NATIONAL TECHNICAL UNIVERSITY
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DEPARTMENT OF PHYSICS

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STUDY GUIDE

“OSCILLATIONS AND WAVES”

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Chapter 1. OSCILLATIONS

Oscillations are periodic changes of any quantity. *Mechanical oscillations (vibrations)* are the motions that repeat themselves over and over again. Oscillations are the periodic motions of the material point or material object.

The *types of oscillations*:

1. *Free (characteristic, natural)* oscillations take place in systems which begin to move after a kick or after upsetting a balance. There are *undamped (continuous, persistent)* oscillations and *damped (convergent, decaying)* oscillations which take place in systems without or with energy loss, correspondingly.

2. *Forced (constrained)* oscillations occur in systems exposed to action of external periodic force.

I. FREE UNDAMPED OSCILLATIONS

1. Simple harmonic motion and its characteristics

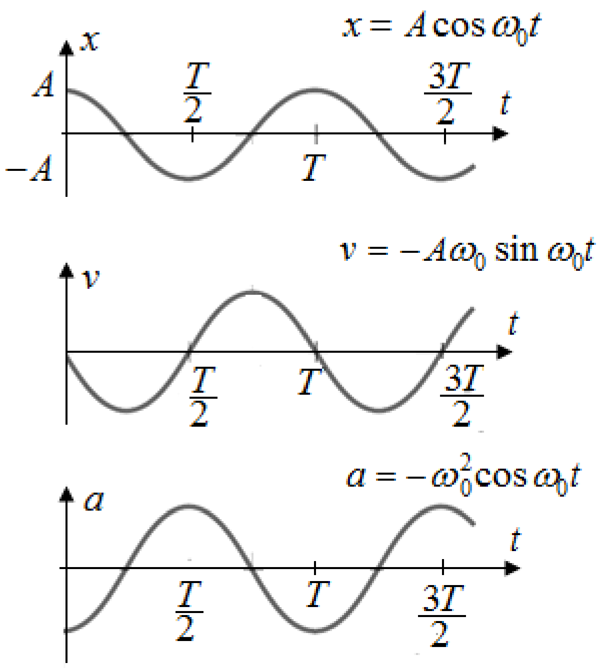
The physical system making oscillations about an equilibrium position is an *oscillator.*

The simple type of oscillations occurring under the law of cosine or sine is *simple harmonic motion* (SHM) and the system in this case is called a *simple harmonic oscillator (SHO).*

The *equation of free undamped simple harmonic motion* is

$$x = A\cos(\omega_0 t + \alpha),$$

where x is the magnitude changing with time periodically (for mechanical oscillations it is the *displacement* of a point from a position of the equilibrium), A is the *amplitude* (the maximum distance from the equilibrium, or the maximum displacement of particle executing SHM), t is time, $(\omega_0 t + \alpha)$ is a *phase* of oscillations, ω_0 is an *own (natural) angular (circular) frequency*, α is *an initial* *phase, or phase constant, or epoch* (it gives information about the initial position).



The time required for one complete vibration (complete to-and-fro movement) is called the *period* (T). The *frequency* (ν) is the number of oscillations completed by particle executing SHM per second.

During one period the phase of oscillation changes by 2π

$$\omega_0\left(t+T\right)+\alpha=\left(\omega_0 t+\alpha\right)+2\pi .$$

Therefore, $\omega_0 T = 2\pi$ and $T=\dfrac{1}{\nu}=\dfrac{2\pi}{\omega}$.

Differentiating the equation of SHM with respect to time t, we get the *velocity*

$$v=\frac{dx}{dt}=\dot{x}=-A\omega_0\sin(\omega_0 t+\alpha)=-v_{\max}\sin(\omega_0 t+\alpha)=A\omega_0\cos(\omega_0 t+\alpha+\frac{\pi}{2}) .$$

In addition, if $x=A\cos(\omega_0 t+\alpha)$, $\cos(\omega_0 t+\alpha)=\dfrac{x}{A}$ and

$$\sin\omega_0 t=\pm\sqrt{1-\left(\frac{x}{A}\right)^2}=\pm\frac{\sqrt{A^2-x^2}}{A},$$

then the velocity v of the particle at position x is

$$v=-A\omega_0\sin(\omega_0 t+\alpha)=\mp\omega_0\sqrt{A^2-x^2} .$$

Differentiating the equation of SHM again with respect to time t, we get *acceleration*

$$a=\frac{dv}{dt}=\frac{d^2x}{dt^2}=\ddot{x}=-A\omega_0^2\cos(\omega_0 t+\alpha)=-a_{\max}\cos(\omega_0 t+\alpha)=$$

$$=A\omega_0^2\cos(\omega_0 t+\alpha+\pi)=-\omega_0^2 x,$$

$$\ddot{x}=a=-\omega_0^2 x .$$

This is a linear second order homogeneous differential equation which solution is the equation of free undamped oscillations.

The *force* acting on the vibrating object is

$$F = ma = m\ddot{x} = -mA\omega_0^2 \cos(\omega_0 t + \alpha) = -F_{\max} \cos(\omega_0 t + \alpha) = -m\omega_0^2 x = -kx.$$

This force is proportional to displacement (*k* is a coefficient of proportionality, in case of a spring pendulum, this is the *spring constant*) and is always directed to a position of an equilibrium. An example of such force is an elastic force. Any other force, inelastic by nature, but satisfying the relation $F = -kx$ is said to be *quasi-elastic force*.

Two forms of energy are involved during vibrations. In standard example of a mass on a spring they are the *potential energy* stored in the spring and the *kinetic energy* of moving mass.

$$W_p = \frac{kx^2}{2} = \frac{m\omega_0^2 A^2 \cos^2\left(\omega_0 t + \alpha\right)}{2}$$

$$W_k = \frac{mv^2}{2} = \frac{m\dot{x}^2}{2} = \frac{m\omega_0^2 A^2 \sin^2\left(\omega_0 t + \alpha\right)}{2}$$

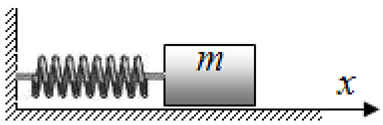
The *total energy* of a simple harmonic oscillator

$$W_{total} = W_k + W_p = \frac{m\omega_0^2 A^2}{2}.$$

The total energy of the particle does not depend on time t and displacement x. Thus total energy of a particle executing SHM remains constant. Besides, it is worthy of note that the energy of SHO is proportional to A^2.

2. Pendulums

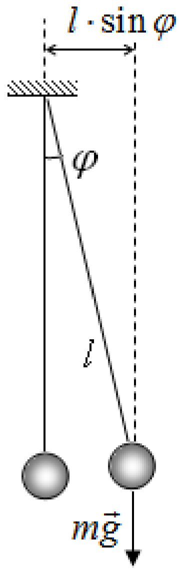
1. *A spring pendulum* is a system consisting of a mass *m* on a spring which mass can be neglected in comparison with a mass *m*. The equation of motion of this system according to Newton's 2nd law is $m\ddot{x} = -kx$, or $\ddot{x} + \omega_0^2 x = 0$, where angular frequency and period are:



$$\omega_0^2 = k/m, \qquad T = 2\pi\sqrt{\frac{m}{k}},$$

The solution of this differential equation is

$$x = A\cos(\omega_0 t + \alpha).$$

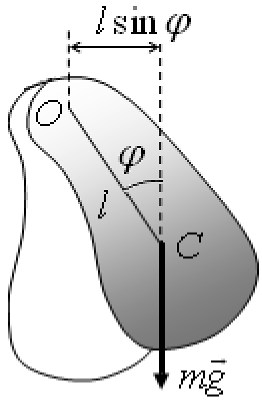


2. A *simple pendulum* is a material point of mass m attached to the end of long weightless inextensible string. Another end of it is attached to fixed point. If the mass is displaced slightly, it oscillates to-and -fro along the arc of a circle in a vertical plane. The equation of motion of this system according to the Newton's 2nd law for rotation is $I\varepsilon = M$, or $ml^2\ddot{\varphi} = -mgl \cdot \sin\varphi$. For the case of small-amplitude oscillations this equation is transformed into $\ddot{\varphi} + (g/l)\varphi = 0$. A solution of it is the function

$$\varphi = \varphi_{\max}\cos(\omega_0 t + \alpha),$$

where $\varphi_{\max}$ is amplitude of oscillations, i.e., the greatest angle through which the pendulum deviates, and angular frequency and the period of oscillations are:

$$\omega_0^2 = g/l, \qquad T = 2\pi\sqrt{\frac{l}{g}}.$$



3. A *physical* (*compound*) *pendulum* is a rigid body oscillating around a horizontal axis passing through the *point of suspension* located above its center of mass. The equation of motion of this system is $I\ddot{\varphi} = -mgx \cdot \sin\varphi$ (I is a moment of inertia of a pendulum about an axis passing through its center of suspension). For the case of small-amplitude oscillations the solution to the equation $\ddot{\varphi} + (mgx/I)\varphi = 0$ is

$$\varphi = \varphi_{\max}\cos(\omega_0 t + \alpha),$$

where angular frequency and period of oscillations are:

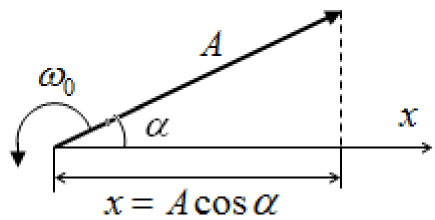
$$\omega_0^2 = \frac{mgx}{I}, \qquad T = 2\pi\sqrt{\frac{I}{mgx}} = \sqrt{\frac{L}{g}}.$$

$\dfrac{I}{mx} = L$ is the length of an equivalent simple pendulum.

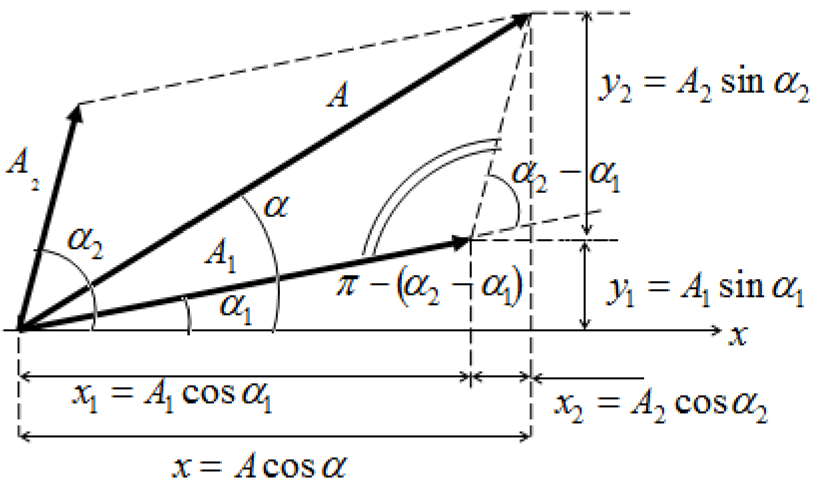
3. Superposition of two SHMs of the same direction and frequency (unidirectional oscillations).

Two vibrations of the same directions and frequency are determined as $x_1 = A_1\cos(\omega_0 t + \alpha_1)$ and $x_2 = A_2\cos(\omega_0 t + \alpha_2)$.

Their analytical addition is complicated procedure; therefore, an elegant mathematical representation to describe harmonic motion may be suggested for its simplification.



We have seen that the phase of the SHM increases linearly with time as the vibration is in progress. The displacement x at any time is proportional to the cosine of this phase. Therefore, this motion can be generated by letting a radius vector of length A rotate anticlockwise uniformly, by projecting the end point onto x-axis. The actual motion, $x = A\cos(\omega_0 t + \alpha)$, is the projection of the radius vector A onto this axis.



Let us use the vector diagram for adding two unidirectional oscillations of the same frequency:

$x_1 = A_1\cos(\omega_0 t + \alpha_1)$

$x_2 = A_2 \cos(\omega_0 t + \alpha_2)$.

As a result of their superposition, we obtain the resulting simple harmonic motion of the same angular frequency ω_0

$x = A\cos(\omega_0 t + \alpha)$,

which amplitude A may be computed according to the cosine law

$$A^2 = A_1^2 + A_2^2 - 2A_1A_2\cos\left[\pi - \left(\alpha_2 - \alpha_1\right)\right] = A_1^2 + A_2^2 + 2A_1A_2\cos\left(\alpha_2 - \alpha_1\right),$$

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\Delta\alpha},$$

and the initial phase α is

$$tg\alpha = \frac{A_1\sin\alpha_1 + A_2\sin\alpha_2}{A_1\cos\alpha_1 + A_2\cos\alpha_2}.$$

Special cases:

1. $\Delta\alpha = 0$, $\Rightarrow$ $\cos\Delta\alpha = 1$. $A = |A_1 + A_2|$
2. $\Delta\alpha = \pm\pi$, $\Rightarrow$ $\cos\Delta\alpha = -1$. $A = |A_1 - A_2|$.
3. $\Delta\alpha = \pm\frac{\pi}{2}$, $\Rightarrow$ $\cos\Delta\alpha = 0$. $A = \sqrt{A_1^2 + A_2^2}$.

4. Superposition of oscillations at right angle

Two simple harmonic motions of same frequency ω_0 having displacements in two perpendicular directions act simultaneously on a particle

$$\begin{cases} x = A\cos\omega_0 t \\ y = B\cos(\omega_0 t + \alpha). \end{cases}$$

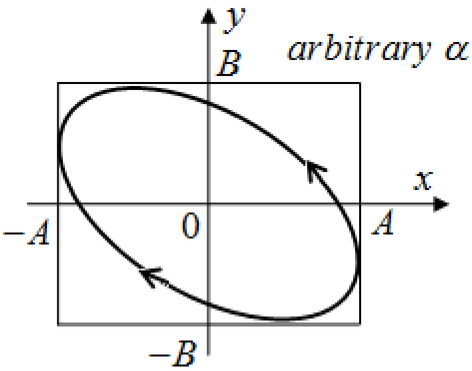
This system of equations determines the coordinates of the vibrating particle in the xy-plane. These expressions are the equation of ellipse in parametrical form. Let us exclude the parameter t and obtain the equation of the path in the classical canonical form.

From the first equation $\cos\omega_0 t = \dfrac{x}{A}$, and $\sin\omega_0 t = \pm\sqrt{1-\cos^2\omega_0 t} = \pm\sqrt{1-\dfrac{x^2}{A^2}}$.

Expand $\cos(\omega_0 t+\alpha)$ of the second equation and substitute $\cos\omega_0 t$ and $\sin\omega_0 t$:

$$\frac{y}{B} = \cos(\omega_0 t+\alpha) = \cos\omega_0 t\cdot\cos\alpha - \sin\omega_0 t\cdot\sin\alpha = \frac{x}{A}\cos\alpha \mp \sqrt{1-\frac{x^2}{A^2}}\cdot\sin\alpha .$$

$$\frac{y}{B} - \frac{x}{A}\cos\alpha = \mp\sqrt{1-\frac{x^2}{A^2}}\cdot\sin\alpha .$$



Square both parts of this equation and obtain

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} - \frac{2xy}{AB}\cos\alpha = \sin^2\alpha .$$

This expression represents the general equation of an ellipse.

Thus, as a result of superposition of two simple harmonic oscillations in mutually perpendicular directions the trajectory of the particle motion is an ellipse whose axes are rotated relative to the coordinate axes.

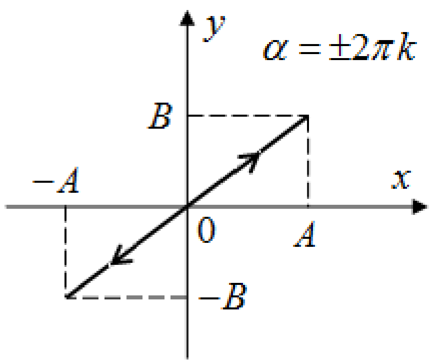
Special cases (in dependence on the initial phase difference α):

1. $\alpha = 0; \pm 2\pi; \pm 4\pi ... = \pm 2\pi k$, where $k = 1, 2, 3,...$ $\cos\alpha = 1$, $\sin\alpha = 0$, and

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} - \frac{2xy}{AB} = 0 ,$$

$$\left(\frac{x}{A} - \frac{y}{B}\right)^2 = 0 , \text{ or}$$

$$y = \frac{B}{A}x .$$



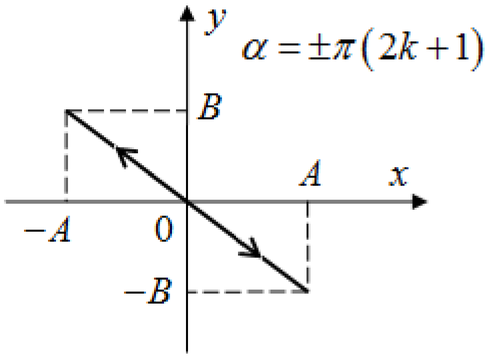
The particle vibrates simple harmonically along the straight line 1 (see figure) with a frequency ω_0 and the amplitude equaled to $\sqrt{A^2+B^2}$.

2. $\alpha = \pm\pi; \pm 3\pi; \pm 5\pi ... = \pm\pi(2k+1)$, where $k = 0, 1, 2, 3,...$ $c\ \alpha\alpha s = -$, $\sin\alpha = 0$, and

$$\frac{x^2}{A^2}+\frac{y^2}{B^2}+\frac{2xy}{AB}=0,$$

$$\left(\frac{x}{A}+\frac{y}{B}\right)^2=0,\ \text{or}$$

$$y=-\frac{B}{A}x.$$

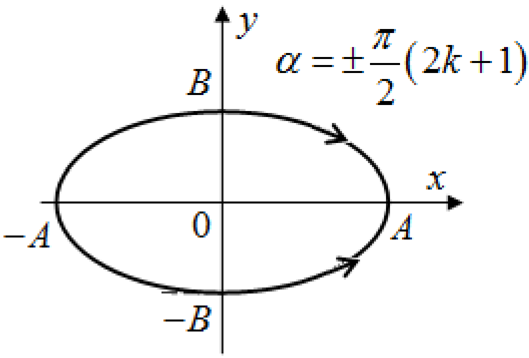


This equation represents a straight line with slope equaled to $\left(-\frac{B}{A}\right)$. The particle vibrates along the straight line 2 see figure) with frequency ω_0 and the amplitude $\sqrt{A^2+B^2}$.

3. $\alpha=\pm\frac{\pi}{2};\pm\frac{3\pi}{2};\pm\frac{5\pi}{2}...=\pm\frac{\pi}{2}(2k+1)$,

where $k=0,1,2,3,...$ $\cos\alpha=$, $\sin\alpha=\pm1$, and

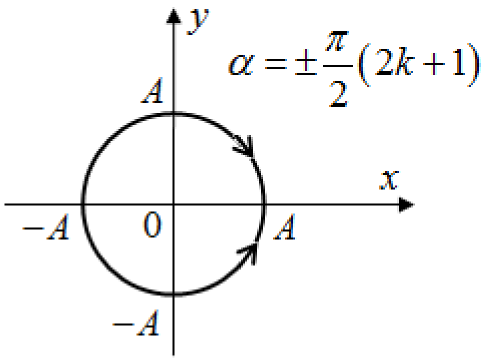
$$\frac{x^2}{A^2}+\frac{y^2}{B^2}=1.$$



The trajectory of the particle is an ellipse with semi major and semi minor axes A and *B*, coinciding with *x*- and *y*- axes, respectively, i.e., an ellipse reduced to the principal axes

Semi-axes of this ellipse are equal to the amplitudes of oscillations *A* and *B.*

If $A=B$, we get the equation of the circle $x^2+y^2=B^2$ with radius B .



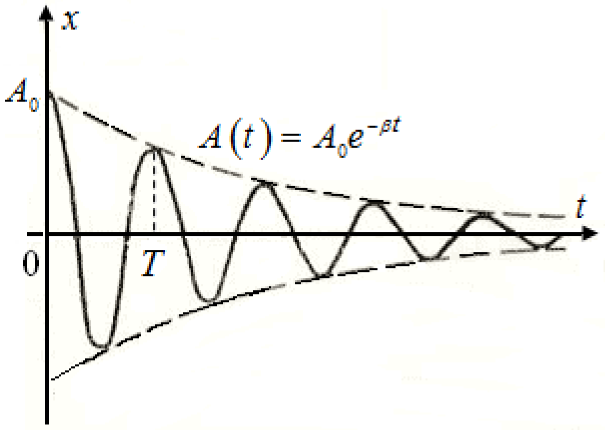
The direction of rotation (clockwise or anticlockwise) of the particle may be obtained from the *x*- and *y*- motions of the particle when *t* is increased gradually

II. FREE DAMPED OSCILLATIONS

The energy of vibration has been losing off a vibrating system for various reasons, for example, such as the conversion to heat via friction. If free oscillations take place in a system with friction, $F_{fr} = -rv$, where r is the *coefficient of friction (resistance of medium)*, the equation of the motion is

$$ma = -kx - F_{fr},$$

$$m\ddot{x} = -kx - r\dot{x}.$$



Reduce this equation to the following form

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0,$$

where $\beta = r/2m$ is the *damping coefficient*, and $\omega_0 = \sqrt{k/m}$ is the natural frequency, i.e., the frequency of the system without friction. This motion equation may be solved in the form: t

$$x = A_0 e^{-\beta t}\cos(\omega t + \alpha),$$

This is the *equation of free damped vibrations*. The effect, called *damping*, will cause the vibrations to decay exponentially unless energy is pumped into the system to replace the loss.

The *amplitude of damped oscillations* decreases exponentially as t advances according to

$$A(t) = A_0 e^{-\beta t}.$$

An *angular frequency of damped oscillations* is

$$\omega = \sqrt{\omega_0^2 - \beta^2}.$$

A *conditional period* of damped oscillations is

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\omega_0^2 - \beta^2}}.$$

A *damping decrement* (or *decrement* of motion) is the ratio of two successive amplitudes of the damped oscillation, corresponding to the instants of time distinguished by the period, i.e.,

$$D = \frac{A(t)}{A(t+T)} = \frac{A_0 e^{-\beta t}}{A_0 e^{-\beta(t+T)}} = e^{\beta T}.$$

A *logarithmic damping decrement* (*damping factor*) is the logarithm of the ratio of two amplitudes which separated by one period

$$\delta = \ln D = \beta T.$$

A *quality factor*

$$Q = \frac{\pi}{\delta}.$$

A quality factor is defined as the number of cycles required for the energy to fall off by factor of 535. (The origin of this obscure numerical factor is $e^{2\pi}$, where e= 2,71828… is the base of natural logarithms). The terminology arises from the fact that friction is often considered a bad thing, so a mechanical device that can vibrate for many oscillations before it loses a significant fraction of its energy would be considered a high-quality device).

The quality factor measures the rate at which the energy decays. Since decay of the amplitude is represented by $A = A_0 e^{-\beta t}$, the decay of energy is proportional to

$$A^2 = A_0^2 e^{-2\beta t},$$

and may written

$$W = W_0 e^{-2\beta t},$$

where W_0 is the energy value at $t = 0$.

If $\frac{W_0}{W} = e^{2\pi}$ (according the definition of the quality factor) and

$$\frac{W_0}{W} = \frac{W_0}{W_0 \cdot e^{-2\beta t}} = e^{2\beta t} = e^{2\beta NT} = e^{2N\delta},$$

$$e^{2\pi} = e^{2N\delta}.$$

$$2\pi = 2N\delta.$$

$$Q = N = \frac{\pi}{\delta}.$$

Therefore, the quality factor is equal to the number of the cycles through which the damped system oscillates as its energy decays by factor $e^{2\pi}$.

A *relaxation time* (*modulus of decay*) τ is the period of time taken for the amplitude to decay to $1/e = 0.368$ of its original value A_0, i.e. it falls off by factor e.

$$e = \frac{A_0}{A_\tau} = \frac{\cancel{A_0}}{\cancel{A_0} e^{-\beta\tau}} = e^{\beta\tau},$$

$$\beta\tau = 1,$$

$$\tau = \frac{1}{\beta} = \frac{2m}{r}.$$

III. FORCED OSCILLATIONS

If the damped oscillator is driven by external periodic force, the equation of motion is

$$m\ddot{x} = -kx - r\dot{x} + F_0 \cos\omega t .$$

Put $\beta = r/2m$, $\omega_0 = \sqrt{k/m}$ and obtain the equation

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = (F_0/m)\cos\omega t ,$$

which solution is the *equation of forced oscillations:*

$$x = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} \cos(\omega t - arctg\, \frac{2\beta\omega}{\omega_0^2 - \omega^2}),$$

where ω is the frequency of a driving force.

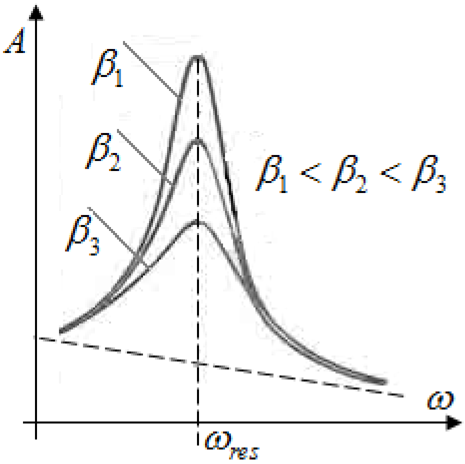
Resonance is the tendency of a vibrating system to respond most strongly to a driving force whose frequency is close to its own (natural) frequency of vibration. Resonance is the dramatic increase in amplitude of a periodic system that occurs when the driving frequency equals to the natural frequency of the system. In an undamped system, the amplitude will be infinite.

For calculation of a resonance frequency let us find the maximum of function $A(\omega)$ by differentiation of a function

$$A(\omega)=\frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}},$$

$$\frac{dA}{d\omega}=\frac{F_0}{m}\left(-\frac{1}{2}\right)\left[\left(\omega_0^2-\omega^2\right)^2+4\beta^2\omega^2\right]^{-\frac{3}{2}}\left[2\left(\omega_0^2-\omega^2\right)(-2)\omega+8\beta^2\omega\right]=0$$

$$4\omega\left(-\omega_0^2+\omega^2+2\beta^2\right)=0.$$



This cubic equation has three roots:

1) $\omega=0$ - the minimal amplitude;

2) $\omega=-\sqrt{\omega_0^2-2\beta^2}$ - physical nonsense;

3) $\omega=\sqrt{\omega_0^2-2\beta^2}$ - this is *resonance frequency*.

Substituting a resonance frequency in the expression for amplitude of forced vibrations we obtain <u>resonance amplitude</u>:

$$A_{res}=\frac{F_0/m}{2\beta\sqrt{\omega_0^2-\beta^2}}.$$

Chapter 2. WAVES

I. TRAVELLING WAVES

1. Classifications of wave motion

Wave motion in a medium is a collective phenomenon that involves local interactions among the particles of the medium. Waves are characterized by: 1) disturbance in space and time; 2) a transfer of energy from one place to another; 3) a non-transfer of material of the medium. A mechanical wave is a disturbance in the equilibrium positions of matter, the magnitude of which is dependent on location and on time.

Oscillations having arisen in one place of an elastic medium are transmitted to the next particles due to interaction and spread with the velocity $\vec{v}$. Therefore, a wave is a disturbance that propagates through the medium. Waves transfer energy, momentum and information, but not mass. So we can classify waves by medium where they spread: mechanical waves (matter is the medium), electromagnetic waves (electric and magnetic fields are the media), etc.

The line indicating a direction of propagation of waves is a *beam*.

Classification of waves by *orientation*:

1. *Transverse waves*: displacements are *perpendicular* to the direction of propagation. All electromagnetic waves are transverse. This includes light. *Crest*: a point of maximal displacement in the positive direction (upward displacement). *Trough*: a point of maximal displacement in the negative direction. Transverse mechanical waves are spread only in *solids*.

2. *Longitudinal waves*: displacements are *parallel* to the direction of propagation. Sound is a longitudinal wave. *Compression* is a region where the medium is under compression. *Rarefaction* is a region where the medium is under tension. Longitudinal waves are spread in *solids, liquids* and *gases*.

Distinction between transverse and longitudinal waves

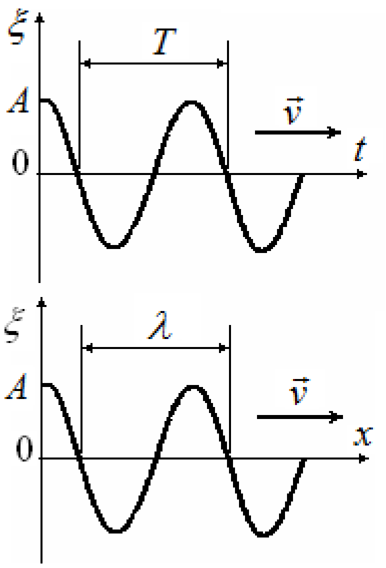
Transverse wave	*Longitudinal waves*
1. In a transverse wave, the motion of the particles of an elastic medium are perpendicular to the direction of the propagating wave.	1. In a longitudinal wave, the motion of the particles of an elastic medium are parallel to the direction of the propagating wave.
2. In a propagating transverse wave, crests and troughs are produced.	2. In a propagating longitudinal wave, condensations and rarefactions are produced.
3. The wavelength is equal to the distance between two consecutive crests or troughs.	3. The wavelength is the distance between the centers of two consecutive compressions or rarefactions.
4. Transverse waves cannot propagate through a gas and liquid.	4. Longitudinal waves can propagate through any material medium.
5. Transverse waves can be polarized.	5. Longitudinal waves cannot be polarized.

If the source or origin of the wave oscillates at a frequency ν, then each point in the medium concerned oscillates at the same frequency. The geometrical place of points which the wave has reached up to a certain time is a *wave-front*. It is unique for the given wave process. The geometrical place of the points that are in the same phase is a *wave surface*. There are a lot of such surfaces.

Waves spread out in all directions from every point on the disturbance that created them. If the disturbance is small, we may consider it as a single point. Then depending on the form of medium where the waves are spread, they may be *one-dimensional* (linear), *two-dimensional* (circular) and *three-dimensional* (spherical) waves. Classifying waves by duration we have to note *episodic* (or pulse) waves when disturbance is momentary and sudden and *periodic* (or harmonic) waves when the disturbance repeats at regular intervals.

Classifying waves by propagation we distinguish *travelling* (progressive) and *standing* waves. *Travelling* waves are the waves that propagate in medium. *Standing* waves don't go anywhere, but they have regions where the disturbance of the wave is quite small, almost zero. These locations are called *nodes*. There are also regions where the disturbance is quite intense, greater than anywhere else in the medium, called *antinodes*.

2. Mathematical description of travelling wave



Waves propagate at a finite speed v (the *wave speed*) that depends upon the type of wave, the composition and the state of the medium. The wave profile moves along at speed of wave. If a snapshot is taken of a travelling wave, it is seen that it repeats at equal distances. The repeat distance is the wavelength λ. *Wavelength* is the distance between any point of a periodic wave and the next point corresponding to the same portion of the wave measured along the path of propagation. Wavelength is measured between adjacent points in phase. If one point is taken, and the profile is observed as it passes this point, then the profile is seen to repeat at equal interval of time. The repeat time is the *period* T. Otherwise, the period is the time between successive cycles of a repeating sequence of events. *Frequency* (υ) is the number of cycles of a repeating sequence of events in a unit interval of time. Frequency and period are reciprocals (or inverses) of one another:

$T = 1/\upsilon$.

The SI units:

$[\lambda] = \text{m}$, $[T] = \text{s}$, $[\upsilon] = \text{Hertz} = \text{Hz} = 1/\text{s} = \text{s}^{-1}$.

Suppose that the wave moves from left to right and that a particle at the origin 0 vibrates according to the equation

$$\xi(0,t) = A\cos(\omega t + \alpha),$$

where t is the time and $\omega = 2\pi\upsilon = 2\pi/T$.

For a particle at the distance x from 0 to the right, the phase of the vibration will be different from that at 0, as the time $\tau = x/v$ is necessary for wave to get to the point (x,t). Hence the displacement of any particle at distance x from the origin is given by

$$\xi(x,t) = A\cos\left[\omega(t-\tau)+\alpha\right] = A\cos\left(\omega t - \omega\frac{x}{v} + \alpha\right) = A\cos(\omega t - kx + \alpha),$$

Equation of a *plane-travelling* (or plane-progressive) *wave* is

$$\xi(x,t) = A\cos\left(\frac{2\pi}{T}t - \frac{2\pi}{\lambda}x + \alpha\right),$$

where $k = \frac{\omega}{v} = \frac{2\pi}{vT} = \frac{2\pi}{\lambda}$ is the *wave number*.

Amplitude (A) is the maximum magnitude of a periodically varying quantity. Amplitude has the unit of the quantity that is changing (in this case – the displacement).

Phase $\left(\frac{2\pi}{T}t - \frac{2\pi}{\lambda}x + \alpha\right)$ is the stage of development of a periodic process. Two points on a wave with the same phase have the same quantity of disturbance (displacement, etc.) and rate of change of disturbance (velocity, etc.). Phase is an angular quantity. Adjacent points *in phase* are separated by one complete cycle. Adjacent points *out of phase* are separated by half a cycle.

The SI unit of phase is the radian.

Generally for a wave propagating in three dimensions, the displacement at point given by position vector $\vec{r}$ at time t is

$$\xi(\vec{r},t) = A\cos\left(\omega t - \vec{k}\vec{r} + \alpha\right),$$

where $\vec{k}$ is the *wave vector*. Its magnitude is equal to the wave number.

The *speed* and *acceleration* of the vibrating particle we can obtain as the first and the second derivatives of $\xi(x,t)$ with respect to time:

$$\dot{\xi} = \frac{\partial \xi}{\partial t} = -A\omega \cdot \sin(\omega t - kx + \alpha),$$

$$\ddot{\xi} = \frac{\partial^2 \xi}{\partial t^2} = -A\omega^2 \cdot \cos(\omega t - kx + \alpha).$$

For two points on the beam of plane-travelling wave separated from each other by the distance Δx (*path-length difference or path-length shift*) the *difference in phase (phase shift)* is

$$\Delta\varphi = \frac{2\pi}{\lambda}\Delta x.$$

3. Wave equation

The wave equation is the linear uniform partial differential equation of the second order describing the propagation of the wave in a medium.

Let's receive this equation using the equation of a wave

$$\xi(\vec{r},t) = A\cos\left(\omega t - \vec{k}\vec{r}\right).$$

Having twice differentiated this equation with respect to time we obtain

$$\frac{\partial \xi}{\partial t} = -A\omega \cdot \sin(\omega t - kr),$$

$$\frac{\partial^2 \xi}{\partial t^2} = -A\omega^2 \cdot \cos(\omega t - kr) = -\omega^2 \cdot \xi.$$

Therefore,

$$\xi = -\frac{1}{\omega^2} \cdot \frac{\partial^2 \xi}{\partial t^2}.$$

Write the equation of a wave as

$$\xi(x,y,z,t) = A\cos\left(\omega t - k_x x - k_y y - k_z z\right),$$

and find the second partial derivates of it with respect to x:

$$\frac{\partial \xi}{\partial x} = Ak_x \sin\left(\omega t - k_x x - k_y y - k_z z\right),$$

$$\frac{\partial^2 \xi}{\partial x^2} = -Ak_x^2 \cos\left(\omega t - k_x x - k_y y - k_z z\right) = -k_x^2 \cdot \xi .$$

We can obtain the second derivatives of the equation of a wave with respect to *y* and *z* in a similar way:

$$\frac{\partial^2 \xi}{\partial y^2} = -Ak_y^2 \cos\left(\omega t - k_x x - k_y y - k_z z\right) = -k_y^2 \cdot \xi ,$$

$$\frac{\partial^2 \xi}{\partial z^2} = -Ak_z^2 \cos\left(\omega t - k_x x - k_y y - k_z z\right) = -k_z^2 \cdot \xi ,$$

$$\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} + \frac{\partial^2 \xi}{\partial z^2} = \Delta \xi = -\left(k_x^2 + k_y^2 + k_z^2\right)\xi = -k^2 \cdot \xi = k^2 \cdot \frac{1}{\omega^2} \cdot \frac{\partial^2 \xi}{\partial t^2} = \frac{1}{v^2} \cdot \frac{\partial^2 \xi}{\partial t^2} .$$

We, therefore, obtain the *wave equation (*Δ is Laplacian)

$$\Delta \xi = \frac{1}{v^2} \cdot \frac{\partial^2 \xi}{\partial t^2} .$$

For a wave propagating in one-dimensional case, we have

$$\frac{\partial^2 \xi}{\partial x^2} = \frac{1}{v^2} \cdot \frac{\partial^2 \xi}{\partial t^2} .$$

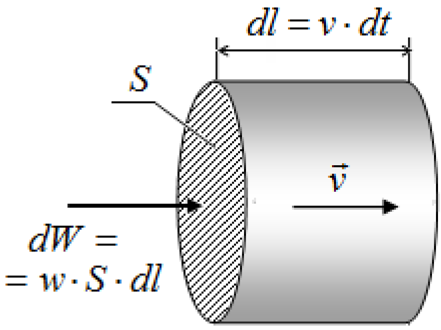
4. Energy transferred by a wave. Umov's vector

Waves, propagating in a medium, transfer energy from one place to another. This energy consists of the kinetic energy of vibrating particles and the potential energy of the deformed areas of the medium. The energy that is transferred by a wave through some area per unit of time is called an *energy flux* through this surface

$$\Phi = \frac{dW}{dt} .$$

$$\left[\Phi\right] = \mathrm{J} \cdot \mathrm{s}^{-1} = \mathrm{W} .$$

Energy flux density or *intensity* (I) of a wave at a place is the energy per second flowing through one square meter held normally at that place in the direction along which the wave travels, i.e., the intensity of any wave is the time averaged rate at which it transmits energy per unit area through some region of space.



$$I = \frac{dW}{S \cdot dt}.$$

$$[I] = \mathrm{J \cdot m^{-2} \cdot s^{-1}} = \mathrm{W \cdot m^{-2}}.$$

Suppose the displacement of wave-front area moving at a speed v for the time Δt is l. If w is *energy density* (i.e. average energy of the particles in unit volume), then the energy transferred through some area per time Δt is $\Delta W = w \cdot S \cdot v \cdot \Delta t$, and the intensity of a wave is

$$I = \frac{w \cdot S \cdot v \cdot \Delta t}{S \cdot \Delta t} = w \cdot v.$$

In the vector form $\vec{I} = w \cdot \vec{v}$.

$\vec{I}$ is *Umov's vector.* It is perpendicular to a wave-front, and indicates the direction of wave propagation. Its magnitude is equal to intensity (energy flux density).

If the energy of each particle is $\frac{m\omega^2 A^2}{2}$ and the number of particles in unit volume is *n*, then $w = n \cdot \frac{m\omega^2 A^2}{2} = \frac{\rho\omega^2 A^2}{2}$, and

$$I = \frac{\rho\omega^2 A^2}{2} \cdot v,$$

where ρ is the density of the medium.

II. STANDING WAVES

1. Superposition of waves. Interference

The *superposition principle*: when two or more waves travel simultaneously through the same medium, each wave proceeds independently as though no other waves were present, and the resultant displacement of any particle is the vector sum of the displacements that the individual waves acting alone would give.

If the oscillations stipulated by separate waves at each point of the medium have a constant phase difference, the waves are the *coherent waves*, i.e., waves whose parts are in phase with each other. When two or more coherent waves of the same frequency overlap, the phenomenon of *interference* occurs.

Consider two waves travelling from two closely situated sources that excite oscillations at the point considerably distant from them. These oscillations are the composition of two unidirectional oscillations. Thus, the amplitude is

$$A^2 = A_1^2 + A_2^2 + 2A_1A_2\cos\Delta\varphi.$$

Intensity I of waves is proportional to A^2, therefore,

$$I = I_1 + I_2 + 2\sqrt{I_1I_2}\cos\Delta\varphi.$$

If $A_1 = A_2$, then $I_1 = I_2$, consequently,

$$I = 2I_0 + 2I_0\cos\Delta\varphi.$$

Let's calculate the intensity at the points where the waves come with such path-length differences for which the cosines of a phase difference are:

1. $$\left.\begin{array}{l} \cos\Delta\varphi = 1; \quad \Delta\varphi = \pm 2\pi k \\ \Delta\varphi = \dfrac{2\pi}{\lambda}\Delta = \pm 2\pi k; \quad \Delta = \pm k\lambda = \pm 2k\dfrac{\lambda}{2} \end{array}\right\} I = 4I_0$$

Constructive interference ($I = I_{max}$)

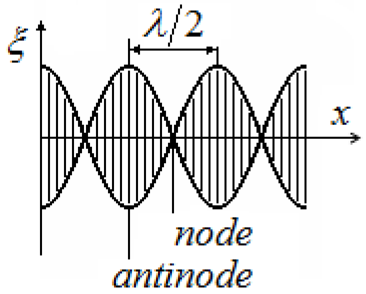
2. $$\left.\begin{array}{l} \cos\Delta\varphi = -1; \; \Delta\varphi = \pm(2k+1)\pi \\ \Delta\varphi = \dfrac{2\pi}{\lambda}\Delta = \pm(2k+1)\pi; \quad \Delta = \pm(2k+1)\dfrac{\lambda}{2} \end{array}\right\} \quad I = 0;$$

Destructive interference ($I = I_{min}$)

Interference is the phenomenon of redistribution of intensity at the superposition of coherent waves.

2. Standing waves

Sometimes when you vibrate a string, or cord, or chain, or cable it's possible to get it to vibrate in such a manner that you're generating a wave, but the wave doesn't propagate. It just sits there vibrating up and down in place. Such a wave is called a *standing wave*.



Standing waves can be formed under a variety of conditions, but they are easily demonstrated in a medium which is finite or bounded. A phone cord begins at the base and ends at the handset. Other simple examples of finite media are a guitar string (it runs from fret to bridge), a drum head (it's bounded by the rim), the air in a room (it's bounded by the walls), the water in lake (it's bounded by the shores), or the surface of the Earth (although not bounded, the surface of the Earth is finite). In general, standing waves can be produced by any two identical waves travelling in opposite directions that have the equal wavelength. In a bounded medium, standing waves occur when a wave meets its reflection. The interference of these two waves produces a resultant wave that does not appear to move.

The equations of two plane-travelling waves propagating along an axis x in opposite directions are:

$$\xi_1 = A\cos(\omega t - kx + \alpha_1) \text{ и } \xi_2 = A\cos(\omega t + kx + \alpha_2).$$

Their superposition is

$$\xi = \xi_1 + \xi_2 = 2A\cos\left(kx + \frac{\alpha_2 - \alpha_1}{2}\right) \cdot \cos\left(\omega t + \frac{\alpha_2 + \alpha_1}{2}\right).$$

Let's choose a reference point on axis x so that $\alpha_2 - \alpha_1 = 0$ and on an axis t so that $\alpha_2 + \alpha_1 = 0$.

Then the *equation of a standing wave* is

$$\xi = \left|2A\cos kx\right| \cdot \cos\omega t .$$

At each point of the standing wave the simple harmonic vibrations take place at the same frequency as the frequency of adding waves. The *amplitude* of the standing wave $\left|2A\cos kx\right|$ depends on coordinate:

1) At the positions which coordinates satisfy the condition $kx = \frac{2\pi}{\lambda}x = \pm\pi m$ the amplitude is $\left|2A\cos kx\right| = \left|2A\right|$. *Antinodes* or loops are the positions $x_{antinodes} = \pm m\frac{\lambda}{2}$ where amplitude reaches the maximum magnitude;

2) At positions which coordinates satisfy the condition $kx = \frac{2\pi}{\lambda}x = \pm(2m+1)\frac{\pi}{2}$ the amplitude is $\left|2A\cos kx\right| = 0$ (this is minimum magnitude). These positions $x_{nodes} = \pm\left(m + \frac{1}{2}\right)\frac{\lambda}{2}$ on a standing wave are *nodes*.

There is no transfer of energy in the standing wave: the total energy of oscillations of each element of volume of the medium limited by the adjacent node and antinode does not depend on time. It only periodically transform from a kinetic energy concentrated basically close to antinode into a potential energy of elastically deformed medium (near the node). Lack of energy transfer is the result of the fact that two identical waves travelling in opposite directions transfer an equal energy.

PROBLEMS

Problem 1

If a particle undergoes SHM $x = 0.2\sin 2\pi t$ *(m) , what is the total distance it travels in one period? Find the angular frequency and period of oscillation.*

Solution

The particle would travel four times the amplitude: from $x = 0$ to $x = A$; then to $x = 0$, then to $x = -A$, and to $x = 0$. So the total distance is equal to $4A = 4 \cdot 0.2 = 0.8$ m.

The angular velocity is $\omega_0 = 2\pi$, and the period of oscillations is

$$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{2\pi} = 1\text{s}.$$

Problem 2

A body oscillates with the simple harmonic motion according to the equation $x = 6\cos\left(3\pi t + \frac{\pi}{3}\right)$ *(m). Calculate the displacement, the velocity, the acceleration, and the phase at the time t = 2 s. Find also the angular velocity, frequency, and the period of motion.*

Solution

The phase is

$$3\pi t + \frac{\pi}{3}\bigg|_{t = 2s} = 3\pi \cdot 2 + \frac{\pi}{3} = \frac{19\pi}{3} = 19.9 \text{ rad.}$$

The displacement is

$$x = 6\cos\left(3\pi t + \frac{\pi}{3}\right)\bigg|_{t=2s} = 6\cos\left(3\pi\cdot 2 + \frac{\pi}{3}\right) = 6\cos\left(\frac{19\pi}{3}\right) = 3\,\text{m}.$$

The velocity is

$$v = \frac{dx}{dt} = -6\cdot 3\pi\cdot\sin\left(3\pi t + \frac{\pi}{3}\right)\bigg|_{t=2s} = -18\pi\sin\left(\frac{19\pi}{3}\right) = -49\,\text{m/s}.$$

The acceleration is

$$a = \frac{dv}{dt} = -6\cdot(3\pi)^2\cdot\cos\left(3\pi t + \frac{\pi}{3}\right)\bigg|_{t=2s} = -54\pi^2\cos\left(\frac{19\pi}{3}\right) = -266.5\ \text{m/s}^2$$

The angular velocity is $\omega_0 = 3\pi$ (rad/s).

Since $\omega_0 = 2\pi\upsilon_0$, the frequency is $\upsilon_0 = \frac{\omega_0}{2\pi} = \frac{3\pi}{2\pi} = 1.5\,\text{Hz}$.

The period $T = \frac{2\pi}{\omega_0} = \frac{1}{\upsilon_0} = \frac{2}{3}\,\text{s}$.

Problem 3

For what part of the period the oscillating point displaces by the half of amplitude, if it started from the equilibrium position?

Solution

Let us choose the trigonometric function for the oscillations description taking into account the given data. The point starts from the equilibrium position, then its displacement on the time instant t = 0 is equal to x = 0, therefore, the equation of oscillating motions is

$$x = A\sin\frac{2\pi}{T}t.$$

If the displacement of a point from a position of the equilibrium is $x = \frac{A}{2}$, than substituting it in the equation of oscillations, we obtain

$$\frac{A}{2} = A\sin\frac{2\pi}{T}t,$$

$$\sin\frac{2\pi}{T}t = \frac{1}{2}.$$

$$\frac{2\pi}{T}t = \arcsin\frac{1}{2} = \frac{\pi}{6}.$$

Hence, the required time is $t = \frac{T}{12}$.

Problem 4

A point oscillates according to the dependence $x = 5\cos\omega_0 t$ (m), where $\omega_0 = 2$ s^{-1}. Find the acceleration of the point when its velocity is equal to 8 m/s.

Solution

The velocity and the accelerations of the vibrating point depends on time as

$$v = \dot{x} = -A\omega_0 \sin\omega_0 t,$$

$$a = \ddot{x} = -A\omega_0^2 \cos\omega_0 t.$$

From the first equation

$$\sin\omega_0 t = -\frac{v}{A\omega_0}.$$

Since we are not to find an instant of time when the velocity is 8 m/s (we have to find nothing but the magnitude of acceleration at this time), we'll calculate only the magnitude of $\cos\omega_0 t$, because the equation for acceleration determination comprises just this trigonometric function. According to the trigonometric identity $\sin^2 \omega t + \cos^2 \omega t = 1$,

$$\cos \omega_0 t = \sqrt{1 - \sin^2 \omega_0 t} = \sqrt{1 - \left(v / A\omega_0 \right)^2} .$$

Consequently,

$$a = -A\omega_0^2 \cos \omega_0 t = -A\omega_0^2 \sqrt{1 - \left(v / A\omega_0 \right)^2} \text{, and}$$

$$a = -5 \cdot 2^2 \sqrt{1 - \left(8/5 \cdot 2 \right)^2} = -12 \,\text{m/s}^2.$$

Problem 5

The maximum speed and acceleration of a particle executing simple harmonic motion are 10 cm/s and 50 cm/s. Find the positions of the particle when the speed is 8 cm/s, if ($x(0) = 0$.

Solution

The maximum speed and accelerations are equal to

$$\begin{cases} v_{\max} = A\omega_0, \\ a_{\max} = A\omega_0^2. \end{cases}$$

Divide the second equation by the first and obtain

$$\frac{a_{\max}}{v_{\max}} = \frac{A\omega_0{}^2}{A\omega_0} = \omega_0 = \frac{50}{10} = 5 \,\text{rad/s}.$$

From the first equation the amplitude of the oscillations is

$$A = \frac{v_{\max}}{\omega_0} = \frac{0.1}{5} = 0.02 \,\text{m}.$$

If $x(0) = 0$, the equation of particle motion is $x = A \sin \omega_0 t$, and its speed depends on time as

$$v = A\omega_0 \cos \omega_0 t .$$

Therefore,

$$\cos\omega_0 t = \frac{v}{A\omega_0} = \frac{0.08}{0.02\cdot 5} = 0.8,$$

and

$$\sin\omega_0 t = \pm\sqrt{1-\cos^2\omega_0 t} = \pm\sqrt{1-0.8^2} = \pm 0.6.$$

Then the desired position is

$$x = A\sin\omega_0 t = \pm 0.02\cdot 0.6 = \pm 0.012\,\text{m}.$$

Problem 6

The maximum velocity of oscillating point is 10 cm/s, and its maximum acceleration is 100 m/s^2. Find the angular frequency and the amplitude of oscillations.

Solution

The equations that describe the simple harmonic motion are

$$x = A\cos(\omega_0 t+\alpha),$$

$$v = -A\omega_0\sin(\omega t+\alpha) = -v_{\max}\sin(\omega_0 t+\alpha),$$

$$a = -A\omega_0^2\cos(\omega_0 t+\alpha) = -a_{\max}\cos(\omega_0 t+\alpha).$$

Then the maximum magnitudes of displacement, velocity and acceleration are

$$\begin{cases} x_{\max} = A, \\ v_{\max} = A\omega, \\ a_{\max} = A\omega^2. \end{cases}$$

Dividing the third equation of system by the second equation we obtain

$$\frac{a_{\max}}{v_{\max}} = \frac{A\omega_0^{\ 2}}{A\omega_0} = \omega_0, \quad \text{and} \quad \omega_0 = 10\,\text{s}^{-1}.$$

The period and the amplitude of oscillations are, respectively,

$$T = \frac{2\pi}{\omega_0} = 0,2\pi = 0,628\,\text{s}, \qquad A = \frac{v_{\max}}{\omega_0} = \frac{0,1}{10} = 0,01\,\text{m}.$$

Problem 7

The equation of motion of a particle started at $t = 0$ *is given by* $x = 5\sin\left(20t + \frac{\pi}{3}\right)$*(cm). When does the particle (a) first come to rest; (b) first have zero acceleration; and (c) first have maximum speed?*

Solution

(a) If the speed of the particle is described by equation

$$v = \frac{dx}{dt} = 5 \cdot 20 \cdot \cos\left(20t + \frac{\pi}{3}\right),$$

the at $v = 0$

$$\cos\left(20t + \frac{\pi}{3}\right) = 0.$$

It gives

$$20t + \frac{\pi}{3} = \frac{\pi}{2}.$$

$$20t = \frac{\pi}{6}.$$

$$t = \frac{\pi}{120} = 0.026.$$

(b) The acceleration of the particle is

$$a = \frac{dv}{dt} = -5 \cdot 20^2 \cdot \sin\left(20t + \frac{\pi}{3}\right).$$

$$\sin\left(20t + \frac{\pi}{3}\right) = 0,$$

Firstly after the beginning of the oscillation process acceleration becomes zero when phase is equal to π:

$$20t + \frac{\pi}{3} = \pi,$$

$$t = \frac{\pi}{30} = 0.105\,\text{s}.$$

(c) $v = 5 \cdot 20 \cdot \cos\left(20t + \frac{\pi}{3}\right)$

Speed is maximum when $\cos\left(20t + \frac{\pi}{3}\right) = 1$, i.e.,

$$20t + \frac{\pi}{3} = \pi.$$

$$t = \frac{\pi}{30} = 0.105\,\text{s}.$$

Note, that we obtained the same result as in (b). It means that when the particle passes the equilibrium position it moves at its maximum speed and has zero acceleration.

Problem 8

At t = 0, the displacement of the point x(0) in a linear oscillator is -8.6 cm, its velocity v(0)= -0.93 m/s and its acceleration a(0)= +48 m/s^2. What are the angular frequency ω and the frequency υ? What is the phase constant? What is the amplitude of the motion?

Solution

The displacement of the particle is given by

$$x(t) = A\cos(\omega_0 t + \alpha).$$

Hence,

$x(0) = A\cos\alpha = -8.6$ cm = - 0.086 m,

$v(0) = -\omega_0 A\sin\alpha = 0.93\text{m/s}$,

$a(0) = -\omega_0^2 A\cos\alpha = 48\,\text{m/s}^2$.

Thus,

$$\omega = \sqrt{-\frac{a(0)}{x(0)}} = \sqrt{\frac{48}{0.086}} = 23.62\,\text{rad/s},$$

$$\upsilon = \frac{\omega}{2\pi} = \frac{23.62}{2\pi} = 3.76\,\text{Hz}.$$

$$\frac{v(0)}{x(0)} = \frac{-\omega_0 \cancel{A}\sin\alpha}{\cancel{A}\cos\alpha} = -\omega_0 \cdot \tan\alpha\,,$$

or,

$$\tan\alpha = -\frac{v(0)}{\omega x(0)} = -\frac{0.93}{23.62 \cdot 0.086} = -0.458\,.$$

Hence $\alpha = 155.4°$ and 335.4° in the range $0 \le \alpha \le 2\pi$. We shall see below how to choose between the two values.

$$A = \frac{x(0)}{\cos\alpha} = \frac{-0.086}{\cos\alpha}.$$

The amplitude of the motion is a positive constant. So, α = 335.4 ° cannot be correct phase, as $\cos 334.5° = 0.909$. We must therefore have $\alpha = 155.4°$, for which $\cos 155.4° = -0.909$:

$$A = \frac{-0.086}{-0.909} = 0.0946\,\text{m}.$$

Problem 9

A point moves with simple harmonic motion whose period is 4 s. If it starts from rest at a distance 4.0 cm from the centre of its path, find the time that elapses *before it has described 2 cm and the velocity it has then acquired. How long will the point take to reach the centre of its path?*

Solution

Amplitude of the oscillations is $A=4\,\text{cm}$ and time period is $T=4\,\text{s}$. An angular velocity is $\omega=2\pi/T=2\pi/4=\pi/2\,\text{rad/s}$. The distance from the centre of the path is $x=4-2=2\,\text{cm}$. Since $x=A\cos\omega t$, we have $\cos\omega t=x/A=2/4=0.5$, $\omega t=\pi/3$.

The time is $t=\dfrac{\pi}{3\omega}=\dfrac{\pi\cdot 2}{3\cdot\pi}=\dfrac{2}{3}\,\text{s}$.

Velocity is $v=\omega\sqrt{A^2-x^2}=\dfrac{\pi}{2}\sqrt{4^2-2^2}=\pi\sqrt{3}$ cm/s.

At the centre of the path $x=0$ and $0=A\cos\omega t$, $\cos\omega t=0$, $\omega t=\pi/2$,

$$t=\frac{\pi}{2\cdot\omega}=\frac{\pi\cdot 2}{2\cdot\pi}=1\ \text{s}.$$

Problem 10

A particle executing SHM on a straight line has a velocity of 4 cm/s when at a distance of 3 m from the mean position, and 3 m/s, when at a distance of 4 m from it. Find the time it takes to travel 2 m from the positive extremity of its oscillation.

Solution

The velocity of the particle executing SHM is derivative of displacement x with respect to time t, consequently, if displacement is $x=A\sin\left(\omega_0 t+a\right)$, then

$$v=\frac{dx}{dt}=A\omega_0\cos\left(\omega_0 t+\alpha\right)=A\omega_0\sqrt{1-\sin^2\left(\omega_0 t+\alpha\right)}=\omega_0\sqrt{A^2-A^2\sin^2\left(\omega_0 t+\alpha\right)},$$

$$v=\omega_0\sqrt{A^2-x^2}\ .$$

Substituting the given data in this expression gives

$$\begin{cases} 4^2 = \omega_0^2\left(A^2 - 3^2\right) \\ 3^2 = \omega_0^2\left(A^2 - 4^2\right) \end{cases}$$

On solving them, we get:

$A = 5\,\text{m}, \;\; \omega_0 = 1\,\text{rad/s}.$

For movement from positive extremity through a distance 2 m, the displacement (from the mean point) is

$x = 5 - 2 = 3$ m.

We use the equation

$x = A\cos\omega_0 t$,

because the oscillation process begins from the extremity point, i.e., at $t = 0$, $x = A$.

Using this expression, we obtain

$$\cos\left(\omega_0 t\right) = \frac{x}{A} = \frac{3}{5} = 0.6,$$

$\omega_0 t = \arccos 0.6 = 53.1^0 = 0.3\pi$ rad,

$$t = \frac{0.3\pi}{\omega_0} = \frac{0.3\cdot 3.14}{1} = 0.94\,\text{s}.$$

Problem 11

A particle is simultaneously subjected to two simple harmonic motions in the same direction, each of frequency 5 Hz. If the amplitudes are 0.05 m and 0.02 m respectively, and phase difference between them is 45^0*, find the amplitude of the resultant displacement and its phase relative to the first component. Write down the expression for the resultant displacement as a function of time.*

Solution

Let the phase constant (an initial phase) α_1 of the first component be zero, then the phase constant α_2 of the second phase is $\frac{\pi}{4}$; the amplitude of the first motion $A_1 = 0.05$ m, the amplitude of the second motion $A_2 = 0.02$ m.

The amplitude of the resultant motion is given by equation

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\alpha_2 - \alpha_1)}$$

Thus

$$A = \sqrt{0.05^2 + 0.02^2 + 2\cdot 0.05\cdot 0.02\cos(\pi/4)} = 6.57\cdot 10^{-2}\,\text{m}.$$

The phase constant α of the resultant motion is given by equation

$$\tan\alpha = \frac{A_1\sin\alpha_1 + A_2\sin\alpha_2}{A_1\cos\alpha_1 + A_2\cos\alpha_2},$$

$$\tan\alpha = \frac{0.05\sin 0^0 + 0.02\sin(\pi/4)}{0.05\cos 0^0 + 0.02\cos(\pi/4)} = 0.22,$$

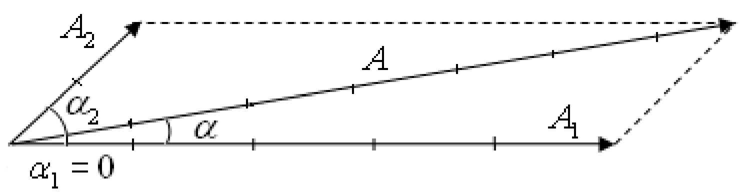
$\alpha = 12.4° \approx 0.07\pi$ rad.

The frequency of each motion is $\upsilon = 5$ Hz, therefore, an angular frequency

$\omega = 2\pi\upsilon = 10\pi$ rad/s.

With these values A, α and ω, the expression for the resultant displacement becomes

$$x = 6.57\cdot 10^{-2}\cos(10\pi t + 0.07\pi)\,\text{m}.$$



The resultant amplitude A and the initial phase α may be obtained by the method of vector addition of amplitudes. The vector diagram is shown below.

The phase constant of the first component is zero, and of the second component is $\frac{\pi}{4}$. Vector A is the resultant of vectors $A_1 = 0.02$ m and $A_2 = 0.05$ m, respectively. The angle α is equal to the phase constant of the resultant motion.

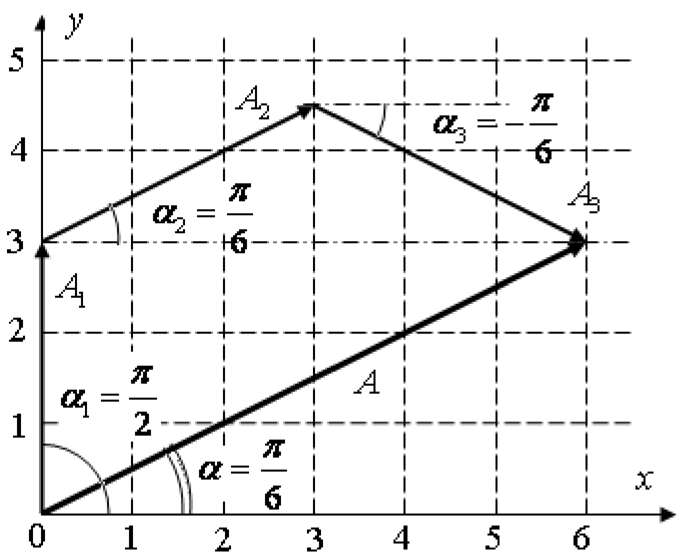
Problem 12

The point is executing three SHM of the same direction simultaneously: $x_1 = 3\cos(5\pi t)$, $x_2 = 3\sin\left(5\pi t + \frac{\pi}{6}\right)$, $x_3 = 3\sin\left(5\pi t - \frac{\pi}{6}\right)$, *where displacements are in centimetres. Find the equation of its resultant motion.*

Solution

This problem may be solved by graphical and analytical methods. But any way the first thing to do is to obtain all equations in identical trigonometric form, for example, using cosine function, therefore, $x_1 = 3\cos(5\pi t) = 3\sin(5\pi t + \pi/2)$ cm.

The first method



Let us construct the diagram using the rule and the graduating arc with scrupulous attention to the scale.

The length of resultant vector A, measured by the rule, is 6 cm. The initial phase, i.e., the angle α measured by the graduating arc is 30^0 or $\pi/6$ rad. Therefore, the equation of the resultant oscillations is

$$x = 6\sin\left(5\pi t + \frac{\pi}{6}\right)\text{cm}.$$

The second method

Lets find the sum of oscillations $x_1 = 3\sin\left(5\pi t + \frac{\pi}{2}\right)$ and $x_2 = 3\sin\left(5\pi t + \frac{\pi}{6}\right)$.

The amplitude of the resultant oscillations is

$$A_{res1} = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\alpha_2 - \alpha_1)} = \sqrt{9 + 9 + 2\cdot 9\cdot\cos\frac{\pi}{3}} = 3\sqrt{3}\ \text{cm},$$

and the initial phase is

$$\tan\alpha_{res1} = \frac{A_1\sin\alpha_1 + A_2\sin\alpha_2}{A_1\cos\alpha_1 + A_2\cos\alpha_2} = \frac{3\sin\frac{\pi}{2} + 3\sin\frac{\pi}{6}}{3\cos\frac{\pi}{2} + 3\cos\frac{\pi}{6}} = \frac{3\cdot 1 + 3\cdot 0.5}{3\cdot 0 + 3\cdot 0.866} = 1.73,$$

$$\alpha_{res1} = \arctan(1.73) = \frac{\pi}{3}.$$

Let us add the obtained result to $x_3 = 3\sin\left(5\pi t - \frac{\pi}{6}\right)$.

$$A = A_{res2} = \sqrt{A_{res1}^2 + A_3^2 + 2A_{res1}A_3\cos(\alpha_{res1} - \alpha_3)} =$$

$$= \sqrt{27 + 9 + 2\cdot 3\sqrt{3}\cdot\cos\left[\frac{\pi}{3} - \left(-\frac{\pi}{6}\right)\right]} = 6\ \text{cm}.$$

$$\tan\alpha_{res2} = \frac{A_{res1}\sin\alpha_{res1} + A_3\sin\alpha_3}{A_{res1}\cos\alpha_{res1} + A_3\cos\alpha_3} =$$

$$= \frac{3\sqrt{3}\sin\frac{\pi}{3} + 3\sin\left(-\frac{\pi}{6}\right)}{3\sqrt{3}\cos\frac{\pi}{3} + 3\cos\left(-\frac{\pi}{6}\right)} = \frac{3\sqrt{3}\cdot 0.866 + 3\cdot 0.5}{3\sqrt{3}\cdot 0.5 + 3\cdot 0.866} = 0.577,$$

$$\alpha = \arctan(0.577) = \frac{\pi}{6}.$$

The equation of the resultant oscillation is $x = 6\sin\left(5\pi t + \frac{\pi}{6}\right)$(cm).

Problem 13

A point mass is subjected to two simultaneous sinusoidal displacement in x*-direction* $x_1 = A\sin\omega t$ *and* $x_2 = A\sin\left(\omega t + \frac{2\pi}{3}\right)$. *Adding a third sinusoidal displacement* $x_3 = B\sin(\omega t + \varphi)$ *brings the mass to a complete rest. Find the values of* B *and* φ.

Solution

The resultant motion of the particle subjected by two simultaneous simple harmonic motions (SHM) is

$$x = x_1 + x_2 = A\sin\omega t + A\sin\left(\omega t + \frac{2\pi}{3}\right) = A_0 \sin(\omega t + \alpha).$$

$$A_0 = \sqrt{A^2 + A^2 - 2A^2 \cos\left(\frac{2\pi}{3} - 0\right)} = A.$$

$$\tan\alpha = \frac{A\sin 0° + A\sin(2\pi/3)}{A\cos 0° + A\cos(2\pi/3)} = 1.732$$

$$\alpha = \arctan 1.732 = \frac{\pi}{3}.$$

The amplitude of the resultant simple harmonic motion is A and the initial phase is $\alpha = \pi/3$.

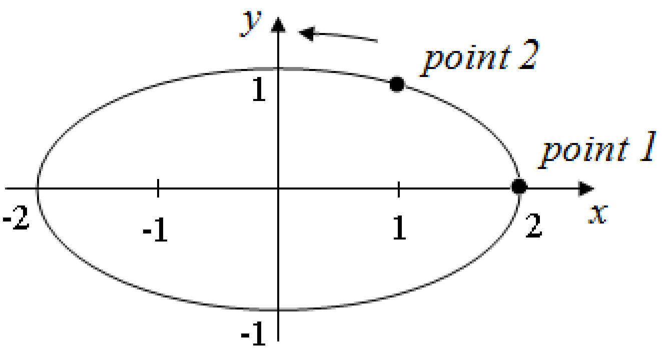
As the particle remains at rest on adding the third simple harmonic motion, $x_3 = B\sin(\omega t + \varphi)$ the amplitude (B) of the third SHM must be A itself, but it must be 180^0 (or, π radian) out of phase. In other words, the initial phase φ of the third SHM must be $\frac{\pi}{3}+\pi=\frac{4\pi}{3}$.

Problem 14

Two vibrations, at right angle to each other, are described by the equations $x=2\cos\frac{\pi}{3}t$ *(cm) and* $y=\sin\frac{\pi}{3}t$ *(cm). Construct the curve for the combined motion and determine the direction of motion.*

Solution

Firstly we have to obtain both equations in identical trigonometric form. For this purpose we transform the second equation from sine function to cosine:



$$\begin{cases} x=2\cos\frac{\pi}{3}t, \\ y=1\cos\left(\frac{\pi}{3}t-\frac{\pi}{2}\right). \end{cases}$$

The phase difference between two oscillations is equal to $\frac{\pi}{2}$, therefore, the equation of trajectory is

$$\frac{x^2}{A^2}+\frac{y^2}{B^2}=1.$$

Thus, the trajectory of the particle is an ellipse with semi major and semi minor axes A and B, coinciding with x- and y- axes, respectively, i.e., it is an ellipse reduced to the principal axes (see Figure).

The direction of rotation (clockwise or anticlockwise) of the particle may be obtained from the x- and y- motions of the particle when t is increased gradually.

Let's find the coordinates of the particle for two close instants of time t_1 and t_2. To estimate their closeness it is necessary to compare $\Delta t = t_1 - t_2$ with the period of oscillations T

$$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{\pi/3} = 6\,\text{s}.$$

Than we may take $\Delta t = 1\,\text{s}$, and for

$$t_1 = 0: \quad \begin{cases} x_1 = 2\cos 0 = 2 \\ y_1 = 1\cos\left(-\dfrac{\pi}{2}\right) = 0 \end{cases} \quad \Rightarrow \text{ point 1 } \ (2;0),$$

$$t_2 = 1\,\text{s}: \quad \begin{cases} x_2 = 2\cos\dfrac{\pi}{3} = 1 \\ y_2 = 1\cos\left(\dfrac{\pi}{3} - \dfrac{\pi}{2}\right) = 0{,}86 \end{cases} \quad \Rightarrow \text{ point 2 } \ (1;0.86).$$

Hence, the particle moves anticlockwise.

Problem 15

Two oscillations, at right angle to each other, are described by the equations $x = 0.02\sin\pi t$ *(m) and* $y = 0.01\cos\left(\pi t + \dfrac{\pi}{2}\right)$ *(m). Construct the trajectory for the combined motion.*

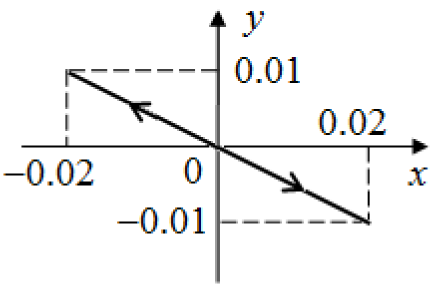
Solution

First it is necessary to transform the second equation

$$y = 0.01\cos\left(\pi t + \frac{\pi}{2}\right) = -0.01\sin\pi t.$$

The combined motion of the object is described by

$$\begin{cases} x = 0.02\sin \pi t, \\ y = -0.01\sin \pi t. \end{cases}$$



Dividing the second equation by the first equation we obtain

$$\frac{y}{x} = -\frac{1}{2}.$$

The equation of the trajectory is $y = -\frac{1}{2}x$. The displacement x is changing from -0.02 m to 0.02 m.

Problem 16

The motion of a 10 g-particle is given by $x = 5\sin\left(\frac{\pi}{5}t + \frac{\pi}{4}\right)$ *(cm). Find the maximum force that acts on the particle, and its total energy.*

Solution

Comparison of the general SHM equation $x = A\sin\left(\omega_0 t + \alpha\right)$ with the equation of particle motion $x = 0.05\sin\left(\frac{\pi}{5}t + \frac{\pi}{4}\right)$ (m) gives the amplitude $A = 5\cdot10^{-2}$ m, the angular frequency $\omega_0 = \pi/5$, and the phase constant (the initial phase of oscillations) $\alpha = \pi/4$.

The acceleration of the particle is

$$a = \ddot{x} = -A\omega_0^2 \sin(\omega_0 t + \alpha).$$

Therefore, according to the Second Newton’s Law, the force that acts on the particle is

$$F = ma = -m\omega_0^2 A\sin(\omega_0 t + \alpha).$$

The maximum force

$F_{\max} = m\omega_0^2 A$.

The total energy of the particle is sum of kinetic and potential energies

$W = W_k + W_p$,

where

$$W_k = \frac{mA^2\omega_0^2}{2}\cos^2(\omega_0 t + \alpha),$$

$$W_p = \frac{mA^2\omega_0^2}{2}\sin^2(\omega_0 t + \alpha).$$

The total energy is

$$W = \frac{mA^2\omega_0^2}{2}\left[\cos^2(\omega_0 t + \alpha) + \sin^2(\omega_0 t + \alpha)\right] = \frac{mA^2\omega_0^2}{2}.$$

Substituting the values, we obtain

$$F_{\max} = m\omega_0^2 A = 10^{-2}\left(\pi/5\right)^2 \cdot 5 \cdot 10^{-2} = 2 \cdot 10^{-4}\,\text{N}.$$

$$W = \frac{mA^2\omega_0^2}{2} = \frac{10^{-2}\pi^2 \cdot \left(5 \cdot 10^{-2}\right)^2}{2 \cdot 25} = 4{,}9 \cdot 10^{-6}\,\text{J}.$$

Problem 17

A 4.5-kg object oscillates on a horizontal spring with amplitude of 3.8 cm. Its maximum acceleration is 26 m/s². Find the force constant k*, the frequency, and the period of the oscillations.*

Solution

The spring constant is $k = \omega^2 \cdot m$. On other side, the maximum acceleration is $a_{\max} = A\omega^2$. Combining these two equations we obtain

$$k = \frac{ma_{\max}}{A} = \frac{4.5 \cdot 26}{0.038} = 3079\,\text{N/m}.$$

$$\upsilon = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{a_{\max}}{A}} = \frac{1}{2\pi}\sqrt{\frac{26}{0.038}} = 4.16\,\text{Hz.}$$

The period of oscillations is

$$T = \frac{1}{\upsilon} = \frac{1}{4.16} = 0.24\,\text{s}$$

Problem 18

Determine the ratio of the kinetic energy W_k of material point participating in SHM, to its potential energy W_p if the phase of oscillations is known.

Solution

The displacement of a point is

$$x = A\cos(\omega_0 t + \alpha)\,,$$

and its velocity is

$$v = \frac{dx}{dt} = -A\omega_0 \sin(\omega_0 t + \alpha)\,.$$

The kinetic and potential energies of the point are

$$W_k = \frac{mv^2}{2} = \frac{mA^2\omega_0^2}{2}\sin^2(\omega_0 t + \alpha)\,,$$

$$W_p = -\int_0^x F dx = -\int_0^x m\omega_0^2 x dx = \frac{mA^2x^2}{2} = \frac{mA^2\omega_0^2}{2}\cos^2(\omega_0 t + \alpha)\,.$$

The ratio is

$$\frac{W_k}{W_p} = \frac{\sin^2(\omega_0 t + \alpha)}{\cos^2(\omega_0 t + \alpha)} = \tan^2(\omega_0 t + \alpha)\,.$$

Problem 19

A *particle executes simple harmonic motion with amplitude of 10 cm. At what distance from the mean position are the kinetic and potential energies equal?*

Solution

Let $x = A\sin\left(\omega_0 t + \alpha\right)$be the distance from the mean position where kinetic and potential energies are equal. The kinetic energy at this instant of time is

$$W_k = \frac{mv^2}{2} = \frac{mA^2\omega_0^2}{2}\cos^2\left(\omega_0 t + \alpha\right),$$

$$W_p = \frac{kx^2}{2} = \frac{mA^2\omega_0^2}{2}\sin^2(\omega_0 t + \alpha).$$

Given that $W_k = W_p$, then

$$\frac{mA^2\omega_0^2}{2}\cos^2\left(\omega_0 t + \alpha\right) = \frac{mA^2\omega_0^2}{2}\sin^2(\omega_0 t + \alpha),$$

$$\cos^2\left(\omega_0 t + \alpha\right) = \sin^2(\omega_0 t + \alpha),$$

$$\tan^2\left(\omega_0 t + \alpha\right) = 1,$$

$$\omega_0 t + \alpha = \mp\frac{\pi}{4},$$

$$x = A\sin\left(\omega_0 t + \alpha\right) = 0.1\cdot\sin\frac{\pi}{4} = \pm\frac{0.1}{\sqrt{2}} = 0.071\text{m}.$$

Problem 20

A simple pendulum has time period T *= 4 s. How the length should be changed so the pendulum may complete 15 oscillations in 30 seconds?*

Solution

The initial period of the pendulum is

$$T = 2\pi\sqrt{\frac{l}{g}}\,.$$

Its length is

$$l = \frac{T^2 g}{4\pi^2} = \frac{16\cdot 9.8}{4\pi^2} = 4\,\text{m}.$$

New period according to the given data is to be $T_1 = \frac{t}{N} = \frac{30}{15} = 2\,\text{s}$, and new length is

$$l_1 = \frac{T_1^2 g}{4\pi^2} = \frac{4\cdot 9.8}{4\pi^2} = 1\text{m}.$$

The length change is $\Delta l = l_1 - l = 4 - 1 = 3$ m.

Problem 21

A simple pendulum of length l *is suspended through the ceiling of an elevator. Find the time period of oscillations if the elevator (a) is going up with acceleration* a_0*; (b) is going down with acceleration* a_0*; and (c) is moving with uniform velocity.*

Solution

The equation of the bob motion according to Newton's 2 Law is

$$m\vec{a}_0 = m\vec{g} + \vec{F}\,,$$

where $m\vec{g}$ is gravity, and $\vec{F}$ is tension.

$$\vec{F} = m\left(\vec{a}_0 - \vec{g}\right).$$

(a) For the upward motion the tension and period of oscillations are

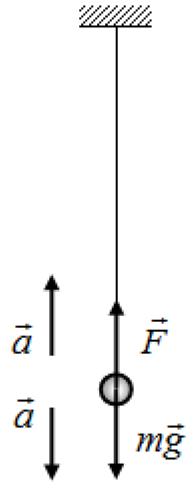
$$F = m\left(a_0 - (-g)\right) = m\left(a_0 + g\right) = ma,$$

$$T = 2\pi\sqrt{\frac{l}{g + a_0}},$$

(b) For downward motion

$$F = m\left(a_0 - g\right) = ma,$$

$$T = 2\pi\sqrt{\frac{l}{g - a_0}}.$$



(c) When elevator is moving at uniform velocity its acceleration is zero and

$$T = 2\pi\sqrt{\frac{l}{g}}.$$

Problem 22

A simple pendulum fixed in a car has a time period of 4 seconds when the car is moving with a uniform velocity on a horizontal road. When the accelerator is pressed, the time period changes to 3.99 seconds. Find the acceleration of the car.

Solution

When car is moving with uniform velocity the period of oscillations of the pendulum is

$$T = 2\pi\sqrt{\frac{l}{g}}.$$

When car accelerates by a_0 the period is

$$T' = 2\pi\sqrt{\frac{l}{\sqrt{g^2 + a_0^2}}} = 2\pi\sqrt{\frac{g}{a}}$$

$$\frac{T}{T'} = \sqrt{\frac{a}{g}},$$

$$a = g\left(\frac{T}{T'}\right)^2 = 9.8 \cdot \left(\frac{4}{3.99}\right)^2 = 9.85\,\text{m/s}^2,$$

$$a^2 = g^2 + a_0^2,$$

$$a_0 = \sqrt{a^2 - g^2} = \sqrt{9.85^2 - 9.8^2} = 0.983\ \text{m/s}^2$$

Problem 23

A mass 8 g is attached to a horizontal spring that requires a force of 0.01 N to extend it to a length 5 cm greater than its natural length. What are the period, the frequency and the angular frequency of the simple harmonic motion of such a system?

Solution

The length that the spring stretches is directly proportional to applied force. The magnitude of this force is

$$F = kx,$$

where k is the force constant. The force constant is

$$k = \frac{F}{x} = \frac{0.01}{0.05} = 0.2\,\text{N/m}.$$

Therefore, the period, the frequency, and the angular frequency are respectively

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{0.008}{0.2}} = 1.26\,\text{s},$$

$$\upsilon = \frac{1}{T} = \frac{1}{1.26} = 0.8\,\text{Hz},$$

$$\omega = 2\pi\upsilon = 2\pi \cdot 0.8 = 1.6\pi = 5.03\,\text{rad/s}.$$

Problem 24

A block suspended from a vertical spring is in equilibrium. Show that the extension of the spring equals the length of an equivalent simple pendulum, i.e., a pendulum having frequency same as that of the block.

Solution

If the mass of the block is m, extension of the spring is x, and the length of the pendulum is l, the angular velocity of the pendulum is

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{l}{g}}\ .$$

At equilibrium position

$$ma + mg = 0\ ,$$

$$mg = -ma = -F = -kx\ ,$$

$$kx = mg\ ,$$

$$x = \frac{mg}{k} = l\ .$$

Problem 25

Find the periods of vertical oscillations of the block suspended with the help of two equal springs if the block is connected to springs 1) in series, or 2) in parallel.

Solution

In the equilibrium point, the force acted on the load is $|F| = kx$, and according to the First Newton’s Law $mg = kx$, the spring extension is

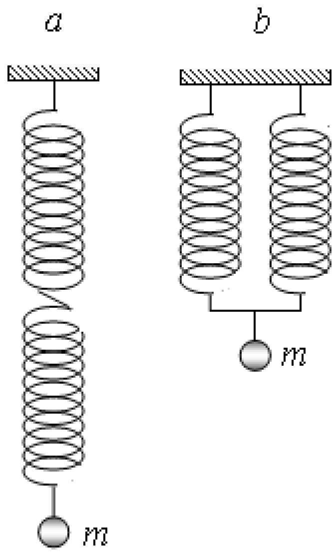
$$x = \frac{mg}{k}.$$

1. When two springs are attached one at the end of the other, their extensions are equal, and the total extension is $x_1 = 2x = \frac{2mg}{k}$. On the other side, $x_1 = \frac{mg}{k_1}$.

Equating these two expressions, we obtain

$$\frac{2mg}{k} = \frac{mg}{k_1}.$$

$$k_1 = \frac{k}{2}.$$



2. If the block is connected to the springs in parallel $k_2 = 2k$.

The periods of oscillations for these two cases are

$$T_1 = 2\pi\sqrt{\frac{m}{k_1}} \quad \text{and} \quad T_2 = 2\pi\sqrt{\frac{m}{k_2}},$$

and their ratio is

$$\frac{T_1}{T_2} = \sqrt{\frac{k_2}{k_1}} = \sqrt{4} = 2.$$

Problem 26

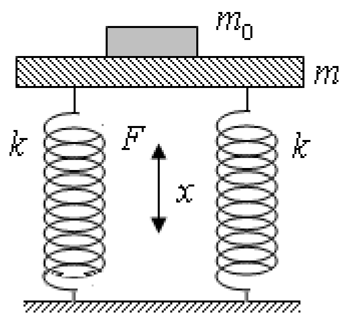
A tray of mass m = 12 kg is supported by two identical springs as shown in Figure. When the tray is pressed down slightly and then released, it executes SHM with a time period of 1.5 s. (a) What is the spring constant of each spring? (b) When the block of mass m_0 is placed on the tray, the period of SHM changes to 3 s, what is the mass of the block?

Solution

(a) Let $m = 12$ kg be the mass of tray and k is the force constant of each spring. When the tray is pressed down slightly it begins to execute SHM. Let x be the downward displacement of the tray at any time t, then each spring exerts a restoring force kx upward.

Net restoring force on tray: $F = -kx - kx = -2kx$.

Clearly, the effective force constant of two springs is $k_0 = 2k$. Time period is



$$T = 2\pi\sqrt{\frac{m}{k_0}} = 2\pi\sqrt{\frac{m}{2k}},$$

$$k = \frac{2\pi^2 m}{T^2}.$$

Given $m = 12$ kg, $T = 1.5$ s,

$$k = \frac{2\pi^2 \cdot 12}{1.5^2} = 105.2 \text{ N/m}.$$

(b) When a block of mass m_0 is placed on the tray, net mass is $m + m_0$. New time period is

$$T_1 = 2\pi\sqrt{\frac{m + m_0}{2k}},$$

$$m + m_0 = \frac{T_0^2 \cdot 2k}{4\pi^2} = \frac{T_0^2 \cdot k}{2\pi^2} = \frac{3^2 \cdot 105.2}{2\pi^2} = 48 \text{kg},$$

$$m_0 = 48 - m = 48 - 12 = 36 \text{ kg}.$$

Problem 27

A rod of the length l = 1 m oscillates about the axis passing through its end. (a) Find the period of oscillations of the rod;(b) Determine the location of the pivot point that provides the maximum frequency of oscillations.

Solution

(a) The rod is a physical (compound) pendulum, and its period of oscillations is determined by $T = 2\pi\sqrt{\frac{I}{mgx}}$, where $x = l/2$ is the distance between the centre of mass and pivot point, and I is the moment of inertia respectively the axis passing through the pivot point. This moment of inertia according to Steiner theorem is

$$I = I_0 + mx^2 = \frac{ml^2}{12} + m\left(\frac{l}{2}\right)^2 = \frac{ml^2}{3}.$$

$$T = 2\pi\sqrt{\frac{I}{mgx}} = 2\pi\sqrt{\frac{ml^2}{3\cdot mg(l/2)}} = 2\pi\sqrt{\frac{2l}{3g}} = 1{,}63 \text{ s.}$$

(b) The frequency of the compound pendulum is

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{mgx}{I}}.$$

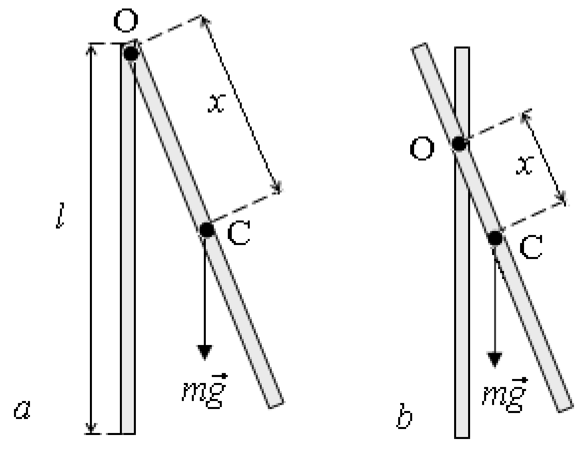
The moment of inertia respectively new pivot point according to Steiner theorem is

$$I = \frac{ml^2}{12} + mx^2,$$

where x is required value.

After that

$$\omega = \sqrt{\frac{mgx}{\frac{ml^2}{12} + mx^2}} = \left(\frac{12gx}{l^2 + 12x^2}\right)^{1/2}.$$



Examination the function $\omega(x)$ for maximum allows finding the required distance x between the pivot point and centre of mass:

$$\frac{d\omega}{dx} = \frac{1}{2}\left(\frac{12gx}{l^2 + 12x^2}\right)^{-\frac{1}{2}}\left[\frac{12g\left(l^2 + 12x^2\right) - 12gx(12\cdot 2x)}{\left(l^2 + 12x^2\right)^2}\right] = \frac{\sqrt{3g}(l^2 - 12x^2)}{\sqrt{x(l^2 + 12x^2)^3}} = 0,$$

$$l^2 - 12x^2 = 0,$$

$$x = \frac{l}{\sqrt{12}} = \frac{1}{2\sqrt{3}} = 0.29 \text{ m}.$$

Problem 28

A mass of 2 kg oscillates on a spring with constant 50 N/m. By what factor does the frequency of oscillation decrease when a damping force with constant r = 12 (coefficient of resistance of medium) is introduced?

Solution

The original angular frequency of oscillation is given by

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{50}{2}} = 5 \text{ rad/s}.$$

The damping coefficient is $\beta = \frac{r}{2m} = \frac{12}{2 \cdot 2} = 3\,\text{s}^{-1}$.

The frequency of damped oscillation is given by:

$$\omega = \sqrt{\omega_0^2 - \beta^2} = \sqrt{5^2 - 3^2} = 4\,\text{rad/s}.$$

Thus the frequency decreases by 1 rad/s, or by 20 percent of its original value.

Problem 29

Amplitude of damped oscillations of the simple pendulum decreased twice during t_1 *= 1 min. By what factor does the amplitude decreased during* t_2 *= 3 min?*

Solution

Amplitude of the damped oscillations depends on the time according to $A_t = A_0 e^{-\beta t}$, where A_0 is initial amplitude. Therefore,

$$\frac{A_0}{A_{t_1}} = \frac{\cancel{A_0}}{\cancel{A_0} \cdot e^{-\beta t_1}} = e^{\beta t_1} = 2,$$

$$\beta t_1 = \ln 2 .$$

The damping coefficient is $\beta = \dfrac{\ln 2}{t_1}$.

For the second time interval

$$\frac{A_0}{A_{t_2}} = \frac{\cancel{A_0}}{\cancel{A_0} \cdot e^{-\beta t_2}} = e^{\beta t_2} = e^{\ln 2 \cdot \frac{t_2}{t_1}} = e^{\frac{0.693 \cdot 3}{1}} = 8 .$$

Thus the amplitude of damped oscillations is decreased by factor of 8 during 3 minutes.

Problem 30

Amplitude of simple pendulum oscillations of the length l = 1m during 10 min was decreased twice. Determine the damping coefficient β, logarithmic decrement δ and the number of oscillations during this time interval. Find the equation of oscillations if initially the pendulum was pulled sideways to a distance of 5 cm and released.

Solution

The amplitude of damped oscillations was decreased twice during $t = 10$ min = 600 s. Then the ratio of the initial amplitude A_0 and the amplitude after time t - A_t is:

$$\frac{A_0}{A_t} = \frac{A_0}{A_0 e^{-\beta t}} = e^{\beta t} = 2,$$

$$\beta t = \ln 2,$$

$$\beta = \frac{\ln 2}{t} = \frac{0.693}{600} = 10^{-3}\,\text{s}^{-1}.$$

For finding the logarithmic decrement we have to know the period of damped oscillations T. Firstly, find the period and angular frequency of simple harmonic motion (without damping):

$$T_0 = 2\pi\sqrt{\frac{1}{g}} = 2\ \text{s}, \qquad \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi\ \text{rad/s}.$$

The angular frequency of damped oscillations is

$$\omega = \sqrt{\omega_0^2 - \beta^2} = \sqrt{\pi^2 - 10^{-6}} \approx \omega_0 .$$

Since the angular frequency ω of damped oscillations is almost equals the own angular frequency ω_0, the period of damped oscillations is $T \cong T_0 = 2$ s. The logarithmic decrement is

$$\delta = \beta T = 2 \cdot 10^{-3}.$$

The number of oscillations N for time t may be found from $\beta t = \beta N T = \ln 2$, as

$$N = \frac{\ln 2}{\beta T} = \frac{0.693}{2 \cdot 10^{-3}} = 346.6 .$$

Since on the initial instant of time the pendulum was displaced by a distance we consider this distance as its initial amplitude $A_0 = 5 \cdot 10^{-2}$ m, and write down the equation of oscillations using the cosine function assuming that the initial phase is equal to zero:

$$x = 5 \cdot 10^{-2} e^{-0.001 \cdot t} \cos \pi t \ \text{(m)}.$$

Problem 31

Harmonic oscillator of the mass $m = 0.25\,kg$ *moves attached to the spring with spring constant* $k = 85\,N/m$ *in the medium with resistance coefficient* $r = 0.07\,kg/s$. *Calculate (a) the period of its oscillation; (b) the number of oscillations in which its amplitude will become half of its original value; (c) the number of oscillations in which its mechanical energy will drop to one-half of its initial value; and (d) the quality factor.*

Solution

(a) The damping coefficient β of this oscillating system is

$$\beta = \frac{r}{2m} = \frac{0.07}{2 \cdot 0.25} = 0.14\,\text{s}^{-1}.$$

The natural angular velocity of this oscillator (without friction) is

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{85}{0.25}} = 18.44\,\text{rad/s}.$$

The angular velocity of this damped oscillator due to $\omega_0 \gg \beta$ is

$$\omega = \sqrt{\omega_0^2 - \beta^2} \approx \omega_0 = \sqrt{18.44^2 - 0.14^2} \approx 18.44\,\text{rad/s}.$$

Then the period of oscillation is

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{18.44} = 0.34\,\text{s}.$$

(b) The ratio of the initial amplitude and the amplitude after N oscillations is

$$\frac{A_0}{A_t} = \frac{A_0}{A_0 \cdot e^{-\beta t}} = e^{\beta t} = e^{\beta N T} = 2.$$

On taking natural logarithm of both sides and rearranging terms, we get

$$\beta N T = \ln 2,$$

$$N = \frac{\ln 2}{\beta T} = \frac{0.693}{0.14 \cdot 0.34} = 14.56.$$

(c) The ratio of the initial energy and the energy after N_1 oscillations is

$$\frac{W_0}{W} = \frac{W_0}{W_0 \cdot e^{-2\beta t}} = e^{2\beta t} = e^{2\beta N_1 T} = 2,$$

$$2\beta N_1 T = \ln 2.$$

$$N_1 = \frac{\ln 2}{2\beta T} = \frac{0.693}{2 \cdot 0.14 \cdot 0.34} = 7.28$$

(d) The quality factor is

$$Q = \frac{\pi}{\delta} = \frac{\pi}{\beta T} = \frac{\pi}{0.14 \cdot 0.34} = 66.$$

Problem 32

A damped oscillator loses 3.5% of its energy during each cycle. What is its Q factor? How much cycles elapse before half of its original energy is dissipated?

Solution

The total energy of oscillator depends on the square of amplitude

$$W = \frac{m\omega_0{}^2 A^2}{2}.$$

Therefore, the energy of damped oscillator is

$$W = \frac{m\omega^2 A^2}{2} = \frac{m\omega^2 A_0^2}{2} \cdot e^{-2\beta t} = W_0 \cdot e^{-2\beta t}.$$

The damped oscillator loses 3.5% of its energy during one cycle. This implies that after the time $t = T$ the energy of oscillator equals to $100\% - 3.5\% = 96.5\%$. Thus

$$\frac{W_0}{W_T} = \frac{\cancel{W_0}}{\cancel{W_0} \cdot e^{-2\beta T}} = e^{2\beta T} = \frac{100}{96.5} = 1.036,$$

$$2\beta T = \ln 1.036 = 0.0356,$$

$$\beta T = \frac{0.0356}{2} = 1.78 \cdot 10^{-2} = \delta,$$

$$Q = \frac{\pi}{\delta} = \frac{\pi}{1.78 \cdot 10^{-2}} = 176.$$

During time t the oscillator losses the half of its energy

$$\frac{W_0}{W_t} = \frac{\cancel{W_0}}{\cancel{W_0} \cdot e^{-2\beta t}} = e^{2\beta t} = 2,$$

$$2\beta t = \ln 2 = 0.693, \quad \text{or} \quad \beta t = \frac{0.693}{2} = 0.3465.$$

On the other hand, $\beta t = \beta NT = N\delta$. Then,

$$N\delta = 0.3465,$$

$$N = \frac{0.3465}{\delta} = \frac{0.3465}{1.78 \cdot 10^{-2}} = 19.5 \approx 20.$$

Problem 33

An oscillator with a period of 1 s has amplitude that decreases by 1% during each complete oscillation. (a) If the initial amplitude is 10.2 cm, what will be the amplitude after 35 oscillations? (b) At what time will the energy be reduced to 46% of its initial value?

Solution

The ratio of initial amplitude and the amplitude after the period is

$$\frac{A_0}{A_T} = \frac{A_0}{A_0 \cdot e^{-\beta T}} = e^{\beta T} = \frac{100}{99} = 1.01.$$

It means that the logarithmic decrement is equal to

$$\beta T = \delta = \ln 1.01 = 0.01.$$

The damping coefficient is

$$\beta = \frac{0.01}{T} = \frac{0.01}{1} = 0.01 \text{ s}^{-1}.$$

If the time for amplitude decrease is t, the ratio of amplitudes is

$$\frac{A_0}{A_t} = \frac{A_0}{A_0 \cdot e^{-\beta t}} = e^{\beta t} = e^{\beta NT} = e^{N\delta},$$

$$A_t = \frac{A_0}{e^{N\delta}} = \frac{10.2}{e^{35 \cdot 0.01}} = 7.18 \text{ cm.}$$

(b)

$$\frac{W_0}{W_\tau} = \left(\frac{A_0}{A_\tau}\right)^2 = \frac{A_0^2}{A_0^2 \cdot e^{-2\cdot\beta\cdot\tau}} = e^{2\cdot\beta\cdot\tau},$$

$$\frac{W_0}{W_\tau} = \frac{W_0}{0.46 \cdot W_0} = \frac{1}{0.46} = 2.17,$$

$$e^{2\cdot\beta\cdot\tau} = 2.17,$$

$$2\beta\tau = \ln 2.17$$

$$\tau = \frac{\ln 2.17}{2\beta} = 38.74 \text{ s.}$$

Problem 34

A compound pendulum with equivalent length of 24.7 cm executes damped oscillations. In what time will the energy of the oscillation become 10% of the initial energy if the logarithmic decrement factor is 0.01.

Solution

The ratio of initial energy and energy after the time interval t is

$$\frac{W_0}{W_t} = \frac{W_0}{W_0 \cdot e^{-2\beta t}} = e^{2\beta t},$$

on other hand,

$$\frac{W_0}{W_t} = \frac{W_0}{0.1 \cdot W_0} = 10.$$

$$e^{2\beta t} = 10.$$

$$2\beta t = \ln 10.$$

$$t = \frac{\ln 10}{2 \cdot \beta}.$$

The natural angular velocity of compound pendulum is

$$\omega_0 = \sqrt{\frac{g}{L}} = \sqrt{\frac{9.8}{0.247}} = 6.3 \text{rad/s}.$$

Since the logarithmic decrement is $\delta = 0.01$ the given oscillator is the system with weak damping. Therefore, we assume that the angular velocity of damping oscillations is equal to the natural angular velocity, i.e., $\omega_0 = \omega$. Moreover, $\delta = \beta T = \frac{2\pi\beta}{\omega}$, hence, $\beta = \frac{\delta \cdot \omega}{2\pi}$ Accordingly to above mentioned,

$$t = \frac{\ln 10}{2 \cdot \beta} = \frac{\ln 10 \cdot 2\pi}{2 \cdot \delta \cdot \omega} = \frac{\ln 10 \cdot \pi}{\delta \cdot \omega} = \frac{2.3 \cdot \pi}{0.01 \cdot 6.3} = 115 \text{ s}.$$

Problem 35

A 145-MHz radio signal propagates along a cable. Measurement shows that the wave crests are spaced 1.25 m apart. What is the speed of the waves on the cable? Compare with the speed of light in vacuum.

Solution

The distance between adjacent wave crests is one wavelength, so the wave speed in the cable is

$$v = \lambda \cdot \upsilon = 1.25 \cdot 145 \cdot 10^6 = 1.81 \cdot 10^8$$

$$\frac{v}{c} = \frac{1.81 \cdot 10^8}{3 \cdot 10^8} = 0.6$$

The desired speed is $v = 0.6c$.

Problem 36

Ultrasound used in a particular medical imager has frequency 4.8 MHz and wavelength 0.31 mm. Find the angular frequency, the wave number and the wave speed.

Solution

The angular frequency related to the frequency as

$\omega = 2\pi\upsilon = 2\pi \cdot 4.8 \cdot 10^6 = 3.02 \cdot 10^7$ rad/s.

The wave number is

$k = \dfrac{2\pi}{\lambda} = \dfrac{2\pi}{0.31 \cdot 10^{-3}} = 2.03 \cdot 10^4\,\text{m}^{-1}$.

The speed of the wave is

$v = \lambda\upsilon = 0.31 \cdot 10^{-3} \cdot 4.8 \cdot 10^6 = 1.49 \cdot 10^3$ m/s.

Problem 37

Write the expression for a harmonic wave that has a wavelength of 2.8 m and propagates with a speed of 13.3 m/s. The amplitude of the wave is 0.12 m, initial phase is zero. Estimate two cases: (a) $\xi(0,0) = 0$*; (b)* $\xi(0,0) = A$.

Solution

The general expression describing the propagating wave depending on the initial conditions will be:

$\xi(x,t) = A\sin\left(\dfrac{2\pi}{T}t - \dfrac{2\pi}{\lambda}x\right)$ for $\xi(0,0) = 0$,

$$\xi(x,t) = A\cos\left(\frac{2\pi}{T}t - \frac{2\pi}{\lambda}x\right) \text{ for } \xi(0,0) = A.$$

The period of oscillations is $\lambda = vT$, therefore, $T = \frac{\lambda}{v} = \frac{2.8}{13.3} = 0.21$ s.

Depending on the initial conditions the equations of the wave are

(a) $\xi(x,t) = 0.12\sin\left(\frac{2\pi}{0.21}t - \frac{2\pi}{2.8}x\right)$,

(b) $\xi(x,t) = 0.12\cos\left(\frac{2\pi}{0.21}t - \frac{2\pi}{2.8}x\right)$.

Problem 38

A transverse wave propagates along a stretched string with the velocity 15 m/s. A period of oscillations of the points of the string is 1.2 s, and amplitude is 2 m. Find the phase, displacement, velocity and acceleration of the point 45 m distant from the vibration source on the instant of time t = 4 s. Determine the maximum velocity and the maximum acceleration of the point. The initial phase is zero.

Solution

The wavelength is $\lambda = vT = 15 \cdot 1{,}2 = 18$ m.

The displacement of the point may be determined using the equation of the travelling wave where the initial phase equals zero (α=0)

$$\xi(x,t) = A\cos\left(\frac{2\pi}{T}t - \frac{2\pi}{\lambda}x + \alpha\right),$$

$$\xi(45,4) = 2\cos\left(\frac{2\pi}{1.2}\cdot 4 - \frac{2\pi}{18}\cdot 45\right) = 1\text{m}.$$

The phase in equation of the wave is

$$\frac{2\pi}{1.2}\cdot 4 - \frac{2\pi}{18}\cdot 45 = \frac{5\pi}{3} = 300° = 5.23 \text{ rad}.$$

The velocity and acceleration of the point may be determined by time differentiating of $\xi(x,t)$ and $\dot{\xi}(x,t)$, respectively:

$$\dot{\xi}(x,t)=\frac{\partial\xi}{\partial t}=-A\cdot\frac{2\pi}{T}\sin\left(\frac{2\pi}{T}t-\frac{2\pi}{\lambda}x\right)=-2\cdot\frac{2\pi}{1.2}\cdot\sin\frac{5\pi}{3}=9.07\,\text{m/s}.$$

$$\ddot{\xi}(x,t)=\frac{\partial^2\xi}{\partial t^2}=-A\cdot\left(\frac{2\pi}{T}\right)^2\cos\left(\frac{2\pi}{T}t-\frac{2\pi}{\lambda}x\right)=-2\cdot\left(\frac{2\pi}{1.2}\right)^2\cos\frac{5\pi}{3}=-27.4\ \text{m/s}^2.$$

The maximum velocity from the first equation is

$$A\cdot\frac{2\pi}{T}=2\cdot\frac{2\pi}{1.2}=10.5\,\text{m/s}.$$

The maximum acceleration from the second equation is

$$A\cdot\left(\frac{2\pi}{T}\right)^2=2\cdot\left(\frac{2\pi}{1.2}\right)^2=54.8\,\text{m/s}^2.$$

Problem 39

Find the wavelength and the phase difference of two oscillating points distant by 10 and 16 m from the vibrating source, respectively. The period is 0.04 s, the velocity of the wave propagation is 300 m/s.

Solution

If the period of oscillation is $T=0.04$ s, and the wave velocity is $v=300$ m/s, the wavelength is

$$\lambda=vT=300\cdot0.04=12\ \text{m}.$$

The equations of the oscillations of the points with the coordinates $x_1=10$ m and $x_2=16$ m at the travelling wave propagation are

$$\xi_1(x_1,t)=A\cos(\omega t-kx_1),$$

$$\xi_2(x_2,t)=A\cos(\omega t-kx_2).$$

The phase difference of the oscillations of these points is

$$\Delta\varphi = |\varphi_1 - \varphi_2| = (\omega t - kx_1) - (\omega t - kx_2) = k(x_2 - x_1),$$

where $k = \frac{2\pi}{\lambda} = \frac{2\pi}{vT}$ is the wave number.

Finally, the phase difference is

$$\Delta\varphi = \frac{2\pi}{vT}(x_2 - x_1) = \frac{2\pi}{300 \cdot 0.04}(16 - 10) = \pi \,.$$

It means that the given oscillations are opposite in phase.

Problem 40

Find the frequency of the sound wave in the tube of the length L =1 m, if its both ends are (a) open; (b) closed, (c) one end is opened and another end is closed. The speed of sound v = 340 m/s.

Solution

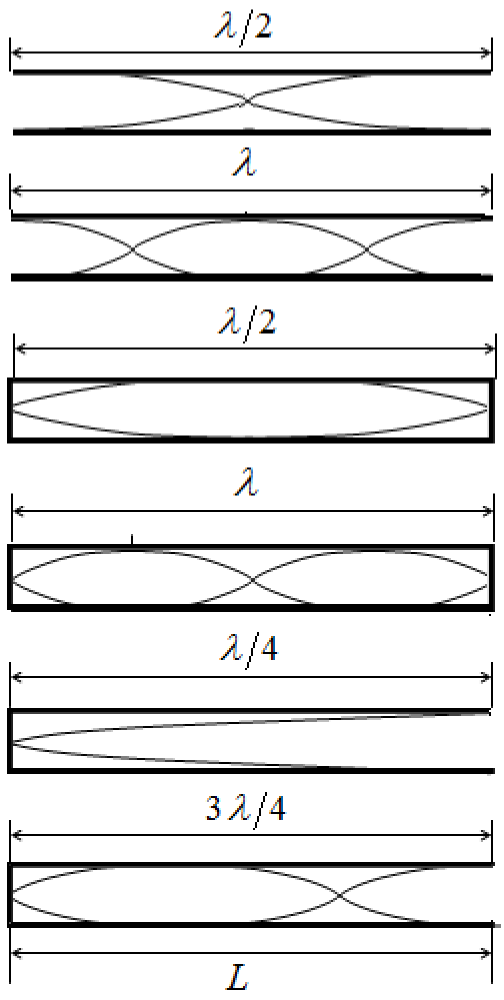
The standing waves are created in all cases of the tubes. The open and closed ends reflect waves differently. The closed end of the tube is an antinode, and the open end is the node. The longest standing wave in a tube of length L with two open ends has displacement antinodes at the both ends and only one node betwee them. The frequency in this case is fundamental frequency.

$$\lambda = 2L\,,\ \upsilon_1 = \frac{v}{\lambda} = \frac{v}{2L} = \frac{340}{2 \cdot 1} = 170 \text{ Hz}.$$

The next standing wave in this tube is second harmonic/. It also has displacement antinodes at each end. $\lambda = L$. The frequency is equal to $\upsilon_2 = \frac{v}{\lambda} = \frac{v}{L} = \frac{340}{1} = 340\,\text{Hz}.$

An integer number of half wavelength have to fit into the tube of length L $L = n\frac{\lambda}{2}$, the wavelength will be $\lambda = \frac{2L}{n}$, and the frequency $\upsilon_n = n\frac{v}{2L}$ (natural frequencies, or harmonics)

For a tube with two closed ends the longest standing wave is $\lambda = 2L$ and fundamental frequency is $\upsilon_1 = \frac{v}{\lambda} = \frac{v}{2L} = \frac{340}{2 \cdot 1} = 170$ Hz. The second natural frequency (at $\lambda = L$) is $\upsilon_2 = \frac{v}{\lambda} = \frac{v}{L} = \frac{340}{1} = 340$ Hz.



The longest standing wave in a tube of length L with one open end and one closed end has a displacement antinode at the open end and a displacement node at the closed end. This is the fundamental. $\lambda = 4L$ and $\upsilon = \frac{v}{\lambda} = \frac{v}{4L} = \frac{340}{4 \cdot 1} = 85$ Hz.

The next frequency will be at $L = 3\lambda/4$, $\lambda = 4L/3$, and $\upsilon = \frac{v}{\lambda} = \frac{3v}{4L} = \frac{3 \cdot 340}{4 \cdot 1} = 255$ Hz

An odd-integer number of quarter wavelength have to fit into the tube of length L.

$$L = n\frac{\lambda}{4}, \quad \lambda = \frac{4L}{n}, \quad \upsilon_n = \frac{v}{\lambda} = n\frac{v}{4L}$$

where n is odd number.

For a tube with one open end and one closed end all frequencies $\upsilon_n = \frac{v}{\lambda} = n\frac{v}{4L} = n\upsilon_1$, with n equal to an odd integer are natural frequencies, i.e. only odd harmonics of the fundamental are natural frequencies.