocity,} \tag{5.6a}$$

$$\mathbf{P} = \sum_j \left(\rho \beta_j \frac{Q_j^2}{A_j} + \bar{p}_j A_j \right) \left(-\hat{\mathbf{n}}_j \right) + \mathbf{F}_{\text{body}}, \text{ in terms of discharge.} \tag{5.6b}$$

It is a simple and surprising result that all contributions to $\mathbf{P}$ from the fluid are in the opposite direction to the unit normal vector. That is, all contributions are in the direction given by $-\hat{\mathbf{n}}_j$, whether the flow is entering or leaving the control volume, and that is also the direction of pressure contributions. Students should remember this when reading other textbooks, where this simplifying result has not been used.

Example 5.1 Consider a jet of water of area A_1 and mean velocity U_1 directed horizontally at a vertical plate. It is assumed that the velocities are so great that the effects of gravity are small and can be neglected. Draw a control surface and calculate the force on the plate, assuming that the fluid after impact travels parallel to the plate.



Jet of water striking a plate and being diverted along the plate

After impact water travels parallel to the plate. Consider the figure. On section 1, $\hat{\mathbf{n}} = -\mathbf{i}$, on surface 2, $\hat{\mathbf{n}} = \mathbf{j}$, surface 3, $\hat{\mathbf{n}} = -\mathbf{j}$, and so those sections play no role in horizontal momentum and force. The force on the plate is $\mathbf{P} = P\mathbf{i}$. There are no contributions to the horizontal ($\mathbf{i}$) momentum flux on the surfaces 2 and 3, and as the whole system can be assumed to be at constant pressure p_a (except on the plate itself), substituting into the momentum equation (5.6a) gives

$$P\mathbf{i} = \rho\beta_1 U_1^2 A_1 \left(-(-\mathbf{i})\right) + 0\mathbf{i}, \text{ giving } P = +\rho\beta_1 U_1^2 A_1.$$

Example 5.2 Repeat, but where after impact, water is diverted back in the other directionNow consider Figure

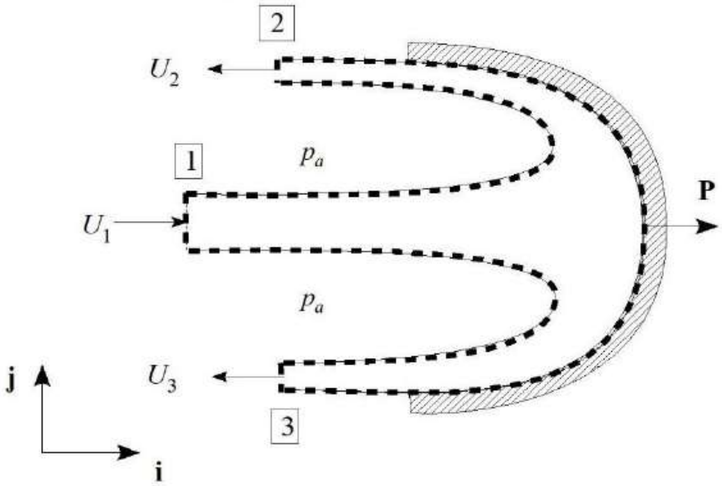


Figure 5-2. Case of jet being diverted back from whence it came

5-2. The crucial point is that while at 1 $\hat{\mathbf{n}}_1 = -\mathbf{i}$ as before, but at 2 and 3, now we have also $\hat{\mathbf{n}}_2 = \hat{\mathbf{n}}_3 = -\mathbf{i}$. Substituting into the momentum equation (5.6a) gives

$$P\mathbf{i} = \rho\beta_1 U_1^2 A_1 \left(-(-\mathbf{i})\right) + \rho\beta_2 U_2^2 A_2 \left(-(-\mathbf{i})\right) + \rho\beta_3 U_3^2 A_3 \left(-(-\mathbf{i})\right) + 0\mathbf{i}.$$

There is no energy loss in this example, and so by conservation of energy the fluid at 2 and 3 will also have a speed of U_1, $U_2 = U_3 = U_1$. By mass conservation

$$-U_1 A_1 + U_2 A_2 + U_3 A_3 = 0, \qquad \text{and} \qquad A_2 + A_3 = A_1.$$

As the problem is symmetrical we can assume $A_2 = A_3 = A_1/2$. We assume $\beta_2 = \beta_3 = \beta_1$, and so we obtain

$$P\mathbf{i} = \rho\beta_1 U_1^2 A_1\,\mathbf{i} + \tfrac{1}{2}\rho\beta_1 U_1^2 A_1\,\mathbf{i} + \tfrac{1}{2}\rho\beta_1 U_1^2 A_1\,\mathbf{i},$$

giving

$$P = 2\rho\beta_1 U_1^2 A_1,$$

twice the force in the first example (a), because the change of momentum has been twice that case. Not only was the jet stilled in the **i** direction, but its momentum was completely reversed.

Now a rather more difficult example is presented.

Example 5.3 Calculation of force on a pipe bend

This is an important example, for it shows us the principles in a general sense. Fluid of density ρ and speed U flows along a pipe of constant area A which has a bend such that the flow is deviated by an angle θ. Calculate the force of the fluid on the bend. It can be assumed that there is little pressure loss in the bend, such that p is constant throughout, and body forces can be ignored. The arrangement is shown on Figure 5-3.It is helpful to use

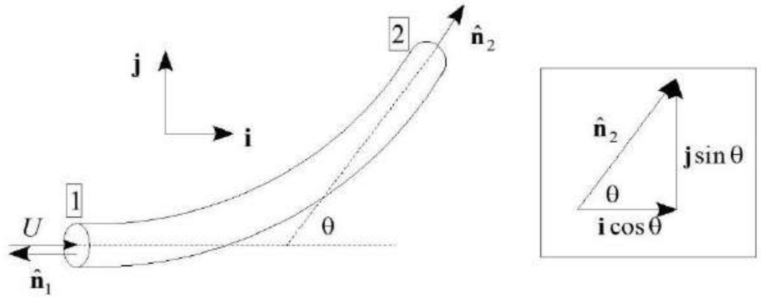


Figure 5-3. Pipe bend

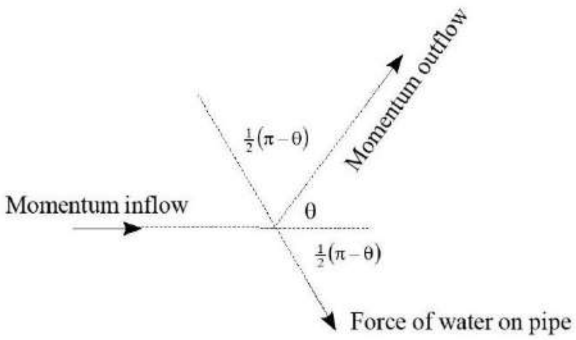


Figure 5-4. Direction of momentum fluxes and force for pipe bend.

unit vectors **i** and **j** as shown. At 1, $\hat{\mathbf{n}}_1 = -\mathbf{i}$, and $\mathbf{u}_1 = U\,\mathbf{i}$, and the contribution to momentum flux in equation (5.6a) is

$$\left(\rho\beta U^2 + p\right) A(-\mathbf{i}).$$

From the construction in the subsidiary diagram, $\hat{\mathbf{n}}_2 = \mathbf{i}\cos\boldsymbol{\theta} + \mathbf{j}\sin\theta$. Hence, at 2 $\mathbf{u}_2 = U\,\hat{\mathbf{n}}_2$, and the momentum flux is

$$\left(\rho\beta U^2 + p\right) A\,\hat{\mathbf{n}}_2 = \left(\rho\beta U^2 + p\right) A\left(\mathbf{i}\cos\theta + \mathbf{j}\sin\theta\right).$$

Using equation (5.3) we have

$$\mathbf{P} = -\left(-\mathbf{i}A\left(\rho\beta U^2 + p\right) + A\left(\rho\beta U^2 + p\right)\left(\mathbf{i}\cos\theta + \mathbf{j}\sin\theta\right)\right) + \underbrace{\mathbf{0}}_{\text{Body force}},$$

from which we obtain

$$\mathbf{P} = A\left(p + \rho\beta U^2\right)\left(\mathbf{i}\left(1 - \cos\theta\right) - \mathbf{j}\sin\theta\right).$$

This answer is adequate, however further insight can be obtained from the trigonometric formulae $1 - \cos\boldsymbol{\theta} = 2\sin^2\left(\theta/2\right)$ and $\sin\theta = 2\sin\left(\theta/2\right)\cos\left(\theta/2\right)$, giving the solution

$$\mathbf{P} = 2A\left(p + \rho\beta U^2\right)\sin\tfrac{\theta}{2}\left(\mathbf{i}\sin\tfrac{\theta}{2} - \mathbf{j}\cos\tfrac{\theta}{2}\right)$$

The magnitude of the force is

$$P = |\mathbf{P}| = 2A\left(p + \rho\beta U^2\right)\sin\tfrac{\theta}{2},$$

and the direction can be obtained a little more simply by considering the unit vector in the direction of the force: $\mathbf{i}\sin(\theta/2) - \mathbf{j}\cos(\theta/2)$. This can be written as

$$\mathbf{i}\cos\left(\tfrac{\theta-\pi}{2}\right) + \mathbf{j}\sin\left(\tfrac{\theta-\pi}{2}\right),$$

in the classical form $\mathbf{i}\cos\phi + \mathbf{j}\sin\phi$, showing that the phase angle of the force is $\theta/2 - \pi/2$. Consideration of the angles involved in Figure 5-4 shows that this seems correct, that the direction of the force of the water is outwards, and symmetric with respect to the inflow and outflow. Other noteworthy features of the solution are that forces on the pipe bend exist even if there is no flow, because the direction of the force due to pressure at inlet is not the same as at outlet. For design purposes the relative contributions of pressure and "inertia" may be quite different. Also in an unsteady state, when there might be pressure surges on the line (waterhammer) there might be large forces. Considering the solution for different values of θ (a) for $\theta = 0$, no deviation, there is no force, (b) the maximum force is when $\theta = \pi$,when the water is turned around completely, and (c) if the water does a complete loop, $\theta = 2\pi$, the net force is zero.

6. Conservation of energy

6.1 The energy equation in integral form

The energy equation in integral form can be written for a control volume CV bounded by a control surface CS, where there is no heat added or work done on the fluid inside the control volume:

$$\underbrace{\frac{\partial}{\partial t}\int_{\mathrm{CV}} \rho e \, dV}_{\text{Rate of change of energy inside CV}} + \underbrace{\int_{\mathrm{CS}} \rho e \, \mathbf{u}.\hat{\mathbf{n}} \, dA}_{\text{Flux of energy across CS}} + \underbrace{\int_{\mathrm{CS}} p \, \mathbf{u}.\hat{\mathbf{n}} \, dA}_{\text{Rate of work done by pressure}} = 0, \tag{6.1}$$

where t is time, ρ is density, dV is an element of volume, ρe is the internal energy per unit volume of fluid, ignoring nuclear, electrical, magnetic, surface tension, and intrinsic energy due to molecular spacing, leaving the sum of the potential and kinetic energies such that the internal energy *per unit mass* is

$$e = gz + \tfrac{1}{2}\left(u^2 + v^2 + w^2\right), \tag{6.2}$$

where the velocity vector $\mathbf{u} = (u, v, w)$ in a cartesian coordinate system, the co-ordinate z is vertically upwards, p is pressure, $\hat{\mathbf{n}}$ is a unit vector with direction normal to and directed outwards from the control surface such that $\mathbf{u}.\hat{\mathbf{n}}$ is the component of velocity normal to the surface at any point, and dS is the elemental area of the control surface.

Here *steady* flow is considered, at least in a time-mean sense, so that the first term in equation (6.1) is zero. The equation becomes, after dividing by density ρ:

$$\int_{\mathrm{CS}} \left(\frac{p}{\rho} + gz + \frac{1}{2}\left(u^2 + v^2 + w^2\right)\right) \mathbf{u}.\hat{\mathbf{n}} \, dA = 0. \tag{6.3}$$

It is intended to consider problems such as flows in pipes and open channels where there are negligible *distributed* energy contributions from lateral flows. It is only necessary to consider flow entering or leaving across finite parts of the control surface, such as a section across a pipe or channel or in the side of the pipe or channel. To do this there is the problem of integrating the contribution over a finite plane denoted by A_j which is also used as the symbol for the cross-sectional area. As energy (a scalar) is being considered and not momentum (a vector), it is *not* necessary to take particular orientations of parts of the control surface to be vertical, especially in the case of pipes. To this end, when the integral is evaluated over a finite plane u will be taken to be the velocity along the pipe or channel, and v and w to be perpendicular to that.

The contribution over a section of area A_j is then $\pm\dot{E}_j$, depending on whether the flow is leaving/entering the control volume, where

$$\dot{E}_j = \int\limits_{A_j} \left(\frac{p}{\rho} + gz + \frac{1}{2}\left(u^2+v^2+w^2\right) \right) u\, dA,$$

Now we consider the individual contributions to this integral.

The pressure distribution in a pipe or open channel (river, canal, drain, *etc.*) is usually very close to hydrostatic (streamlines are very close to all being parallel), so that $p/\rho + gz$ is constant over a section through which flow passes, and we try to ensure that at all points at which flow crosses the control surface that this is true. Hence we can take the first two terms of the integral outside the integral sign and use the result that $\int_A \mathbf{u}\cdot\hat{\mathbf{n}}\, dA = Q$ to give

$$\dot{E}_j = (p+\rho g z)\, Q + \frac{\rho}{2}\int\limits_{A_j} \left(u^2+v^2+w^2\right) u\, dA, \tag{6.4}$$

where p and z are the pressure and elevation at any point on a particular section.

Now, in the same spirit as for momentum, when we introduced a coefficient β to allow for a non-constant velocity distribution, we introduce a coefficient α such that it allows for the variation of the kinetic energy term across the section

$$\int\limits_{A} \left(u^2+v^2+w^2\right) u\, dA = \alpha U^3 A = \alpha \frac{Q^3}{A^2}. \tag{6.5}$$

Textbooks, strangely, take just the first component under the integral sign and write

$$\int\limits_{A} u^3 dA = \alpha U^3 A,$$

where α is the *Coriolis*[8] coefficient, for which a typical value is $\alpha \approx 1.25$. We think that the more general coefficient defined in equation (6.5) should be used, and it will be in this course. With equation (6.4) and this definition of α:

$$\text{Rate of energy transport across } j\text{th part of the control surface} = \dot{E}_j = (p_j + \rho g z_j)\, Q_j + \alpha_j \frac{\rho}{2}\frac{Q_j^3}{A_j^2}.$$

This can be written in a factorised form

$$\dot{E}_j = \underbrace{\rho Q_j}_{\text{Mass rate of flow}} \times \left(\underbrace{\frac{p_j}{\rho} + g z_j + \frac{\alpha_j}{2}\frac{Q_j^2}{A_j^2}}_{\text{Energy per unit mass}} \right).$$

The energy per unit mass has units of $\mathrm{J\,kg^{-1}}$. It is common in civil and environmental engineering problems to factor out gravitational acceleration and to write

$$\dot{E} = \rho g Q_j \times \left(\underbrace{\frac{p_j}{\rho g} + z_j + \frac{\alpha_j}{2g}\frac{Q_j^2}{A_j^2}}_{\text{Mean total head of the flow}} \right), \tag{6.6}$$

where the quantity in the brackets has units of length, corresponding to elevation, and is termed the *Mean total head of the flow* H, the mean energy per unit mass divided by g. The three components are:

$$\begin{aligned} \frac{p}{\rho g} &= \text{pressure head, the height of fluid corresponding to the pressure } p, \\ z &= \text{elevation, and} \\ \frac{\alpha}{2g}\frac{Q^2}{A^2} &= \text{kinetic energy head, the height which a flow of mean speed } Q/A \text{ could reach.} \end{aligned}$$

This form is more convenient, because often in hydraulics elevations are more important and useful than actual energies. For example, the height of a reservoir surface, or the height of a levee bank on a river might be known and govern design calculations.

Now, evaluating the integral energy equation (6.3) using these approximations over each of the parts of the control surface through which fluid flows, numbered 1, 2, 3, ... gives an energy equation similar to equation (4.3) for the mass conservation equation

$$\pm\dot{E}_1 \pm \dot{E}_2 \pm \dot{E}_3 \pm \ldots = \pm\rho_1 g Q_1 H_1 \pm \rho_2 g Q_2 H_2 \pm \rho_3 g Q_3 H_3 \pm \ldots = 0, \tag{6.7}$$

where the positive/negative sign is taken for fluid leaving/entering the control volume. In almost all hydraulics problems the density can be assumed to be constant, and so dividing through by the common density and gravity, we have the

Energy conservation equation for a control surface

$$\pm Q_1 H_1 \pm Q_2 H_2 \pm Q_3 H_3 \pm \ldots = 0. \tag{6.8}$$

We will be considering this more later when we consider systems of pipes. For the moment, however, we will consider a length of pipe or channel in which water enters at only one point and leaves at another.

6.2 Application to simple systems

For a length of pipe or channel where there are no other entry or exit points for fluid, using equation (6.8) and the definition of head in equation (6.6):

$$-Q_{\text{in}}\left(\frac{p}{\rho g} + z + \frac{\alpha}{2g}\frac{Q^2}{A^2}\right)_{\text{in}} + Q_{\text{out}}\left(\frac{p}{\rho g} + z + \frac{\alpha}{2g}\frac{Q^2}{A^2}\right)_{\text{out}} = 0, \tag{6.9}$$

and the continuity equation (4.3) gives $-Q_{\text{in}} + Q_{\text{out}} = 0$, so $Q_{\text{in}} = Q_{\text{out}} = Q$, so that the discharge is a common factor in equation (6.9), and we can divide through by Q:

$$\left(\frac{p}{\rho g} + z + \frac{\alpha}{2g}\frac{Q^2}{A^2}\right)_{\text{in}} = \left(\frac{p}{\rho g} + z + \frac{\alpha}{2g}\frac{Q^2}{A^2}\right)_{\text{out}}. \tag{6.10}$$

In this form the equation is very useful and we can solve a number of useful hydraulic problems. However, to give more accurate practical results, an empirical allowance can be made for friction, and in many applications the equation is used in the form

$$H_{\text{in}} = H_{\text{out}} + \Delta H,$$

where ΔH is a head loss. In many situations it is given as an empirical coefficient times the kinetic head, as will be seen below.

6.2.1 Hydrostatic case

Notice that if there is no flow, $Q = 0$, the energy equation becomes

$$\left(\frac{p}{\rho g} + z\right)_1 = \left(\frac{p}{\rho g} + z\right)_2,$$

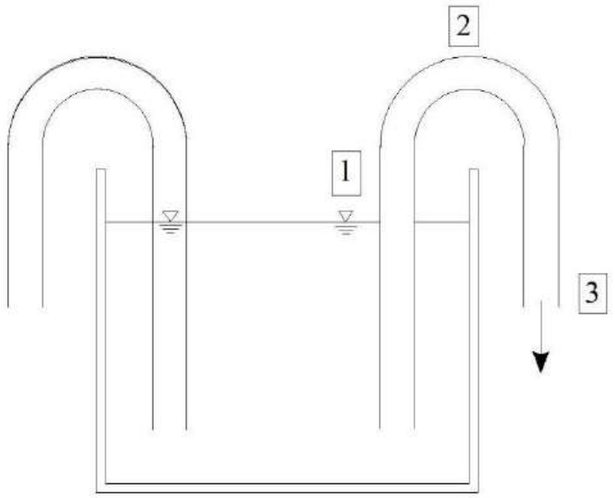
which is the Hydrostatic Pressure Equation.

6.2.2 A physical deduction

Note that as the quantity $p/\rho g + z + U^2/2g$, where $U = Q/A$, is conserved, then for the same elevation, a region of high/low velocity actually has a low/high pressure. This may come as a surprise, as we associate fast-flowing air with larger pressures. In fact, what we feel in a strong wind as high pressure is actually caused by our body bringing the wind to a low velocity locally. We will use this principle later to describe the Venturi meter for measuring flow in pipes.

Example 6.1 A Siphon

Consider the tank shown. On the left side is a pipe with one end under the water and the other end below the level of the water surface. However the pipe is not full of water, and no flow can occur. The pipe on the right has been filled with water (possibly by submerging it, closing an end, and quickly bringing that end out of the tank and below the surface level). Flow is possible, even though part of the pipe is above the water surface. This is a siphon.



(a) Calculate the velocity of flow at 3 in terms of the elevations of points 1 and 3. Ignore friction losses.

(b) calculate the pressure at 2 in terms also of its elevation.

(a) We consider a control volume made up of the solid surfaces through which no flow can pass, an open face that of the free surface in the tank, and the other open face that at 3. As the velocity with which the surface at 1 drops we will ignore it. Equation (6.10) gives:

$$\begin{aligned}\left(\frac{p}{\rho g}+z+\frac{\alpha}{2g}\frac{Q^2}{A^2}\right)_1 &= \left(\frac{p}{\rho g}+z+\frac{\alpha}{2g}\frac{Q^2}{A^2}\right)_3\\ \frac{0}{\rho g}+z_1+0 &= \frac{0}{\rho g}+z_3+\frac{\alpha}{2g}U_3^2,\end{aligned}$$

which gives

$$U_3=\sqrt{\frac{2g}{\alpha}\left(z_1-z_3\right)}.$$

Note that the factor α does play a role. The velocity of flow is reduced by a factor $1/\sqrt{\alpha}$, which if $\alpha \approx 1.3$ is 0.88.

(b) Now consider the control volume with the entry face across the pipe at 2. Equation (6.10) gives:

$$\begin{aligned}\left(\frac{p}{\rho g}+z+\frac{\alpha}{2g}\frac{Q^2}{A^2}\right)_2 &= \left(\frac{p}{\rho g}+z+\frac{\alpha}{2g}\frac{Q^2}{A^2}\right)_3\\ \frac{p_2}{\rho g}+z_2+\frac{\alpha}{2g}U_2^2 &= \frac{0}{\rho g}+z_3+\frac{\alpha}{2g}U_3^2,\end{aligned}$$

now solving the equation and using the fact that if the pipe has the same cross-sectional area, then mass conservation gives us $U_2 = U_3$,

$$p_2=\rho g\left(z_3-z_2\right).$$

As $z_3 - z_2$ is negative, so is this pressure, relative to atmospheric. Flow is possible provided that point 2 is not too high. If p_2 drops to the vapour pressure of the water, then the water boils, vapour pockets develop in it, and the flow will stop. At 20°C the vapour pressure of water is $2.5\,\mathrm{kPa}$, *i.e.* a gauge pressure of roughly $-100+2.5=-97.5\,\mathrm{kPa}$, corresponding to $z_3-z_2=p_2/\rho g=-97.5\times 10^3/1000/9.8\approx -10\,\mathrm{m}$, or point 2 being about $10\,\mathrm{m}$ above point 3. Of course, near 100°C the vapour pressure of water is close to atmospheric pressure, so that practically no elevation difference is possible without the water boiling.

6.3 Bernoulli's equation along a streamline

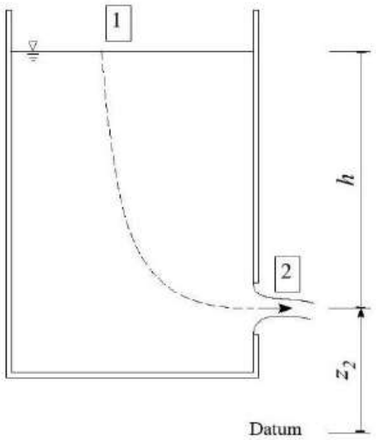
Most presentations of the energy principle in hydraulics actually use Bernoulli's[10] equation, which is an energy-like result obtained from non-trivial momentum considerations for flow through an infinitesimal streamtube (see, for example, §3.7 ofWhite 2003). The result that H is constant along a streamline. This is not as useful as widely believed, as in general this Bernoulli "constant" varies across streamlines, a point that is not always emphasised. Nevertheless, it is in the Bernoulli form that many tutorial and practical problems involving pipes and channels are analysed, and almost all of them ignore the fact that the kinetic energy density varies across a flow. In this course we consider primarily the energy equation in integral form, which allows for cross-stream variation.

There are, however, certain local problems where using *Bernoulli's equation* is easier than the energy conservation equation. It can be stated:

In steady, frictionless, incompressible flow, the head $H = \dfrac{p}{\rho g} + z + \dfrac{V^2}{2g}$ *is constant along a streamline,*

where V is the fluid speed such that $V^2 = u^2 + v^2 + w^2$. As it is written for frictionless flow along a streamline only, Bernoulli's equation is often not particularly useful for hydraulic engineering, as in both pipes and open channel flows we have to consider the situation where the energy per unit mass varies across the section. It does, however, give simple answers to some simple problems. Nevertheless in many textbooks the application of energy conservation is often described as being application of Bernoulli's equation.

Example 6.3 The surface of a tank is h above a hole in the side. Calculate the velocity of flow through the hole.



The situation is substantially friction-free. We apply Bernoulli's theorem between two points on any streamline:

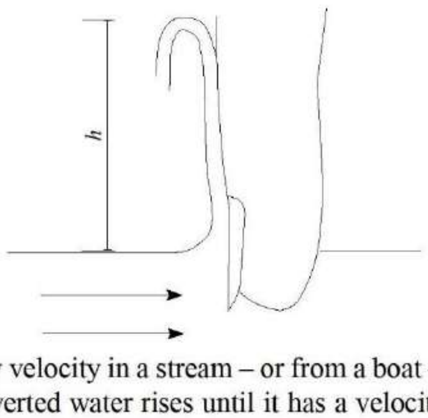
Point 1 which is on the surface, where the elevation relative to an arbitrary datum is $z_2 + h$, the pressure is atmospheric, and where we ignore the small velocity at which the surface drops as water flows out, and

Point 2 just outside the hole, where the elevation is z_2, the streamlines are parallel and the pressure is atmospheric,

$$\begin{aligned} \left(\frac{p}{\rho g} + z + \frac{V^2}{2g}\right)_1 &= \left(\frac{p}{\rho g} + z + \frac{V^2}{2g}\right)_2 \\ \frac{p_a}{\rho g} + z_2 + h + \frac{0^2}{2g} &= \frac{p_a}{\rho g} + z_2 + \frac{V_2^2}{2g}, \end{aligned}$$

from which $V_2 = \sqrt{2gh}$ is quickly obtained. Note that this is the same velocity that a particle dropped from that height would achieve. Also notice that the atmospheric pressure cancelled, so we may as well take it to be zero, and the z_2 cancelled, so that we may as well take the datum at one of the points considered.

Example 6.4 The Do-It-Yourself Velocity Meter



One simple way of measuring the flow velocity in a stream – or from a boat – is to put your finger in the water, and estimate the height h to which the diverted water rises until it has a velocity close to zero. Applying Bernoulli's theorem along the streamline between a point on the surface of the water where the velocity to be measured is U, and the highest point to which the water rises, both of which are at atmospheric pressure, gives

$$\frac{0}{\rho g} + 0 + \frac{U^2}{2g} = \frac{0}{\rho g} + h + \frac{0^2}{2g},$$

giving $U = \sqrt{2gh}$.

7.4.1 Reynolds number

The number was first proposed by Reynolds in 1883.In any problem involving viscosity, with a discharge Q and length scale D such that there is a velocity scale $U = Q/D^2$, a dimensionless viscosity number appears as in the preceding examples:

$$\frac{\mu D}{Q\rho} = \frac{\nu D}{Q} = \frac{\nu}{UD},$$

which is the ratio of viscosity to inertial forces. In most hydraulics literature this quantity is written upside down and is called the Reynolds number:

$$R = \frac{UD}{\nu}.$$

It would have been more satisfying if the number quantifying the relative importance of viscosity had been directly proportional to viscosity! With the traditional definition, high Reynolds number flows are those which are large and/or fast such that the effects of viscosity are small. In hydraulics problems, with, say, a typical length scale of $1\,\mathrm{m}$, a velocity scale of $1\,\mathrm{m\,s^{-1}}$, and a typical value for water at 20° of $\nu = 1 \times 10^{-6}\,\mathrm{m^2\,s^{-1}}$, a typical value of $R = 10^6$ is obtained. Flows which may be laminar for small Reynolds number become turbulent at a Reynolds number of the order of 10^3, so it can be seen that nearly all flows in hydraulic engineering are turbulent, and the effects of viscosity are not so important. However, we will see that, for example, the resistance coefficient for pipe flows shows significant variation with Reynolds number in commonly-experienced conditions.

8. Flow in pipes

The main problem in studying flow in pipes is to obtain the relationship between flow rate and energy loss, so that if frictional characteristics are known, then flow rates can be calculated, given the available head difference between two points.

8.1 The resistance to flow

8.1.1 Weisbach's equation

for the head loss ΔH in a pipe of length L and diameter D with a flow rate Q, where the roughness size on the pipe wall is d, and the physical quantities g, μ, and ρ are as usual. This is the basis for Weisbach[15]'s equation for pipes, introduced in the 1840s, which states

$$\Delta H = \lambda \frac{L}{D}\frac{U^2}{2g}, \tag{8.1}$$

where U is the mean velocity, and λ is a dimensionless friction coefficient, for which the symbol f or $4f$ is sometimes used, which is a function of relative roughness $\varepsilon = d/D$ and pipe Reynolds number UD/ν. The equation is widely used to calculate friction in pipes, and was at the core of hydraulics research in the first half of the 20th century. Often the equation is called the Darcy-Weisbach equation; some authors do not bother with a name for the equation or for the coefficient λ. For pipes the gravitational acceleration g plays no role other than that of multiplying ΔH to give the energy loss per unit mass. The total power loss in the pipe is $\rho g Q \Delta H$.

8.2 Practical single pipeline design problems

Case	Known	To find
A	Discharge Q and diameter D known	Head loss ΔH
B	Head loss ΔH and diameter D known	Discharge Q
C	Head loss ΔH and discharge Q known	Diameter D

Table 8-2. Three cases of pipeline design problems

Here we consider the three types of simple pipe problems, so-called because they are problems where pipe friction is the only loss. Six variables are involved: Q, L, D, ΔH, ν, and d. In general the quantities which are known are: length L, the temperature and hence the viscosity ν, and d the roughness of the pipe material being considered. the simple pipe problems may then be treated as three types, as shown in Table 8-2.We also have the Weisbach equation (8.1) in terms of more practical quantities

$$\Delta H = \lambda \frac{L}{D} \frac{U^2}{2g} = \lambda \frac{8LQ^2}{g\pi^2 D^5}. \tag{8.8}$$

8.3 Minor losses

Losses which occur in pipelines because of bends, elbows, joints, valves, *etc.* are called *minor losses*, although

they are just as important as the friction losses we have considered. In almost all cases the minor loss is determined by experiment. A significant exception is the head loss due to a sudden expansion which we will treat below. All such losses are taken to be proportional to the square of the velocity and are expressed as a coefficient times the kinetic head:

$$\Delta H = K \frac{U^2}{2g}. \tag{8.14}$$

8.3.1 Pipeline fittings

Values of K for various pipeline fittings are given in Table 8-3, taken from Streeter & Wylie (1981, p245).

Fitting	K
Globe valve (fully open)	10.0
Angle valve (fully-open)	5.0
Swing check valve (fully open)	1.0
Gate valve (fully open)	0.2
Close return bend	2.2
Standard T	1.8
Standard elbow	0.9
Medium sweep elbow	0.8
Long sweep elbow	0.6

Table 8-3. Head-loss coefficients K for various fittings

Minor losses may be neglected when they comprise 5% or less of the head losses due to pipe friction. The friction factor, at best, is known to about 5% error, and it is meaningless to try to specify great accuracy. In general, minor losses may be ignored when there is a length of some 1000 diameters between each minor loss.

8.3.3 Sudden contraction

The head loss at the entrance to a pipeline from a tank or reservoir is usually taken as $K = 0.5$ if the opening is square edged. For well-rounded entrances $K = 0.01$ to 0.05 and may usually be neglected. For re-entrant openings, such as when the pipe extends into the tank beyond the wall, $K = 1$.

8.4 Pipeline systems

8.4.1 Introduction

One of the most common problems faced by an hydraulic engineer is the analysis and/or design of pipeline systems. In this section we will bring together some sections considered above and apply them to practical problems. In particular, Section 6, the energy equation; Section 8, flow in pipes; and Section 11, fluid machinery. Complex flow problems will be investigated, including systems that incorporate different elements such as pumps and multiple pipeline networks.

There are some warnings to make here:

- Some American books still use the Hazen-Williams equation for pipe friction. It was superseded in the 1930s by the work of the German school.
- For the Weisbach friction factor American books use the symbol f. British books tend to use $4f$ instead. To overcome the ambiguity, in this course we have used the European symbol λ and Weisbach's formula in the form $\Delta H = \lambda L/D \times U^2/2g$, with no factor of 4 in front of it.

8.4.2 Summary of useful results from above for calculating pipe friction

Continuity equation: For a control surface, equation (4.3) is

$$\pm Q_1 \pm Q_2 \pm \ldots = 0, \tag{8.17}$$

where the Q_i are the discharges crossing the control surface, where the positive/negative sign is taken for fluid leaving/entering the control volume.

Integral energy theorem: The integral energy theorem (equation 6.8) can be written

$$\pm Q_1 H_1 \pm Q_2 H_2 \pm Q_3 H_3 \pm \ldots = 0. \tag{8.18}$$

where the H_i are the corresponding *total head of the flow*, the mean energy per unit mass divided by g:

$$H = \frac{p}{\rho g} + z + \frac{\gamma}{2g}\frac{Q^2}{A^2}.$$

Simple pipe or channel: For a length of pipe or channel where there are no other entry or exit points for fluid, equation (6.10), explicitly including head loss ΔH, gives:

$$\left(\frac{p}{\rho g} + z + \frac{\gamma}{2g}\frac{Q^2}{A^2}\right)_{\text{in}} = \left(\frac{p}{\rho g} + z + \frac{\gamma}{2g}\frac{Q^2}{A^2}\right)_{\text{out}} + \Delta H \tag{8.19}$$

Head loss formulae: Weisbach's equation is

$$\Delta H = \lambda \frac{L}{D}\frac{U^2}{2g}, \tag{8.20}$$

where ΔH is the head loss in a pipe of length L and diameter D, U is the mean velocity, and λ is a dimensionless friction coefficient.

Minor losses: Losses which occur in pipelines because of bends, elbows, joints, valves, *etc.* were described in

#8.3. All such losses are taken to be proportional to the square of the velocity and are expressed as a coefficient times the kinetic head:

$$\Delta H = K\frac{U^2}{2g}. \tag{8.23}$$

Values of K for various pipeline fittings and situations are given in Table 8-5.

Fitting/Situation	K
Globe valve (fully open)	10.0
Angle valve (fully-open)	5.0
Close return bend	2.2
Standard T	1.8
Elbows	0.6 – 0.9
Gate valve (fully open)	0.2
Expansion	$\left(1 - \frac{A_1}{A_2}\right)^2$
Sharp contraction	0.5
Re-entrant contraction	1.0
Rounded contraction	0.05 – 0.1

Table 8-5. Head-loss coefficients K for various fittings and expansions and contractions

8.5 Total head, piezometric head, and potential cavitation lines

8.5.1 General considerations

Consider the mean total head at a section across a pipe:

$$H = \frac{p}{\rho g} + z + \frac{\alpha}{2g}\frac{Q^2}{A^2}. \tag{8.24}$$

The pressure p and elevation z will usually vary linearly over the section, such that the *piezometric head* $h = p/\rho g + z$ is a constant over the section, and hence so is H. This total head is usually known at control points such as the surface of reservoirs. The amounts lost due to friction and local losses in the system can be calculated.

If p_v is the absolute vapour pressure of the water in the pipe, cavitation will *not* occur if the absolute pressure $p + p_a \geqslant p_v$, where p_a is the atmospheric pressure. Hence, from equation (8.24), for *no* cavitation

$$\frac{p + p_a}{\rho g} = H - z + \frac{p_a}{\rho g} - \frac{\alpha}{2g}\frac{Q^2}{A^2} \geqslant \frac{p_v}{\rho g},$$

giving the condition

$$z \leqslant H + \Delta h_c, \tag{8.25}$$

where Δh_c might be called the *cavitation height*

$$\Delta h_c = \frac{p_a}{\rho g} - \frac{p_v}{\rho g} - \frac{\alpha}{2g}\frac{Q^2}{A^2}. \tag{8.26}$$

That is, as z is the elevation of any part of the cross-section, no part of the pipe can have an elevation larger than the magnitude of the local head plus a distance Δh_c. The limiting condition is for the highest fluid in the pipe at any section, which is just below the soffit (the top of the inside) of the pipeline, such that we can write

> Cavitation will not occur at a section if the soffit of the pipe is lower than a point Δh_c above the local total head elevation.

Now consider the quantities which make up the cavitation height Δh_c in equation (8.26):

- Atmospheric pressure head $p_a/\rho g$ – typical atmospheric pressures are roughly $990 - 1010\,\mathrm{m\,bar}$, and the density of fresh water is $1000\,\mathrm{kg\,m^{-3}}$ at 5° and $958\,\mathrm{kg\,m^{-3}}$ at 100°, so that $p_a/\rho g$ will vary between 10.1

and 10.8 m. The standard atmosphere of 760 mm of mercury with a value of 10.34 m of water could be assumed without much error.

- Vapour pressure – this shows rather more variation, primarily with temperature. As shown in Figure 8-5 for atmospheric temperatures it is always less than a metre, and for moderate temperatures could be ignored, however as the temperature approaches 100°C it approaches 10.33 m, and $p_a - p_v$ goes to zero such that boiling occurs at atmospheric pressure.

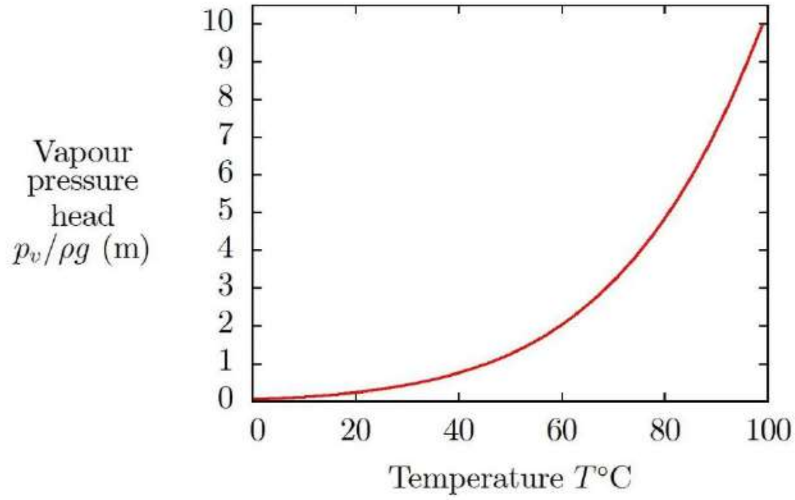


Figure 8-5. Vapour pressure head and its variation with temperature

- Kinetic head – this varies from situation to situation and from pipe to pipe in the same network. For this to be as much as 1 m, the mean fluid velocity in the pipe would be roughly $\sqrt{2 \times 9.8/1.2} \approx 4.0\,\mathrm{m\,s^{-1}}$, which is large.

The manner in which a pipeline is examined for the possibility of cavitation is essentially to plot a graph of $H + \Delta h_c$ (equation 8.25) against horizontal distance along the pipe. If the soffit of the pipe were found to pass above that line, then the absolute pressure in the water would be equal to the vapour pressure there, and it would start to cavitate and disrupt the flow. The calculation is usually done in three stages, giving three lines which are significant to the problem:

Total head line or "energy grade line": First calculate the total head H at points such as reservoirs from knowledge of the elevation, then at other points from knowledge of both friction and local losses due to structures, pumps, expansions or contractions. Plotting this line often reveals details of the flow, such as where energy might be reduced due to turbulent mixing and losses destroying kinetic energy. After plotting the total head points, the total head line can be plotted by connecting the known points with straight lines.

Piezometric head line or "hydraulic grade line": This is often calculated and plotted, which often helps understanding a problem, but it is not strictly necessary to do so to calculate $H + \Delta h_c$. The piezometric head line joins a series of points showing the piezometric head for each point along a pipe, plotted against horizontal distance. As can be obtained from equation (8.24), the piezometric head h:

$$h = H - \frac{\alpha}{2g}\frac{Q^2}{A^2},$$

showing that it is simply plotted a vertical distance equal to the kinetic head $\alpha/2g \times Q^2/A^2$ *below* the total head line. The piezometric head line is parallel to the total head line for each pipe element.

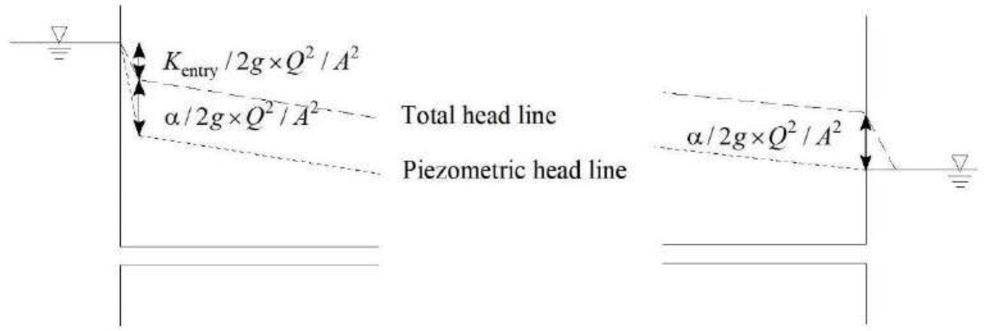
Potential cavitation line: Equation (8.25) shows that for no cavitation, $z \leqslant H + \Delta h_c$, where z refers to the soffit of the pipe. This condition can be written

$$z \leqslant \underbrace{H - \frac{\alpha}{2g}\frac{Q^2}{A^2}}_{\text{Piezometric head}} + \underbrace{\frac{p_a - p_v}{\rho g}}_{\text{Almost constant on any one day}}$$

This shows that where points and joining lines are drawn, displaced vertically a distance $(p_a - p_v)/\rho g$ above the piezometric head line, then if the soffit of the pipe ever passes above this "potential cavitation line", cavitation will occur. The additive quantity is a function of atmospheric pressure and water temperature, but for most hydraulic engineering problems varies little.

8.5.2 Entry and exit points

Entry point: Immediately the flow enters a pipe the total head reduces by an amount $K\,U^2/2g$ (above we saw



(a) Entry to pipe (b) Exit from pipe to reservoir

Figure 8-6. Behaviour of total and piezometric head lines at entry and exit from pipe into a reservoir

that the head loss coefficient for a square-edged inlet was $K \approx 0.5$). The pressure will further drop by an amount $\alpha\,U^2/2g$, as the kinetic contribution to the head has increased by this amount. This means that the local total head line and piezometric head lines look as in 8-6(a).

Exit point: As the flow leaves the pipe and enters a much larger body of water such as a reservoir, all the kinetic energy is destroyed. This can be explained by the theory above leading to equation (8.16) for the case $A_2 = \infty$ or $U_2 = 0$, giving

$$\Delta H = \frac{\alpha U_1^2}{2g}.$$

This means that the piezometric head becomes equal to the head in the reservoir at the exit, and the pressure is the same just inside the pipe as at a point of the same elevation in the reservoir, as shown in Figure 8-6(b).

8.5.3 A typical problem

Figure 8-7 shows an example where a pipeline passes from one reservoir to another over a hill, and shows a number of features of total and piezometric head lines. As it is drawn, cavitation will occur, and the actual pressure distribution will be very different. Here, the example is included to demonstrate how one checks for potential cavitation. In practice, if it were found to be likely, then a more complete analysis would be done, incorporating its effects.

- Considering the flow just after the entrance to the pipe, the total head line immediately shows a slight drop because of the losses associated with the sudden contraction, something like 0.1 or 0.5 times $U^2/2g$, according to equation (8.14) and Table 8-1.
- The piezometric line shows an even larger sudden drop because as the fluid has been accelerated into the pipe and now has a finite velocity, the distance between the two curves $\alpha U^2/2g$ is finite.
- As velocity is constant along the pipe, so is the distance between the two curves.
- As the flow approaches the downstream reservoir, the piezometric head in the pipe will be the same as in the reservoir, but the total head contains the kinetic component which is excess – the flow enters the reservoir as a turbulent jet until all the kinetic head is destroyed, as shown in the figure, after which both total and piezometric heads are now equal to the elevation of the still reservoir surface.
- The potential cavitation line can now be simply drawn as a line everywhere a constant distance $(p_a - p_v)/\rho g$ above

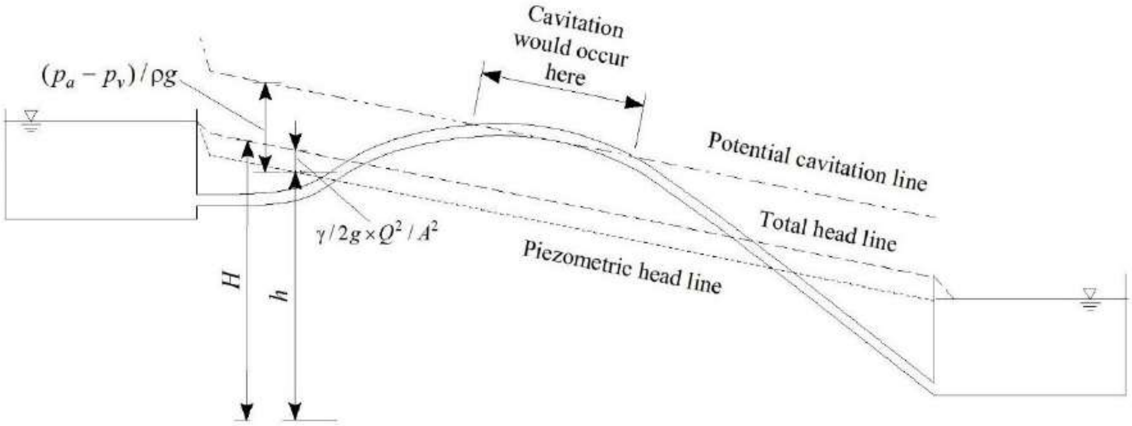


Figure 8-7. Pipeline between two reservoirs passing over a hill showing hydraulic lines and region of cavitation.

the piezometric head line.

- It can be seen that the soffit of the pipe passes above this line, and cavitation will occur wherever it is above. The flow situation will, in fact change, in a manner to be described further below. For present purposes we have shown that cavitation is a problem in this case.

Example 8.1 (Example #10.1 Streeter & Wylie 1981)

Consider the pipeline emerging from a tank, passing through a valve, and finally the flow emerges into the atmosphere *via* a nozzle. Determine the elevation of hydraulic and energy grade lines at points A-E. The energy loss due to the nozzle is $0.1 \times U_E^2/2g$. In this case we take a value of $\alpha = 1.3$.

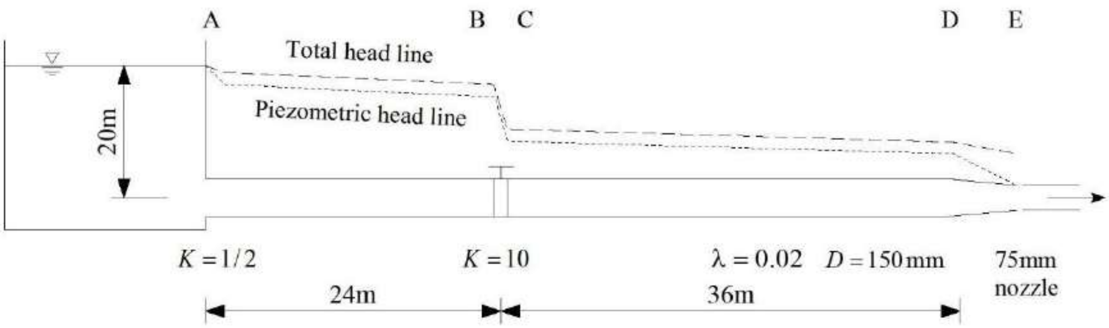


Figure 8-8. Tank, pipeline and nozzle, with total head and piezometric head lines

First we apply the integral energy equation for a control surface around all the sides of the tank, the pipe, and across the nozzle.

$$\left(\frac{p}{\rho g}+z+\frac{\alpha}{2g}\frac{Q^2}{A^2}\right)_{\text{in}}=\left(\frac{p}{\rho g}+z+\frac{\alpha}{2g}\frac{Q^2}{A^2}\right)_{\text{out}}+\Delta H$$

which gives

$$\begin{aligned}0+20+0 &= 0+0+\frac{\alpha}{2g}\underbrace{\frac{Q^2}{\left(\pi\left(D/2\right)^2/4\right)^2}}_{\text{Note: Diameter of nozzle}}+\frac{Q^2}{2g\left(\pi D^2/4\right)^2}\left(\underbrace{\frac{1}{2}}_{K_A}+\underbrace{10}_{K_{BC}}+\underbrace{\frac{0.02\times 60}{0.15}}_{\text{Friction: } fL/D}\right)\\ &\quad+0.1\times\frac{Q^2}{2g\times\underbrace{\left(\pi\left(D/2\right)^2/4\right)^2}_{D/2 \text{ diameter of nozzle}}}\end{aligned}$$

$$20 = \frac{Q^2}{2g\left(\pi D^2/4\right)^2}\left(\alpha\times 16+\tfrac{1}{2}+10+8+0.1\times 16\right)$$

With $\alpha = 1.3$, we obtain a discharge of $Q = 0.0547$, while with $\alpha = 1$, $Q = 0.0582$. The difference of 6% seems enough to warrant using the more realistic value. Hence, our basic quantity is

$$\frac{U^2}{2g} = \frac{Q^2}{2g\left(\pi D^2/4\right)^2} = \frac{0.0547^2}{2\times 9.8\times\left(\pi 0.15^2/4\right)^2} = 0.489,$$

to be used in local loss formulae, while the kinetic head term to be used is

$$\alpha\frac{U^2}{2g} = 1.3\times 0.489 = 0.636.$$

Now we apply the energy equation between the surface of the tank and just after A, and then between each of the subsequent points:

$H_A = 20 - 0.5\times 0.489 = 19.76$ $\qquad h_A = 19.76 - 0.64 = 19.12$
$H_B = 19.76 - \frac{0.02\times 24}{0.15}\times 0.489 = 18.20$ $\qquad h_B = 18.20 - 0.64 = 17.56$
$H_C = 18.2 - 10\times 0.489 = 13.31$ $\qquad h_C = 13.31 - 0.64 = 12.67$
$H_D = 13.31 - \frac{0.02\times 36}{0.15}\times 0.489 = 10.96$ $\qquad h_D = 10.96 - 0.64 = 10.32$
$H_E = 10.96 - 16\times 0.1\times 0.489 = 10.18$ $\qquad h_E = 10.18 - 16\times 0.64 = -0.06$

The final value of the piezometric head at E should have been zero, but accumulated round-off error by working to two places only, has caused the error. These results are plotted in Figure 8-9.

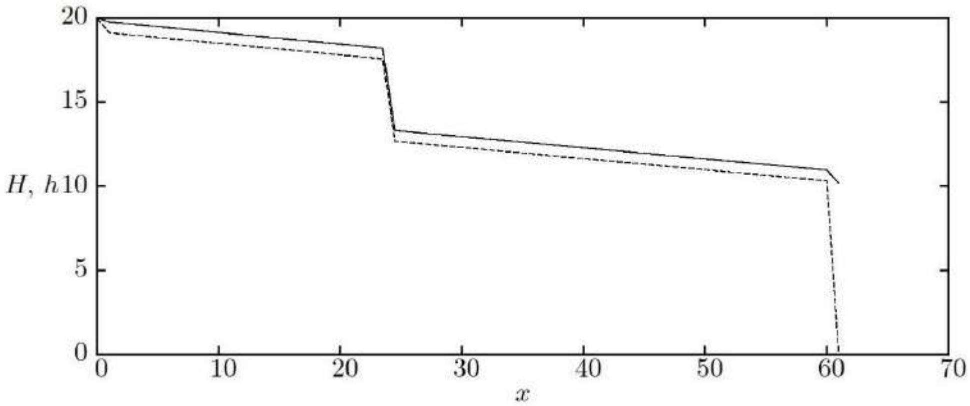


Figure 8-9. Total and piezometric head lines for example

Example 8.2 Consider a single pipe of length $L = 1000\,\mathrm{m}$ connecting two reservoirs with a head difference $\Delta H = 10\,\mathrm{m}$. It has a diameter of $D = 300\,\mathrm{mm}$, roughness height $d = 0.3\,\mathrm{mm}$; it has a square-edged entry, $K_0 = 0.5$, and with $\alpha = 1.3$ in the pipe. Calculate the flow in the pipe.

$$0 + \Delta H + 0 = 0 + 0 + 0 + \frac{Q^2}{2g\left(\pi D^2/4\right)^2}\left(K_0 + \frac{\lambda L}{D} + \alpha\right)$$

We will solve the problem by iteration using Haaland's approximation for λ, which is a weakly-varying function of Q, so that the equation is an implicit equation for Q:

$$Q = \frac{\pi D^2/4}{\sqrt{K_0 + \lambda L/D + \alpha}}\sqrt{2g\Delta H}, \tag{8.27}$$

and we will start with the fully-rough approximation for λ.

For $\varepsilon = d/D = 0.3/300 = 0.001$,

$$\lambda = \frac{1}{1.8^2}\frac{1}{\left(\log_{10}\left(\left(\frac{0.001}{3.7}\right)^{10/9}\right)\right)^2} = 0.0196.$$

Hence from (8.27)

$$Q = \frac{\pi \times 0.3^2/4}{\sqrt{0.5 + 0.0196 \times 1000/0.3 + 1.3}} \sqrt{2 \times 9.8 \times 10} = 0.121\,\mathrm{m}^3\,\mathrm{s}^{-1},$$

This is our first estimate. Here we calculate R as a function of Q:

$$R = \frac{UD}{\nu} = \frac{QD}{A\nu} = \frac{4Q}{\pi D\nu} = \frac{4}{\pi \times 0.3 \times 10^{-6}} Q \quad \text{or} \quad \frac{1}{R} = \frac{2.36 \times 10^{-7}}{Q}$$

and so from Haaland, equation (8.7)

$$\lambda = \frac{1}{1.8^2} \frac{1}{\left(\log_{10}\left(\left(\frac{0.001}{3.7}\right)^{10/9} + 6.9/R\right)\right)^2} = \frac{1}{1.8^2} \frac{1}{\left(\log_{10}\left(\left(\frac{0.001}{3.7}\right)^{10/9} + \frac{1.63 \times 10^{-6}}{Q}\right)\right)^2},$$

and so, using $Q = 0.121\,\mathrm{m}^3\,\mathrm{s}^{-1}$ we get $\lambda = 0.0202$. Substituting again into (8.27)

$$Q = \frac{\pi \times 0.3^2/4}{\sqrt{0.5 + 0.0201 \times 1000/0.3 + 1.3}} \sqrt{2 \times 9.8 \times 10} = 0.119\,\mathrm{m}^3\,\mathrm{s}^{-1},$$

And repeating, we get $\lambda = 0.0202$, hence it has converged sufficiently. The discharge is $0.119\,\mathrm{m}^3\,\mathrm{s}^{-1}$.

8.5.4 Compound pipe systems

Pipes in series: When two pipes of different sizes or roughnesses are so connected that fluid flows through

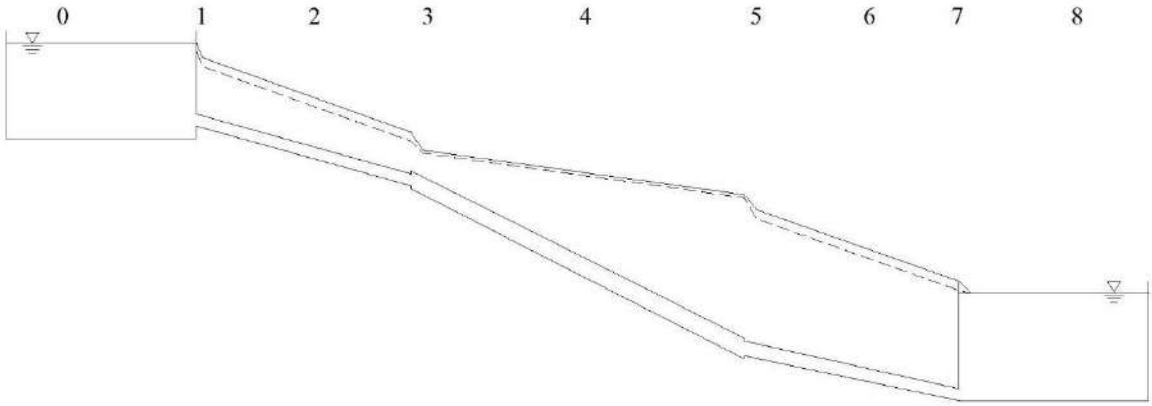


Figure 8-10. Compound pipe system – in series, joining two reservoirs

one pipe and then through the other. Consider Figure 8-10, where we apply the energy integral equation between the surface of the two reservoirs:

$$\begin{aligned}\left(\frac{p}{\rho g} + z + \frac{\gamma}{2g}\frac{Q^2}{A^2}\right)_{\text{in}} &= \left(\frac{p}{\rho g} + z + \frac{\gamma}{2g}\frac{Q^2}{A^2}\right)_{\text{out}} + \Delta H \\ 0 + z_0 + 0 &= 0 + z_8 + 0 + K_1 \frac{Q^2/A_1^2}{2g} + \lambda_2 \frac{L_2}{D_2}\frac{Q^2/A_2^2}{2g} + \ldots + \lambda_6 \frac{L_6}{D_6}\frac{Q^2/A_6^2}{2g} + \gamma \frac{Q^2/A_7^2}{2g}\end{aligned}$$

Hence we obtain

$$z_0 - z_8 = \frac{Q^2}{2g}\left(\frac{K_1}{A_1^2} + \frac{\lambda_2 L_2/D_2}{A_2^2} + \ldots + \frac{\lambda_6 L_6/D_6}{A_6^2} + \frac{\gamma}{A_7^2}\right) = \frac{Q^2}{2g}\sum_{i=1}^{N}\frac{K_i}{A_i^2}, \tag{8.28}$$

where for the friction loss in a length of pipe, $K_i = \lambda_i L_i/D_i$, and usually, if the kinetic head is destroyed at the end of the pipe, $K_N = \gamma$. The Colebrook-White equation

$$\frac{1}{\sqrt{\lambda}} + 2\log_{10}\left(\frac{\varepsilon}{3.7} + \frac{2.51}{\mathbf{R}\sqrt{\lambda}}\right) = 0$$

shows us that λ_i is an implicit function of relative roughness $\varepsilon_i = d_i/D_i$ of a pipe and the Reynolds number of the pipe flow in the form

$$\frac{1}{\mathbf{R}_i} = \frac{\nu}{U_i D_i} = \frac{\nu\pi D_i}{4Q}.$$

Substituting gives

$$\frac{1}{\sqrt{\lambda_i}} + 2\log_{10}\left(\frac{\varepsilon_i}{3.7} + \frac{1.97\nu D_i}{Q}\frac{1}{\sqrt{\lambda_i}}\right) = 0. \tag{8.29}$$

We now have a system of nonlinear equations to solve: equation (8.28) is an equation for Q and all of the λ_i (and any K_i which depend on the Reynolds number), while for each pipe segment, equation (8.29) is a single equation in the common Q and a single λ_i. There are precisely as many equations as unknowns, and a solution seems possible. Any one of a number of different methods such as Newton's method for a system of equations, or optimisation methods are possible, which might be quite complicated. In this case, however, the structure of the system is simple, because in each of the Colebrook-White equations the term involving λ_i is weakly varying, which suggests using a *direct iteration method*.

As noted in Section 8.2, there are three types of practical pipeline design problems:

8.5.5 Parallel pipelines

If the flow is divided among two or more pipes and then is joined again, it is a *parallel-pipe system*. In pipes in series the same fluid flows through all the pipes and the head losses are cumulative, but in parallel pipes the head losses are the same in any of the lines and the discharges are cumulative.

Two types of problem occur:

- With the heads known at the ends of the pipes, find the discharge. This is just the separate solution of simple pipe problems for discharge, since the head loss is known for each.
- With the discharge known, find the distribution of flow and the head loss. This is more complicated, as the losses are nonlinear, and for different flows the fraction of flow in each pipe will change.

  Consider the case where there are three pipes. The equations are, from (8.28) but assuming just frictional resistance,

  $$H = \lambda_1 \frac{L_1}{D_1}\frac{Q_1^2}{2gA_1^2} = \lambda_2 \frac{L_2}{D_2}\frac{Q_2^2}{2gA_2^2} = \lambda_3 \frac{L_3}{D_3}\frac{Q_3^2}{2gA_3^2}, \text{ and } Q = Q_1 + Q_2 + Q_3,$$

  and adding a Colebrook-White equation for each pipe gives us a total of seven equations in the seven unknowns H, λ_1, Q_1, λ_2, Q_2, and λ_3, Q_3. They are nonlinear. It would seem simplest to use *Solver* again.

8.5.6 Branching pipelines

There are two different cases:

The dividing flow type: this is the more usual case, where a pipeline bifurcates or enters a manifold, from which flow leaves *via* a number of different exits. Examples include the bifurcating nature of a water supply system and diffusers for disposal of sewage or heated effluents into large bodies of water. There is no reason to expect unusual losses, and a simple energy balance equation can be written, including conventional coefficients associated with bends *etc.* at the bifurcation.

The combining flow type: in this case where two or more pipes combine, it is possible that there will be some finite energy losses associated with two different streams of different energies per unit mass and different mass flow rates combining and leaving with the same energy per unit mass. This seems not to be treated in text books, but it is possible that an analysis based on momentum might yield results.

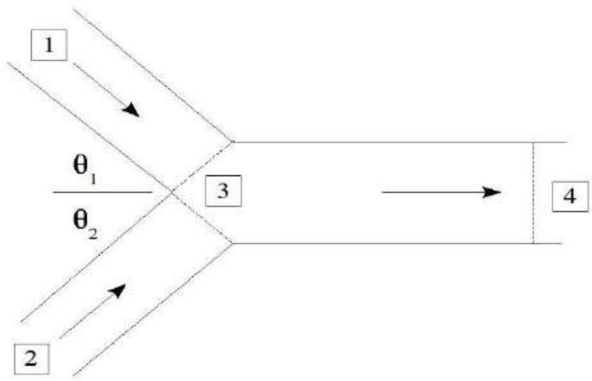


Figure 8-12. Pipe junction, showing control surface dashed

Consider two pipes joining as in Figure 8-12, where a flow 1 and flow 2, with the assumed directions of flow shown, join at 3, and it is assumed that in general some losses will occur before a station 4. The control surface is shown by a dashed line. The mass conservation equation gives

$$-Q_1 - Q_2 + Q_4 = 0, \tag{8.33}$$

the momentum conservation equation gives, where it is assumed that at the entry to the control volume, the pressure in both pipes is p_3, and that frictional forces between 3 and 4 are negligible,

$$-\cos\theta_1\left(\frac{p_3}{\rho}A_1 + \frac{\beta_1 Q_1^2}{A_1}\right) - \cos\theta_2\left(\frac{p_3}{\rho}A_2 + \frac{\beta_2 Q_2^2}{A_2}\right) + \frac{p_4}{\rho}A_4 + \frac{\beta_4 (Q_1+Q_2)^2}{A_4} = 0, \tag{8.34}$$

where it is been assumed that due to turbulent mixing of the streams, that $\beta_4 = 1$ and $\gamma_4 = 1$. The energy conservation equation gives

$$-Q_1\left(\frac{p_1}{\rho} + \frac{\gamma_1}{2}\left(\frac{Q_1}{A_1}\right)^2\right) - Q_2\left(\frac{p_2}{\rho} + \frac{\gamma_2}{2}\left(\frac{Q_2}{A_2}\right)^2\right) + Q_4\left(\frac{p_4}{\rho} + \frac{1}{2}\left(\frac{Q_4}{A_4}\right)^2\right) - Q_4 g\,\Delta H = 0, \tag{8.35}$$

where $-g\Delta H$ is the change in head between 3 and 4 such that ΔH is a positive quantity. Eliminating p_4 between equations (8.34) and (8.35) gives

$$g\Delta H = \frac{p_3}{\rho}\left(1 - \frac{A_1}{A_4}\cos\theta_1 - \frac{A_2}{A_4}\cos\theta_2\right) - \frac{\beta_1 Q_1^2}{A_4 A_1}\cos\theta_1 - \frac{\beta_2 Q_2^2}{A_4 A_2}\cos\theta_2$$
$$+\frac{1}{2}\frac{(Q_1+Q_2)^2}{A_4^2} + \frac{1}{2}\frac{\gamma_1 Q_1^3}{(Q_1+Q_2)A_1^2} + \frac{1}{2}\frac{\gamma_2 Q_2^3}{(Q_1+Q_2)A_2^2}. \tag{8.36}$$

This has an important counterpart in the hydraulic jump, the turbulent phenomenon in open channels which takes a shallow high-speed stream to a deep low-speed state with no momentum loss, but with intense turbulence causing a loss in energy.

It is interesting that in this case, unlike losses due to fittings and bends, the pressure also plays a role.

To obtain an idea of the magnitude of the losses, consider two pipes of the same area $A_1 = A_2$ carrying the same discharge $Q_1 = Q_2$, which enter a larger pipe at angles of $\pm 45°$. From the Weisbach equation it can be shown that, for the same head gradient, that discharge is proportional to the pipe diameter to the power $5/2$, hence we can say that $D_4/D_1 = (Q_4/Q_1)^{2/5} = 2^{2/5}$, or that $A_4/A_1 = 2^{4/5}$. As a first approximation we will set all the β and γ to 1, and so substituting into equation (8.36),

$$\begin{aligned} g\Delta H &= \frac{p_3}{\rho}\left(1 - 2\times 2^{-4/5}\,2^{-1/2}\right) - 2\times\frac{Q_1^2}{2^{4/5}A_1^2}\frac{1}{\sqrt{2}} + \frac{1}{2}\frac{(2Q_1)^2}{2^{8/5}A_1^2} + \frac{2}{2}\frac{Q_1^2}{2\,A_1^2} \\ &= 0.188\,\frac{p_3}{\rho} + 0.348\,\frac{Q_1^2}{A_1^2}. \end{aligned} \tag{8.37}$$

Now we calculate the effective local loss coefficient K_{junction} such that $\Delta H = K_{\text{junction}} \times U_4^2/2g$, giving

$$K_{\text{junction}} = \frac{2g\Delta H}{(Q_4/A_4)^2} = \frac{2g\Delta H}{(2Q_1/A_1/2^{4/5})^2} = 1.52\times\frac{g\Delta H}{(Q_1/A_1)^2},$$

and substituting back into (8.37)

$$K_{\text{junction}} = 1.52\left(0.188\,\frac{p_3}{\rho U_1^2} + 0.348\right) = 0.284\frac{p_3}{\rho U_1^2} + 0.527,$$

hence for negligible pressure, we obtain a value of 0.53, typical of such local losses. However the pressure contribution might be quite large. Let us take $p_3/\rho g \approx 100\,\mathrm{m}$ and $U_1 = 1\,\mathrm{m\,s^{-1}}$, giving

$$K_{\text{junction}} = 0.284\,\frac{9.8\times 100}{1} + 0.527 \approx 280,$$

which seems a remarkably high value, and may cause a re-evaluation of the role of junctions in a network.