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LECTURE NOTES

“DYNAMICS”

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I. NEWTON'S LAWS OF MOTION

1. The basic quantities of dynamics

The mathematical description of motion that includes the quantities that affect motion – mass and force – is called dynamics. *Dynamics* is the study of *why* things move as they do.

Mass is the quantitative measure of inertia of a body. Objects change their motion in response to actions from external objects, and mass is the amount of opposition to changes in motion that an object possesses.

Inertia of a body is its reluctance to start moving, and its reluctance to stop once it has begun moving. Thus an object at rest begins to move only when it is pushed or pulled, i.e., when force acts on it. In general, inertia is resistance to changing. In mechanics, inertia is the resistance to changing in velocity or, if you prefer, the resistance to acceleration.

Mass is a scalar quantity associated with matter. When a system is composed of several objects it is the total mass that matters.

$[m] = \text{kilogram} = \text{kg}$.

Force is a physical quantity that can affect the motion of an object. In mechanics, a force is an interaction that causes a change in velocity or an interaction that causes acceleration. Force is a vector quantity associated with an interaction. Since force is a vector quantity we use geometry instead of arithmetic when combining forces. When several forces act on a system it is the *net (total)* external force that matters.

$[F] = \text{Newton} = \text{N} = \text{kg} \cdot \text{m/s}^2$.

Many forces are *contact forces*; hey act only while two objects are physically touching. Other forces are *action–at–a–distance* forces for which no physical contact is necessary. Examples are gravitational, electrical, and magnetic forces.

Linear momentum (or *momentum*) of material objects, i.e., piece of matter, is a vector quantity defined as

$\vec{p} = m \cdot \vec{v}$.

It is the product of the object's mass and its velocity vector.

$[p] = \mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-1}$.

and there is unfortunately no abbreviation for this clumsy combination of units.

2. Inertial frames of reference. Galileo's principle of relativity

Among a lot of reference frames used to describe the motion there are such reference frames the motion respectively which is described most simply, i.e., it is uniform and rectilinear motion. They are *inertial frames of reference*. An example of an inertial frame of reference is heliocentric one in which the origin of coordinates is the Sun and the axes are directed towards the distant stars.

Galileo's principle of relativity: absolute linear motion at a constant velocity cannot be detected, nor can the absolute rest. All motion is relative to a frame of reference. It is impossible to distinguish motion with a constant velocity from rest. All constant velocity frames of reference are equivalent (including frames of reference that appear to be at rest –after all, a prolonged state of rest is motion with a constant speed of zero).

3. Newton's laws of motion

The First Law: An object at rest tends to remain at rest and an object in motion tends to continue moving with constant velocity unless compelled by a net external force to act otherwise.

The Second Law: The change of momentum per second is proportional to the applied force and the momentum change takes place in the direction of the force.

$$\frac{d\vec{p}}{dt} = \vec{F},$$

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt} = m\frac{d\vec{v}}{dt} = m\vec{a}.$$

The Third Law: Action and reaction are always equal and opposite.

$\vec{F}_{12} = -\vec{F}_{21}$.

It is necessary to note that these forces do not counterbalance each other since they are applied to different objects.

All Newton's laws of are valid only in inertial frames of reference.

II. FORCES IN DYNAMICS

All the forces can be explained in terms of the following four fundamental interactions.

1. *Gravity* – the interaction between objects due to their masses.
2. *Electromagnetism* – the interaction between objects due to their charges.
3. *Strong nuclear interaction* – the interaction between subatomic particles with "color" (an abstract quantity that has nothing to do with human vision). This is the force that holds protons and neutrons together in the nucleus and holds quarks together in the protons and neutrons. It cannot be found outside of the nucleus.
4. *Weak nuclear interaction* – the interaction between subatomic particles with "flavor" (an abstract quantity that has nothing to do with human taste). This force, which is many times weaker than the strong nuclear interaction, is involved in certain forms of radioactive decay. It also cannot be found outside of the nucleus.

1. Gravitational interaction. Gravity. Weight

Every object in the universe attracts every other object in the universe with the gravitational force. The magnitude of the gravitational force between two objects is directly proportional to the product of their masses and inversely proportional to the square of the distance between their centers.

Gravitational force of attraction between objects of mass m_1 and m_2 separated by the distance r is given by *Universal Law of Gravitation* or *Newton's Law of*

Gravity

$$\vec{F} = \gamma \frac{m_1 \cdot m_2}{r^2} \cdot \vec{e}_r ,$$

where γ = 6,67·10^{-11} m^{3}s^{-2}kg^{-1} is the *gravitational constant,* $\vec{e}_r$ is a unit vector directed along a vector $\vec{r}$.

Free fall occurs whenever an object is acted upon by gravity only.

$$m\vec{a} = \gamma \frac{m \cdot M}{R^2} \cdot \vec{e}_R = m\vec{g} ,$$

where M and R are the mass and the radius of the Earth, respectively.

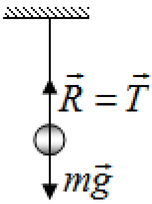
An object in free fall experiences the *acceleration due to gravity* $\vec{g}$.

$$\vec{a} = \vec{g} = \gamma \frac{M_{earth}}{R_{earth}^{\;2}} \cdot \vec{e}_R .$$

The acceleration due to gravity is independent on mass.

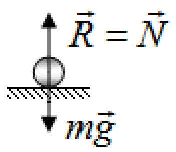
The acceleration due to gravity varies with location. On the Earth this value varies with latitude and altitude. The acceleration due to gravity is greater at the poles than at the equator and greater at sea level than atop Mount Everest. There are also local variations that depend upon geology. The value of 9,8 m/s^2 is thus merely a convenient average over the entire surface of the Earth.

If a body is suspended (1) or put on a base (2) the *gravity* $m\vec{g}$ is counterbalanced by force $\vec{R}$ (*reaction force*).



1). *Tension* $\vec{T}$: this reaction force is the force of a string or rope on an object to which the string or rope is attached. The direction of the tension is always along the rope or string and *away* from the surface of the object to which the rope or string is attached. Tension forces can only *pull* the objects they act on. It is important to remember that we always assume a non–stretching string or rope (unless explicitly told otherwise) so that the magnitude of the tension is constant along the string or rope. This enormously simplifies the mathematics of using the tension by adding important constraints to the solution of the problem. One of the most important constraints is that the length of the string or rope is constant.

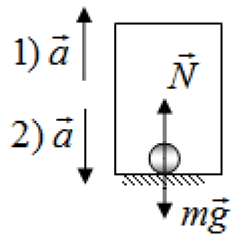
2). *Normal force* $\vec{N}$: these are the reaction forces that result from contact between two objects. Normal forces are always directed perpendicularly *away* from the surface which exerts the normal force. Their magnitude depends on some external agent which maintains the contact between objects. To determine their direction for any situation, note that a normal force can only *push* the object it acts on.



The *weight* $\vec{G}$ of an object is defined as the force acting on it due to gravitational pull, or gravity. So the weight of an object can be measured by attaching it to a spring–balance. In a motionless state $\vec{G} = m\vec{g}$. The motion breaks this equation.

$$m\vec{a} = m\vec{g} + \vec{N}. \quad \vec{N} = m(\vec{a} - \vec{g}).$$

$$\vec{G} = -\vec{N}, \text{ hence, } \vec{G} = m(\vec{a} - \vec{g}).$$



Then,

1) if vectors $\vec{g}$ and $\vec{a}$ are oppositely directed, $G = m(g-(-a)) = m(g+a) \Rightarrow$ *overload*;

2) if these vectors are identically directed, $G = m(g-a) \Rightarrow$ *apparent weightlessness*.

2. Spring forces. Hooke's law

Deformation is a change of the form and the size of an object. The force arising in an object under deformation (typically extension or compression) and returning it to its original shape when released (like a spring or elastic band), is the *spring force*. The deformation in this case is an *elastic deformation*.

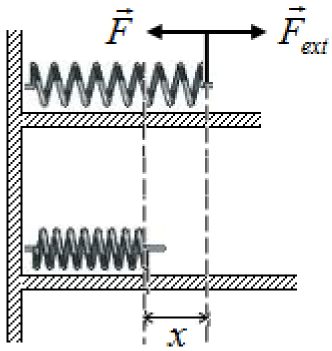
An elongation of the object is proportional to the magnitude of the external force.

$$x = \frac{1}{k} F_{ext}.$$

As $F_{ext} = -F$, then

$$x = -\frac{1}{k} F,$$

where F is the spring force, k is the spring constant, x is the amount by which the spring is stretched ($x > 0$) or compressed ($x < 0$).



Hooke's law is

$$F = -kx,$$

i.e., extension is directly proportional to force. The minus sign indicates that the direction of the force is *opposite* to the direction of pull or push on the spring.

$$\sigma = E\varepsilon.$$

The *stress* $\left(\sigma = F/S_{\perp}\right)$ applied to any solid is proportional to the *strain* $\left(\varepsilon = \Delta x/x\right)$ it produces within the elastic limit for that solid. The constant of that proportionality $\left(E\right)$ is the *Young modulus* for that substance.

3. Friction

Friction is the force between two surfaces in contact that resists their sliding along each other. These forces always *resist* the motion occurring or the motion that would occur if friction were not present. Friction is directed opposite to the direction of the relative motion or the intended direction of motion of either of the surfaces.

There are two types of friction: external (contact) and internal (viscosity). *External* friction arises at the interface of two adjoining surfaces (*sliding friction* and *rolling friction*) – *kinetic friction* or during attempts to cause such transition (*friction of rest or starting friction*) – *static friction*. External friction depends on the nature of the materials in contact and the smoothness of their surfaces and is independent on the area of contact. Starting friction is usually greater than sliding friction. Friction may be dry and with greasing.

It was Guillaume Amontons who first established a proportional relationship between friction force and the mutual pressure (or normal force) between the bodies in contact. The relationship when we divide friction force by normal force can identify the quotient (the *coefficient of friction*).

Amontons–Coulomb's law for friction is

$$F_{fr} = \mu N ,$$

where μ is the coefficient of sliding friction, N is the normal force pressing the surfaces together.

The friction force we have discussed so far acts when two surfaces are in contact. The force that tends to reduce the velocity of objects moving through air is very similar to the friction force; this force is *drag force*. The drag force acting on an object moving through air is given by

$$D = \frac{C \cdot \rho \cdot S}{2} \cdot v^2 ,$$

where S is the effective cross–sectional area of the object, ρ is the density of air, v is the speed of the object, C is a dimensionless drag coefficient that depends on the shape of the object; its value generally is in the range between 0,5 and 1,0. The direction of the drag force is opposite to the direction of the velocity.

Internal friction arises between stratums of liquid or stratums of gas and depends on a velocity: for rather small velocities $F_{fr} \sim v$, and for big velocities $F_{fr} \sim v^2$.

III. THE CONSERVATION OF LINEAR MOMENTUM

1. Basic definitions

Mechanical system is the material objects chosen for analyzing. The objects may interact with each other (*internal* forces) and with the objects outside the system (*external* forces).

Closed (isolated) system is the system on which no external forces act.

2. The law of conservation of linear momentum

Let the system consist of N particles. $\vec{F}_{ik}$ is the internal force with which *k*–particle acts on *i*– particle, $\vec{F}_i$ is the net external force acting on *i*– particle.

The equations of motion for all particles of the system are:

$$\dot{\vec{p}}_1 = \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14} + ... + \vec{F}_{1N} + \vec{F}_1$$

$$\dot{\vec{p}}_2 = \vec{F}_{21} + \vec{F}_{23} + \vec{F}_{24} + ... + \vec{F}_{2N} + \vec{F}_2$$

....

$$\dot{\vec{p}}_N = \vec{F}_{N1} + \vec{F}_{N2} + \vec{F}_{N3} + ... + \vec{F}_{N,N-1} + \vec{F}_N$$

$$\frac{d\vec{p}}{dt} = \frac{d(\vec{p}_1 + \vec{p}_2 + ... + \vec{p}_N)}{dt} = (\vec{F}_{12} + \vec{F}_{21}) + (\vec{F}_{13} + \vec{F}_{31}) + ... + (\vec{F}_{1N} + \vec{F}_{N1}) + \sum_{i=1}^{N} \vec{F}_i .$$

All sums in the brackets are equal to zero (according to Newton’s third law). Hence, $\vec{p}$ (the *total momentum of the system*) depends on the net external force.

The rate of change of linear momentum of a particle is equal to the net force acting on the object, and is pointed in the direction of the force.

$$\frac{d\vec{p}}{dt} = \sum_{i=1}^{N} \vec{F}_i .$$

If the net force acting on an object is zero (the closed system), its linear momentum is constant.

$$\sum_{i=1}^{N} \vec{F}_i = 0 \quad \Rightarrow \quad \vec{p} = const$$

This is the mathematical expression for the *law of conservation of linear momentum*: in any closed system, the vector sum of all momenta remains constant.

3. Center of mass of mechanical system and the theorem of its motion

Assume a system of N discrete masses each of mass is m_i, and its location is determined by positional vector $\vec{r}_i$. The position of the center of mass is defined by vector $\vec{r}_c$:

$$\vec{r}_C = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + ... + m_N\vec{r}_N}{m_1 + m_2 + ... + m_N} = \frac{1}{m}\sum_{i=1}^{N} m_i\vec{r}_i ,$$

where m is the total mass of the system.

Differentiating this equation with respect to time shows

$$\dot{\vec{r}}_C = \frac{1}{m}\sum_{i=1}^{N} m_i\dot{\vec{r}}_i = \frac{1}{m}\sum_{i=1}^{N} m_i\vec{v}_i = \frac{1}{m}\sum_{i=1}^{N} \vec{p}_i = \frac{\vec{p}}{m} .$$

Momentum of the system is equal to the product of the system total mass and the velocity of its center of mass

$$\vec{p} = m\vec{v}_C .$$

Once again differentiating this expression with respect to time we can obtain

$$\frac{d\vec{p}}{dt} = \frac{d(m\vec{v}_C)}{dt} = m\frac{d\vec{v}_C}{dt} = m\vec{a}_C = \sum_{i=1}^{N} \vec{F}_i .$$

This equation shows that the motion of the center of mass is only determined by the *external forces*. The forces exerted by one part of the system on other parts of the system are called *internal forces*. According to Newton's third law, the sum of all internal forces cancels out (for each interaction there are two forces acting on two parts: they are equal in magnitude but pointing in opposite direction and cancel if we take the vector sum of all internal forces).

The *theorem of center of mass motion*: the center of mass of a system of particles acts like a particle of mass *m*, and reacts like a particle when the system is exposed to external forces:

$$m\vec{a}_C = \sum_{i=1}^{N} \vec{F}_i .$$

This equation shows that when the net external force acting on the system is zero:

$$\sum_{i=1}^{N} \vec{F}_i = 0 \quad \Rightarrow \quad m\vec{a}_C = 0,$$

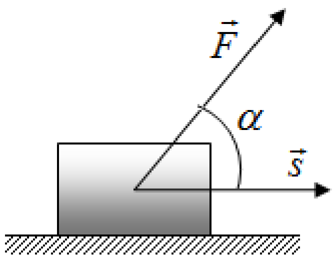
i.e., the velocity of the center of mass is constant. Consequently, the center of mass of a closed system moves uniformly in straight line or stays at rest.

Therefore, it is convenient to choose it as the origin of inertial frame of reference.

IV. THE CONSERVATION OF ENERGY

1. Work. Power

Let's assume that the constant force $\vec{F}$ acts on the object during its motion. Both the force and displacement are vectors that are not necessarily pointing in the same direction. The *work* done by the force $\vec{F}$ as it undergoes a displacement $d\vec{s}$ is defined as



$$dA = (\vec{F}, d\vec{s}) = F \cdot ds \cdot \cos\alpha = F_s ds .$$

where ds is a magnitude of displacement.

The work done by the force $\vec{F}$ is zero if the displacement is equal to zero or if $\alpha = 90^0$, i.e., the force is perpendicular to the displacement.

According to definition, the work is a scalar. The work done by the force can be positive or negative depending on α. If $\alpha < 90^0$, the work is positive; if $\alpha > 90^0$, the work is negative.

When a system does work on its environment, $A > 0$; that is, the total energy of the system *decreases*. Work is done *by* the system.

When the environment does work on a system, $A < 0$; that is, the total energy of the system *increases*. Work is done *on* the system.

When a variable force $\vec{F}(s)$ is acting on an object, the work is

$$A = \int_1^2 dA = \int_1^2 (\vec{F}, d\vec{s}) = \int_1^2 F_s ds .$$

where $\vec{F} = \vec{F}(s)$.

If several forces act on the object and $\vec{F}$ is the net force is

$$A = (\vec{F}, d\vec{s}) = (\sum_{i=1}^{N} \vec{F}_i, d\vec{s}) = (\vec{F}_1, d\vec{s}) + (\vec{F}_2, d\vec{s}) + ... + (\vec{F}_N, d\vec{s}) = A_1 + A_2 + ... A_N .$$

$$[A] = \text{Joule} = \text{J} = \text{kg} \cdot \text{m}^2 \cdot \text{s}^{-2}$$

According to definition, the *power* is the rate at which work is done. But it is also the dot product of force and velocity.

$$P = \frac{dA}{dt} = \frac{(\vec{F}, d\vec{s})}{dt} = (\vec{F}, \frac{d\vec{s}}{dt}) = (\vec{F}, \vec{v}).$$

$$[P] = \text{Watt} = \text{W} = \text{J/s} = \text{kg} \cdot \text{m}^2 \cdot \text{s}^{-3}.$$

2. Energy. Kinetic energy

The work is closely related to energy. The work causes a change in energy and energy characterizes the ability to do the work, i.e., work is a quantitative measure of changes in energy (*the Work–Energy Theorem*).

In mechanics, it is possible to define energy as an ability to do work and to consider it as the maximum work which any object can do under the given conditions. But generally it is not so. For example, thermal energy cannot be transformed completely into the work.

A system possesses energy if it has the ability to do work. Energy is a scalar quantity which is given meaning through calculation.

The *energy* is the uniform quantitative measure of various forms of substance motion; it is a measure of transition of substance motion from one form into another. Energy can exist in different forms.

If the force $\vec{F}$ acts on mass m changing its velocity from v_1 to v_2, the work done by this force is

$$dA = F_s ds = m \cdot \frac{dv}{dt} \cdot ds = m \cdot v \cdot dv.$$

According to work–energy theorem, the work causes a change in energy, therefore,

$$A = \int_1^2 dA = \int_{v_1}^{v_2} mvdv = m\int_{v_1}^{v_2} vdv = \left.\frac{mv^2}{2}\right|_{v_1}^{v_2} = \frac{mv_2^2}{2} - \frac{mv_1^2}{2} = W_{k2} - W_{k1} = \Delta W_k.$$

We obtained the expression for energy associated with motion and called the *kinetic energy:*

$$W_k = \frac{mv^2}{2}.$$

3. Potential energy

1. Let us find the work of gravity during the material point motion along a curvilinear path (acceleration due to gravity is nearly constant and height change is small compared with the distance between centers of the Earth and the particle).

$$\Delta A = mg \cdot \Delta s_k \cos\alpha = mg\Delta h_k$$

According to work–energy theorem, that work causes a change in energy

$$A = \sum_k \Delta A_k = mg \sum_k \Delta h_k = mgh = mg(h_1 - h_2) = mgh_1 - mgh_2 = W_{p1} - W_{p2} = -\Delta W_p .$$

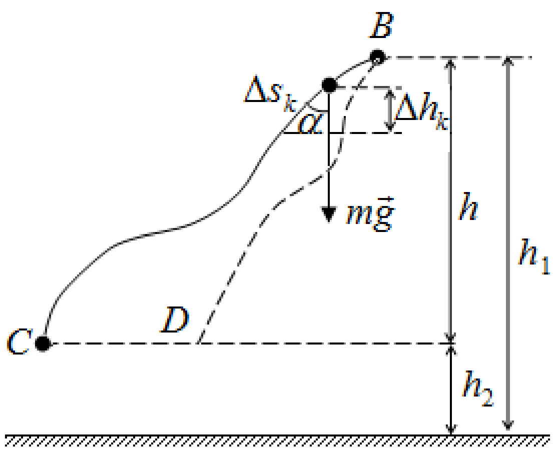
We obtained the expression for energy associated with position and called the *potential energy*.

The *potential energy due to gravitational force* is

$$W_p = mgh .$$

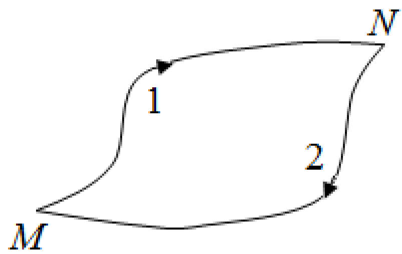
We see that in evaluating the work done by the force during the motion, no mention is made of the actual path taken by the particle.

A force is *conservative* if the work done by it on a particle that moves between two points is the same for all paths connecting these points; otherwise, it is *non–conservative.*



If a force is conservative, the *work* it does on a particle that moves *through a round trip* is zero.

Let's assume that the work done for the round trip from M to N and back to M is zero. This means that



$$A^1_{MN} + A^2_{NM} = 0 \text{ or } A^1_{MN} = -A^2_{NM}.$$

The work done by the force on each segment reverses sign if we reverse the direction

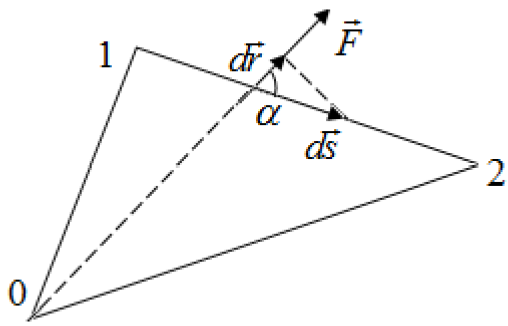
$$A^2_{MN} = -A^2_{NM}.$$

This relation than can be used to show that

$$A^1_{MN} = A^2_{MN}.$$

This is exactly what definition of conservative force states (the work done by the force acting on the object depends only on the initial and final positions of the object and not on the path taken).

2. The potential energy is closely connected to existence of field of forces (gravitational, electric). The field, in which a particle moves under the influence of a force that acts on the particle in such a way that it is always directed towards a single point (the center of force), is so–call *central–force field.* The magnitude of any force depends on the distance from this center. The force is directed either to the force center or from the center. An example of such a field is gravitational field of the Earth.



Let us find the work in central–force stationary (independent on time) field.

$$dA = \vec{F}(r)d\vec{s} = F(r)ds \cdot \cos\alpha = F(r)dr.$$

$$\left.\begin{aligned} F(r) &= \gamma \frac{m_1 \cdot m_2}{r^2} \\ F(r) &= k \frac{q_1 \cdot q_2}{r^2} \end{aligned}\right\} \quad \text{Put } \beta = \begin{cases} \gamma \cdot m_1 \cdot m_2 \\ k \cdot q_1 \cdot q_2 \end{cases}, \text{ then } F(r) = \frac{\beta}{r^2} \text{ and } dA = \frac{\beta}{r^2} dr.$$

Integrating along the path 1–2, we'll find the work done by force

$$A=\int_1^2 dA=\int_1^2 \frac{\beta}{r^2}dr=-\frac{\beta}{r}\bigg|_{r_1}^{r_2}=\frac{\beta}{r_1}-\frac{\beta}{r_2}=W_{p1}-W_{p2}.$$

The *potential energy of interaction* (*mutual potential energy*) is inversely proportional to the distance between the objects.

For gravitational field:

$$W_p=\gamma\frac{m}{r};$$

for electric field:

$$W_p=k\frac{q}{r}.$$

The potential energy is defined with the accuracy to constant of integrating as any integral. But the nature of potential energy is that the *zero point* is arbitrary; it can be set like the origin of a coordinate system. That is not to say that it is insignificant; once the zero of potential energy is set, then every value of potential energy is measured with respect to that zero. Another way of saying it is that it is the *change* in potential energy which has physical significance.

When we use the gravitational potential energy $W_p=mgh$, the assumption is usually made is that the zero of gravitational potential energy is on the surface of the earth and that the potential energy is proportional to the height above the earth's surface. This is an approximation which is only valid near the surface of the earth, but it is suitable for the common applications of gravitational potential energy. If you are in a room, it is logical to just call the floor the zero of gravitational potential energy, and measure the energy of an elevated object with respect to the floor.

When you use the more general form of the gravitational potential energy, including the fact that it drops off with distance from the earth, $F(r)=\gamma\frac{m_1\cdot m_2}{r^2}$, then the logic of the choice of zero potential is different. In this case, we generally choose the zero of gravitational potential energy at infinity, since the gravitational force approaches zero at infinity. This is a logical way to define the zero since the potential energy with respect to a point at infinity tells us the energy with which an object is bound to the earth. (This more general case is similar to what is done with the zero of electrical potential, since it is logical to define the zero of voltage far away from any charges).

3. Let us calculate the *potential energy due to elastic forces* (for example, for the spring).

The force exerted by a spring on a mass m can be calculated using Hooke's law $F = -kx$.

$$dA = \vec{F} \cdot d\vec{x} = (\vec{F} \parallel d\vec{x} \Rightarrow \cos\alpha = 1) = F \cdot dx = -kxdx,$$

$$A = \int_1^2 dA = -\int_{x_1}^{x_2} kxdx = -\frac{kx^2}{2}\Bigg|_{x_1}^{x_2} = \frac{kx_1^2}{2} - \frac{kx_2^2}{2} = W_{p1} - W_{p2}.$$

Potential energy due to elastic forces is

$$W_p = \frac{kx^2}{2}.$$

The potential energy of the spring in its relaxed position is defined as zero.

4. The law of conservation of energy

The *total mechanical energy* is the sum of the energy associated with motion and the energy associated with position, i.e., the sum of kinetic and potential energies:

$$W_{total} = W_k + W_p.$$

Let us consider the system consisting of particles exerting forces on each other and estimate the work done at the displacement from one position to another, accompanied by the modification of configuration of the system.

The work of external conservative forces is

$A_{12}^{'} = W_{p1}^{'} - W_{p2}^{'}$.

The work of internal conservative forces is

$A_{12}^{''} = W_{p1}^{''} - W_{p2}^{''}$.

The work of non–conservative forces is A_{12}^{*}.

The total work of all forces is expended on an increment in kinetic energy of a system:

$$(W_{p1}^{'} - W_{p2}^{'}) + (W_{p1}^{''} - W_{p}) + A_{12}^{*} = W_{k2} - W_{k1},$$

$$\left(W_{k2} + W'_{p2} + W''_{p2}\right) - \left(W_{k1} + W'_{p1} + W''_{p1}\right) = A_{12}^{*},$$

$$W_{total2} - W_{total1} = A_{12}^{*}.$$

In the absence of nonconservative forces $A_{12}^{*} = 0$, hence, $W_{total} = const$.

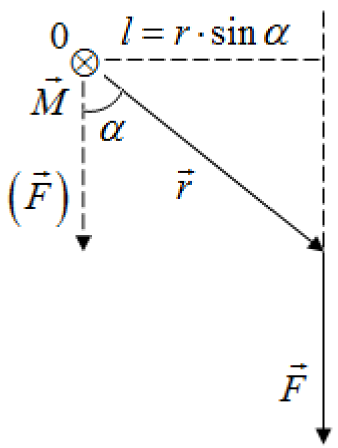
The total mechanical energy of the system where the conservative forces acts remains constant.

The law of conservation of energy is: energy may be transformed from one type into another type in an isolated system but it can not be created or destroyed; the total energy of a closed system always remains the same.

In the presence of non–conservative forces, mechanical energy is converted into internal energy or thermal energy.

V. CONSERVATION OF ANGULAR MOMENTUM

1. Torque. Couple of forces



Torque (*moment of a force*) is a measure of how much a force $\vec{F}$ acting on an object causes that object to rotate. The object rotates about a point, which we will call the *pivot point* (or *fulcrum*), and will label O. The distance from the pivot point to the point where the force acts is called the *moment arm*, and is denoted by

$\vec{r}$. Note that this distance, $\vec{r}$, is also a vector, and points from the axis of rotation to the point where the force acts. Torque is defined as

$$\vec{M} = \left[\vec{r}, \vec{F}\right],$$

The magnitude of torque is

$$M = r \cdot F \cdot \sin\alpha = F \cdot l .$$

The *arm of force l* is the length of the perpendicular dropped from a point O to a straight line along which the force acts.

Vector $\vec{M}$ is a pseudo vector since its direction gets out conditionally using the *right hand rule*: if we put our fingers in the direction of $\vec{r}$, and turn them to the direction of $\vec{F}$, the thumb points the direction of the torque.

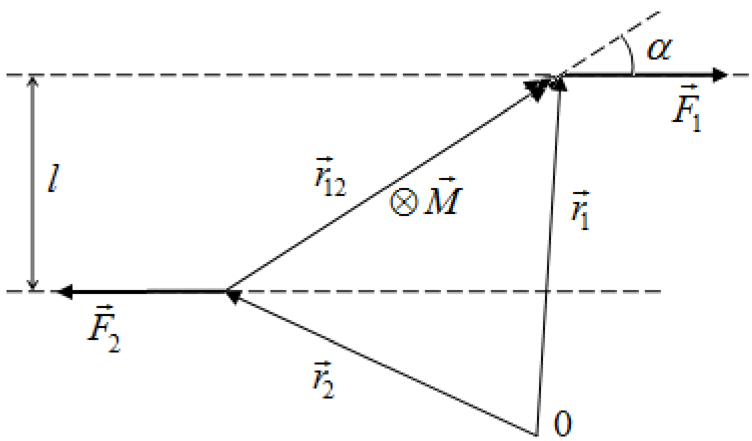
$$\left[M\right] = \mathrm{N} \cdot \mathrm{m} = \mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-2} .$$

The SI unit of torque is the Newton·meter, which is also a way of expressing a Joule (the unit for energy). However, torque is **not** energy. So, to avoid confusion, we will use the units N·m, and not J. The distinction arises because energy is a scalar quantity, whereas torque is a vector.

The projection of torque vector to an arbitrary axis z containing the point O is the *torque about the axis*:

$$\vec{M}_Z = \left[\vec{r}, \vec{F}\right]_Z .$$

A *force couple* (*coupled forces*) is two forces of equal magnitudes acting in opposite directions in the same plane but not same point. These two forces always have a turning effect, or moment, called a torque.



Perpendicular distance l between forces is an *arm of couple*.

Let's find the torque of the coupled forces about a point O.

$$\vec{M} = \left[\vec{r}_1, \vec{F}_1\right] + \left[\vec{r}_2, \vec{F}_2\right] = \left\{\left\{\vec{F}_1 = -\vec{F}_2\right\}\right\} = \left[\vec{r}_1, \vec{F}_1\right] - \left[\vec{r}_2, \vec{F}_1\right] = \left[\left(\vec{r}_1 - \vec{r}_2\right), \vec{F}_1\right] = \left[\vec{r}_{12}, \vec{F}_1\right];$$

$$\vec{M} = [\vec{r}_{12}, F_1];$$

$$M = r_{12} \cdot F \cdot \sin\alpha = F \cdot l.$$

The torque of the coupled forces does not depend on a choice of the point O.

Rotational equilibrium is analogous to *translational equilibrium,* where the sum of the forces is equal to zero. In rotational equilibrium, the sum of the torques is equal to zero**.** In other words, there is no net torque on the object.

For any system of particles the sum of the moments of all internal forces is equal to zero since they are force couples with "zero" arms.

$$\sum_i \left(\vec{M}_{inernal}\right)_i = 0.$$

2. Angular momentum

An *angular momentum* of a particle with respect to origin O is defined as

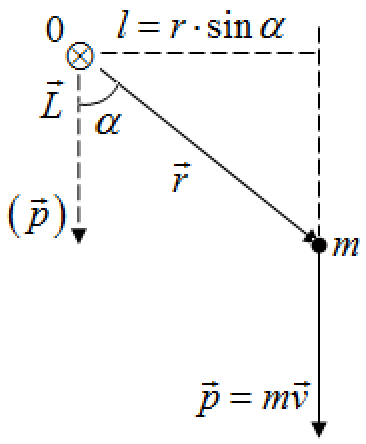
$$\vec{L} = [\vec{r}, \vec{p}],$$

where $\vec{r}$ is a position vector of the particle having linear momentum $\vec{p} = m\vec{v}$.

Its magnitude is

$$L = r \cdot p \cdot \sin\alpha = r \cdot m \cdot v \cdot \sin\alpha = mvl.$$

This definition implies that if the particle is moving directly away from the origin, or directly towards it, the angular momentum associated with this motion is zero. A particle will have a different angular momentum if the origin is chosen at a different location.



The projection of the vector $\vec{L}$ on any axis z containing the point O is the *angular momentum about the axis* z:

$$\vec{L} = [\vec{r}, \vec{p}]_Z.$$

$$[L] = \text{kg} \cdot \text{m}^2 \cdot \text{s}^{-1}.$$

The change in the angular momentum of the particle can be obtained by differentiating of the expression for $\vec{L}$:

$$\frac{d\vec{L}}{dt}=\frac{d}{dt}[\vec{r},m\vec{v}]=\left[\frac{d\vec{r}}{dt},m\vec{v}\right]+\left[\vec{r},m\frac{d\vec{v}}{dt}\right]=\left[\frac{d\vec{r}}{dt},m\vec{v}\right]+\left[\vec{r},\vec{F}\right]=\cancel{[\vec{v},m\vec{v}]}_0+\left[\vec{r},\vec{F}\right]=\vec{M}$$

$$\frac{d\vec{L}}{dt}=\vec{M}\ .$$

The rate of change of the angular momentum depends on the total torque (net moment of forces applied to the object).

3. The law of conservation of angular momentum

Let's consider a system, consisting of *k* particles, where both internal and external forces are applied. An angular momentum of this system with respect to the point O is the vector sum of the angular momenta of all particles of a system:

$$\vec{L}=\sum_k \vec{L}_k=\sum_k[\vec{r}_k,\vec{p}_k].$$

The derivative of this expression with respect to time is

$$\frac{d\vec{L}}{dt}=\sum_k\frac{d\vec{L}_k}{dt}.$$

For each of particles to which N_1 internal and N_2 external forces are applied, it is possible to write the equation

$$\frac{d\vec{L}_k}{dt}=\vec{M}=\sum_{i=1}^{N_1}\left(\vec{M}_i\right)_{inernal}+\sum_{j=1}^{N_2}\left(\vec{M}_j\right)_{external},$$

where $\sum_{i=1}^{N_1}\left(\vec{M}_i\right)_{inrnal}, \sum_{j=1}^{N_2}\left(\vec{M}_j\right)_{external}$ are the total torques of internal and external forces applied to the particle.

Therefore, the total angular momentum of the system is

$$\frac{d\vec{L}}{dt}=\sum_k\frac{d\vec{L}_k}{dt}=\sum_k\sum_{i=1}^{N_1}\left(\vec{M}_i\right)_{inernal}+\sum_k\sum_{i=1}^{N_1}\left(\vec{M}_i\right)_{external}=\vec{M}_{external}$$

Thus the rate of change of the angular momentum of a system depends on the total external torque.

In closed system $\vec{M}_{external}=0$, so, $\frac{d\vec{L}}{dt}=0$ and $\vec{L}=const$.

If no external forces act on a system of particles or if the external torque is equal to zero, the total angular momentum of the system is conserved. The net angular momentum of closed system remains constant, no matter what changes take place within the system. It is the law of *conservation of angular momentum*.

VI. MOMENT OF INERTIA. NEWTON'S 2ND LAW FOR ROTATION

1. Newton's 2nd law for rotation. Moment of inertia of a point mass

Let the point m move with acceleration $\vec{a}$ along a curvilinear path under the action of force $\vec{F}$. Using Newton's 2nd law $\vec{F} = m\vec{a}$ and relationship between linear and angular accelerations $\vec{a} = [\vec{\varepsilon}, \vec{r}]$, we can write the torque

$$\vec{M} = [\vec{r}, \vec{F}] = [\vec{r}, m\vec{a}] = [\vec{r}, m[\varepsilon, \vec{r}]] = m[\vec{r}, [\varepsilon, \vec{r}]] = m\left\{\vec{\varepsilon}(\vec{r}, \vec{r}) - \vec{r}\,\cancel{(\vec{r}, \vec{\varepsilon})}_{0}\right\} = mr^2\vec{\varepsilon} = I\vec{\varepsilon}$$

$$\vec{M} = I\vec{\varepsilon}.$$

This expression is said to be *Newton's 2nd law for rotation*.

The quantity

$$I = mr^2$$

tells us how the mass of the rotating body is located relatively the axis of rotation. This quantity is called the *moment of inertia* (or *rotational inertia*) of a *point mass*. It is the rotational analog of mass.

$$[I] = \text{kg} \cdot \text{m}^2.$$

3. Moment of inertia of a material body. The parallel axis theorem

For a set of point masses the moment of inertia is the sum of their momenta of inertia.

Moment of inertia of a composite object can be obtained by superposition of the moments of its constituents:

$$I = \sum_i \Delta m_i \cdot r_i^2$$

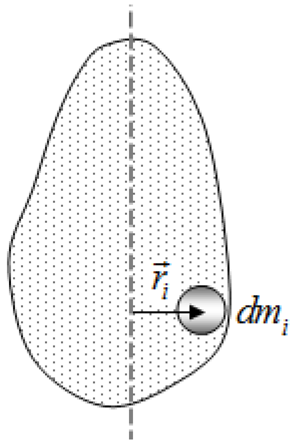
Since the moment of inertia of a point mass is defined by $I = mr^2$, the moment of inertia contribution of an infinitesimal mass element *dm* has the same form. This mass element is called differential element of mass and its moment of inertia is given by

$$dI = r^2 dm.$$

Note that the differential element of the moment of inertia dI must always be defined with respect to a specific rotation axis.

The sum over all these mass elements (an integral over all differential elements of mass) is a *moment of inertia of a solid*

$$I = \int r^2 dm.$$

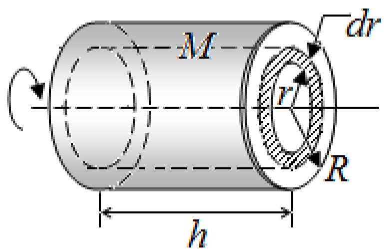


Usually, the mass element *dm* will be expressed in terms of the geometry of the object, so that the integration can be carried out over the object as a whole (for example, over a long uniform rod or uniform disc)

Having called this a general form, it is probably appropriate to point out that it is a general form only for axes which may be called "principal axes", a term which includes all axes of symmetry of objects. A *principal axis* may be simply defined as one about which no net torque is needed to maintain rotation at a constant angular velocity. The concept of moment of inertia for general objects about arbitrary axes is a much more complicated subject.

The calculation of moments of inertia

1. *Solid uniform cylinder* about its axis of symmetry going through its center of mass



Distribution of mass throughout the rigid body is characterized by density which is generally determined as

$$\rho = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V} = \frac{dm}{dV}.$$

The mass element dm can be expressed in terms of an infinitesimal radial thickness dr by $dm = \rho dV = \rho \cdot 2\pi rh \cdot dr$.

Substitution gives an integral

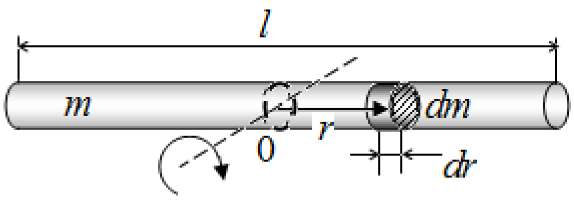
$$I = \int r^2 dm = \int_0^R r^2 \cdot \rho \cdot 2\pi rh \cdot dr = 2\pi\rho h \int_0^R r^3 dr = 2\pi\rho h \frac{r^4}{4}\bigg|_0^R = \frac{\pi\rho h R^4}{2} = \frac{mR^2}{2}.$$

For the cylinder (disk) of mass m and radius R: $I = \dfrac{mR^2}{2}$.

2. *Moment of inertia of uniform rod* with negligible thickness about its center of mass.

The moment of inertia calculation for a uniform rod involves expressing any mass element in terms of a distance element dr along the rod. Since the total length l has mass m, then m/l is the proportion of mass to length and the mass element can be expressed as

$$dm = \frac{m}{l} dr.$$

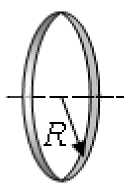


Integrating from $-l/2$ to $+l/2$ from the center includes the entire rod. The integral is

$$I = \int_{-\frac{l}{2}}^{\frac{l}{2}} r^2 \frac{m}{l} dr = \frac{m}{l} \cdot \frac{r^3}{3}\bigg|_{-\frac{l}{2}}^{\frac{l}{2}} = \frac{m}{3l}\left(\frac{l^3}{8} - \frac{-l^3}{8}\right) = \frac{ml^2}{12}.$$

For the uniform rod of mass m and length l:

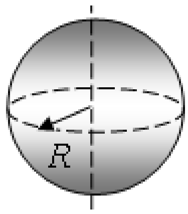
$$I = \frac{ml^2}{12}.$$



3. For the *thin hoop and hollow (thin-walled) cylinder* of mass m and radius R

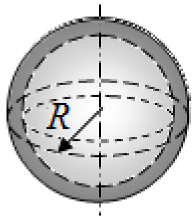
$$I = mR^2.$$

4. For the *sphere* of mass *m* and radius *R*:



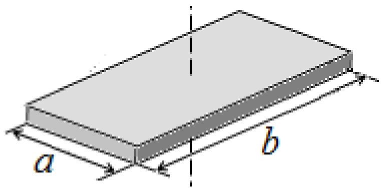
$$I = \frac{2}{5} mR^2 .$$

5. For the *thin-walled hallow sphere* of mass *m* and radius *R*:



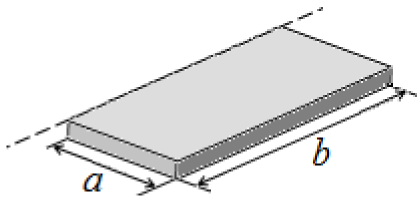
$$I = \frac{2}{3} mR^2$$

6. Rectangular plate, axis through center



$$I = \frac{m(a^2 + b^2)}{12}$$

7. Rectangular plate, axis along edge



$$I = \frac{ma^2}{3}$$

The parallel axis theorem (Steiner's theorem) states that the moment of inertia of an object about a specified axis of rotation equals the moment of inertia about a parallel axis through the centre of mass plus the mass of the object times the square of the distance between the axes,

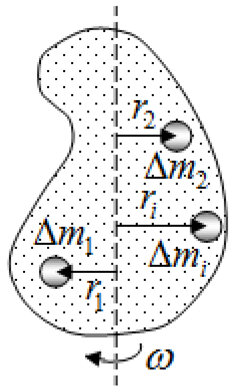
$$I_x = I + mx^2 .$$

The perpendicular axis theorem for planar objects states that the moment of inertia about axis perpendicular to the plane is the sum of the moments of inertia of two perpendicular axes trough the same point in the plane of the object.

$$I_Z = I_X + I_Y$$

The utility of this theorem goes beyond that of calculating momenta of strictly planar objects. It is a valuable tool in building up of the momenta of inertia of three dimensional objects such as cylinders by breaking them up into planar disks and summing the momenta of inertia of the composite disks.

4. Rotational kinetic energy. Work done by torque



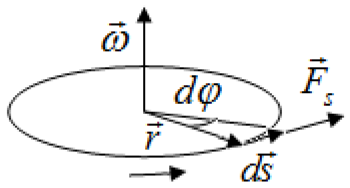
Let's consider the kinetic energy of a rotating object. If the angular velocity of a mass Δm_i is ω and its distance from the axis of rotation is r, the velocity of this mass is $v_i = \omega \cdot r_i$. Therefore, the rotational kinetic energy of the elementary mass is

$$(\Delta W_k)_i = \frac{\Delta m_i \cdot v^2}{2} = \frac{\Delta m_i \cdot \omega^2 \cdot r_i^2}{2}.$$

The rotational kinetic energy of the object about point O is sum of kinetic energy of all elementary masses: 5

$$W_k = \sum_i (\Delta W_k)_i = \frac{\Delta m_i \cdot \omega^2 \cdot r_i^2}{2} = \frac{\omega^2}{2} \sum_i \Delta m_i \cdot r_i^2 = \frac{I\omega^2}{2}.$$

The *rotational kinetic energy* is



$$W_k = \frac{I\omega^2}{2}.$$

Note that some moving objects may take part both in linear and rotational motions. The wheel of moving automobile turns around its axis, and the axis moves along parallel to the road. The kinetic energy of such an object is *the sum* of the kinetic energy due to linear motion and the kinetic energy due to rotational motion.

$$W_k = \frac{mv^2}{2} + \frac{I\omega^2}{2}.$$

If the external force $\vec{F}$ is applied to the body at the point at distance r from the axis of rotation and turns this body through an angle $d\varphi$, the *work done by the torque* is

$$dA = F_S ds = F_S r \cdot d\varphi = M_Z d\varphi = M_\omega d\varphi,$$

$$dA = M_\omega d\varphi.$$

5. Rotational–linear parallels

Transitional motion	Rotational motions
Distance s	Angular position φ
Linear velocity $v = \frac{ds}{dt}$	Angular velocity $\omega = \frac{d\varphi}{dt}$
Linear acceleration $a = \frac{dv}{dt}$	Angular acceleration $\varepsilon = \frac{d\omega}{dt}$
Mass m	Moment of inertia $I = mr^2$
Linear momentum $p = m\vec{v}$	Angular momentum $\vec{L} = I\vec{\omega}$
Force $\vec{F}$	Torque $\vec{M}$
Newton's 2nd law $\vec{F} = \frac{d\vec{p}}{dt} = m\vec{a}$	Newton's 2nd law for rotation $\vec{M} = \frac{d\vec{L}}{dt} = I\vec{\varepsilon}$
Kinetic energy $W = \frac{mv^2}{2}$	Kinetic energy $W = \frac{I\omega^2}{2}$
Work $dA = F_S ds = F_v ds$	Work $dA = M_{\omega} d\varphi$
Power $P = Fv$	Power $P = M\omega$