NATIONAL TECHNICAL UNIVERSITY
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STUDY GUIDE

“ELECTROMAGNETIC OSCILLATIONS”

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9.5. Electromagnetic oscillations in the circuits

In a circuit containing inductor and capacitor, the energy is stored in two different ways: 1) when a current flows in an inductor, energy is stored in the magnetic field, and 2) when a capacitor is charged, energy is stored in the static electric field.

The magnetic field in the inductor is built by the current, which gets provided by the discharging capacitor. Similarly, the capacitor is charged by the current produced by collapsing magnetic field of the inductor and this process continues on and on, causing electrical energy to oscillate between the magnetic field and the electric field. This forms a harmonic oscillator for current. In *RLC* circuit, the presence of resistor causes the oscillations to decay over the period of time and it is called as the damping effect of the resistor.

Consider the circuit (Fig. 9.5). By the Kirchhoff's Voltage Law

$$IR + (\varphi_1 - \varphi_2) = \mathrm{E}_S + \mathrm{E}. \qquad (9.31)$$

Substitution of $I = \dfrac{dq}{dt}$, $\varphi_1 - \varphi_2 = \dfrac{q}{C}$, and $\mathrm{E}_S = -L\dfrac{dI}{dt} = -L\dfrac{d^2q}{dt^2}$ in (9.31) gives the equation of *RLC* circuit (or ***oscillatory circuit,*** or ***oscillator***):

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = \mathrm{E}. \qquad (9.32)$$

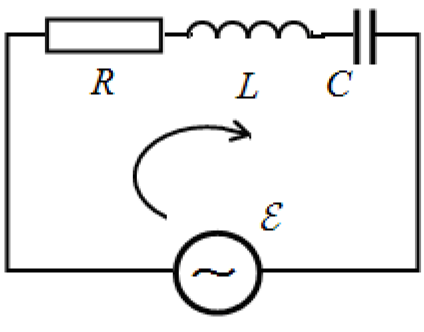

Figure 9.5 – *RLC* series AC circuit

The solution of this second order differential equation depends on the presence of electrical elements in the circuit.

9.5.1. Free undamped oscillation in LC circuit

LC circuit consists of the capacitor (to produce the electric field and to store an electric energy) and inductor (to produce the magnetic field) (Fig. 9.6). The capacitor was charged up by connecting to emf. When the capacitor was fully charged the source was disconnected.

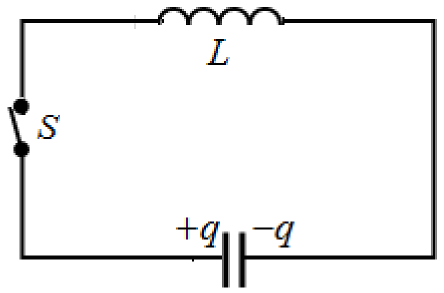

Figure 9.6 – *LC* series circuit

Since $R=0$ and $\mathrm{E}=0$, equation (9.32) takes on the form

$$\frac{d^2q}{dt^2}+\frac{1}{LC}q=0, \qquad (9.33)$$

$$\ddot{q}+\omega_0^2 q=0. \qquad (9.34)$$

The solution of (9.34) is the equation of ***harmonic oscillations***

$$q=q_m\cos(\omega_0 t+\alpha), \qquad (9.35)$$

where q is the instantaneous charge, q_m is the maximum charge (amplitude of the charge), $\omega_0=\frac{1}{\sqrt{LC}}$ is the natural angular frequency, and α is the initial phase (epoch angle).

Thomson's formula gives the ***period*** of free undamped oscillations

$$T_0=\frac{2\pi}{\omega_0}=2\pi\sqrt{LC}. \qquad (9.36)$$

Dividing the both sides of (9.35) by C gives the equation for instantaneous voltage across a capacitor:

$$U=U_m\cos(\omega_0 t+\alpha), \qquad (9.37)$$

where the maximum voltage is

$$U_m=\frac{q_m}{C}. \qquad (9.38)$$

On differentiating (9.35) we get the instantaneous current

$$I = \frac{dq}{dt} = -q_m \omega_0 \sin(\omega_0 t + \alpha) = -I_m \sin(\omega_0 t + \alpha). \qquad (9.39)$$

The energy of *LC* circuit is initially stored in the electric field of the charged capacitor (Fig. 9.7, a). The relative graphs of charge versus time and current

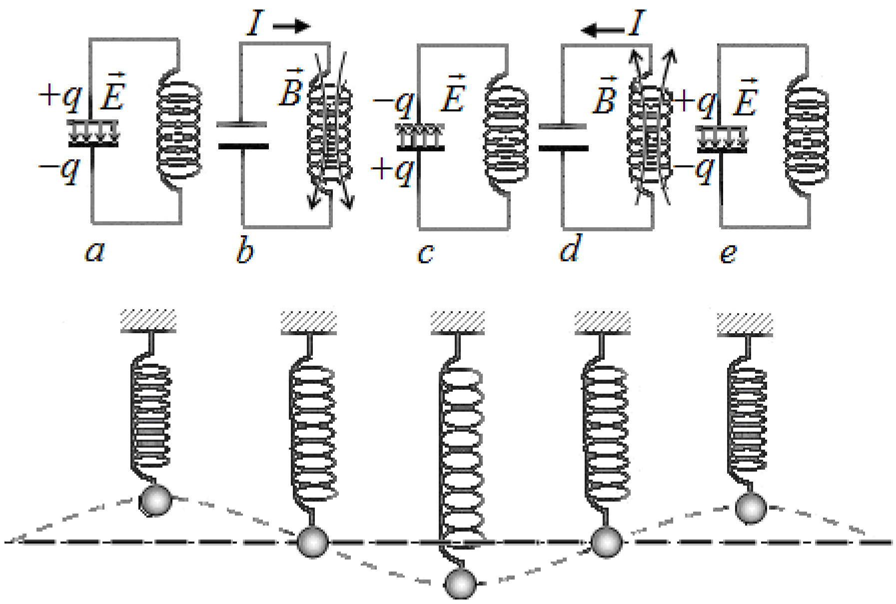

Figure 9.7 – *LC* circuit and its mechanical analog

versus time are shown in Fig. 9.8, a,b. The plots of the time variations of U_C and U_L are shown in Fig. 9.8, c,d. The sum of U_C and U_L is equal to the U_m.

$$W_{el} = W_m = \frac{CU_m^2}{2} \qquad (9.40)$$

Then the capacitor begins to discharge, producing a current in the circuit. Electric energy depends on time as

$$W_{el} = \frac{CU^2}{2} = \frac{CU_m^2}{2}\cos^2(\omega_0 t + \alpha) = W_m \cos^2(\omega_0 t + \alpha). \qquad (9.41)$$

The current, in turn, creates a magnetic field in the inductor. The net effect of this process is a transfer of energy from the capacitor, with its diminishing electric field, to the inductor, with its increasing magnetic field.

$$W_{mag} = \frac{LI^2}{2} = \frac{LI_m^2}{2}\sin^2(\omega_0 t + \alpha) = W_m \sin^2(\omega_0 t + \alpha), \qquad (9.42)$$

When the capacitor is completely discharged, all the energy is stored in the magnetic field of the inductor (Fig. 9.7, b). At this instant, the current is at its maximum value I_m and the energy in the inductor is

$$W_{mag} = W_m = \frac{LI_m^2}{2}. \tag{9.43}$$

Since there is no resistance in the circuit, no energy is lost through Joule heating; thus, the maximum energy stored in the capacitor is equal to the maximum energy stored at a later time in the inductor:

$$W_m = \frac{CU_m^2}{2} = \frac{LI_m^2}{2}. \tag{9.44}$$

At an arbitrary time when the capacitor charge is q and the current is I the total energy in the circuit is given by

$$W_{total} = W_{el} + W_m \tag{9.45}$$

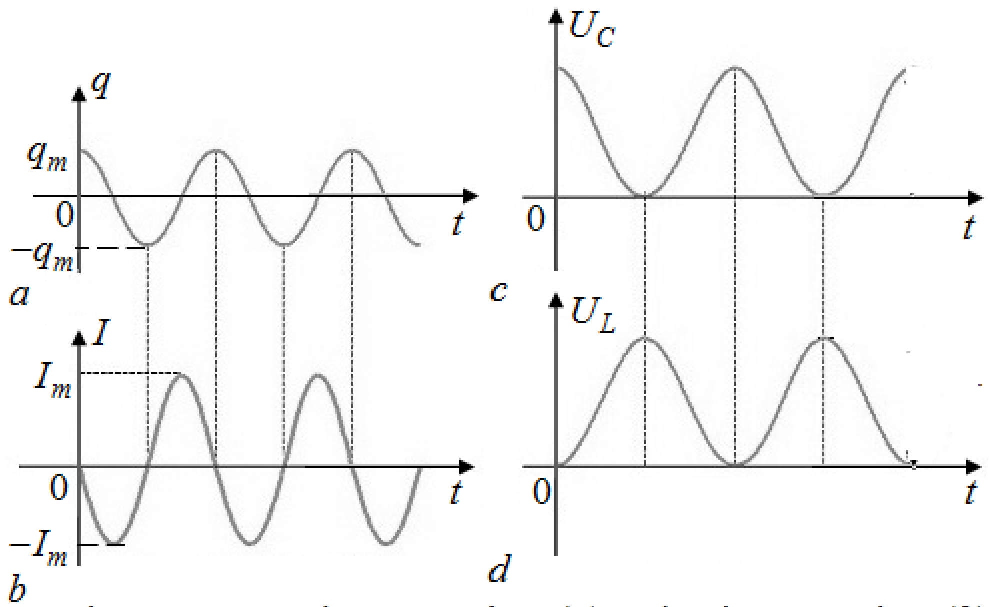

Figure 9.8 – Charge vs. time (a) and voltage vs. time (b), U_C vs. time (c) and U_L vs. time (d) for LC circuit

After reaching its maximum I_m, the current I continues to transport charge between the capacitor plates, thereby recharging the capacitor. Since the inductor resists a change in current, current continues to flow, even though the capacitor is discharged. This continued current causes the capacitor to charge with opposite polarity. The electric field of the capacitor increases while the magnetic field of the inductor diminishes and the overall effect is a transfer of energy from the inductor back to the capacitor. From the law of energy conservation, the maximum charge that the capacitor re-acquires is q_m. However, the capacitor plates are charged opposite to what they were initially (Fig. 9.7, c).

When fully charged, the capacitor once again transfers its energy to the inductor until it is again completely discharged (Fig. 9.7, d). Then, in the last part of this cyclic process, energy flows back to the capacitor, and the initial state of the circuit is restored (Fig. 9.7, e).

The electromagnetic oscillations are analogous to the mechanical oscillations of a mass at the end of a spring (Fig. 9.7). In this latter case, energy is transferred back and forth between the mass, which has kinetic energy $KE = \frac{mv^2}{2}$, and the spring, which has potential energy $PE = \frac{kx^2}{2}$. With the absence of friction in the mass-spring system, the oscillations would continue indefinitely. Similarly, the oscillations of an LC circuit with no resistance would continue forever if undisturbed.

9.5.2. Free damped oscillation in RLC circuit

When $R \neq 0$ and $\mathrm{E} = 0$, equation (9.32) takes on the form

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q = 0. \qquad (9.46)$$

If we denote $\frac{R}{L} = 2\beta$ and $\frac{1}{LC} = \omega_0^2$ the equation (9.46) can be expressed as

$$\ddot{q} + 2\beta\dot{q} + \omega_0^2 q = 0. \qquad (9.47)$$

This equation has the general solution at the initial condition of $q(t=0) = q_0$ and at weak damping:

$$q = q_0 e^{-\beta t}\cos(\omega t + \alpha). \qquad (9.48)$$

The equation (9.48) describes a sinusoidal oscillation with an exponentially decaying ***amplitude*** $q_0 e^{-\beta t}$.

In this equation,

$$\beta = \frac{R}{2L} \tag{9.49}$$

is a ***damping*** or ***decay constant*** (***damping factor***) that describes the damping rate, or how apparent resistance is in the system. A large β means that oscillations are stopped rapidly; while β approaching zero gives an infinitely-oscillating system (LC circuit).

The other parameter, ω_0, is the natural frequency of the system; that is, if the damping is reduced to almost zero, the system would oscillate with this frequency. When with damping, the oscillation frequency is an ***angular frequency of the damped oscillations***

$$\omega = \sqrt{\omega_0^2 - \beta^2} = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} . \tag{9.50}$$

Since L, C and R are the constant magnitudes for a certain circuit, therefore, a ***period of damped oscillations***

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\omega_0^2 - \beta^2}} = \frac{2\pi}{\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}} \tag{9.51}$$

is assumed to constant. It is possible at a condition of the weak damping due to the small resistance. As a result, the period T is a conditional period unlike the period (9.36) $T_0 = 2\pi\sqrt{LC}$, which is a natural period of undamped oscillations.

The ***voltage*** across a capacitance is

$$U = U_0 e^{-\beta t} \cos(\omega t + \alpha) . \tag{9.52}$$

The ***current*** in an inductor is

$$I = \frac{dq}{dt} = q_0 e^{-\beta t}\left[-\beta \cos(\omega t + \alpha) - \omega \sin(\omega t + \alpha)\right] = \\ = \omega\, q_0 e^{-\beta t} \cos(\omega t + \alpha + \phi), \tag{9.53}$$

where δ is introduced according to $\cos\phi = -\beta/\omega_0$ and $\sin\phi = \omega/\omega_0$.

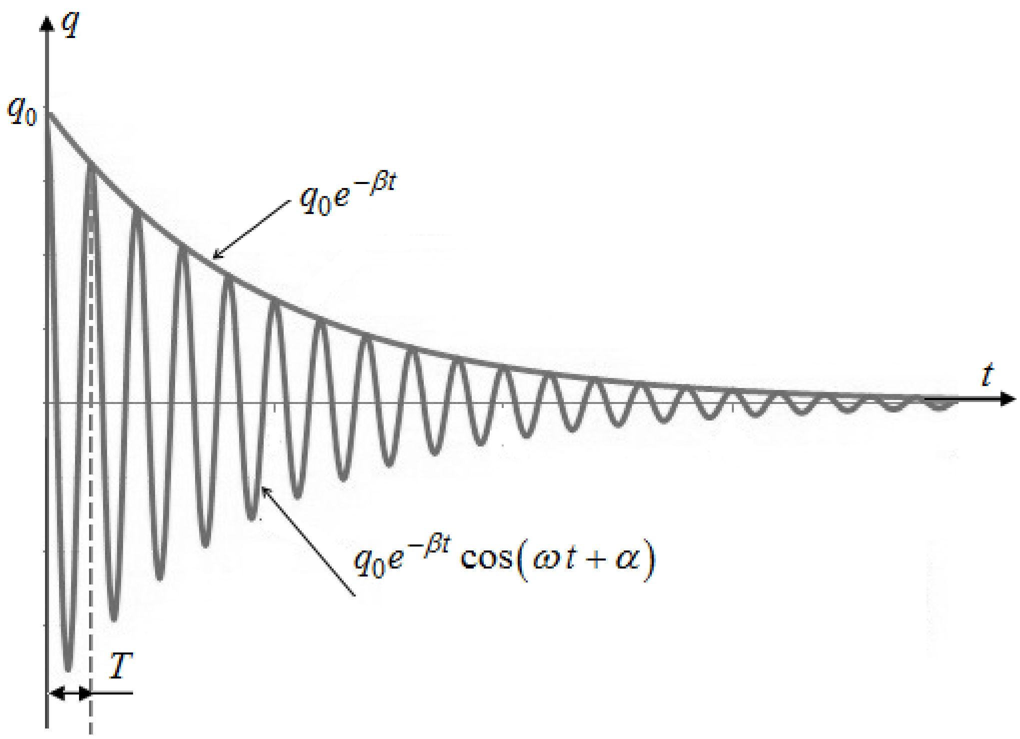

Figure 9.9 – Damped oscillations in *RLC* circuit

Because the charge is still decaying the decrement D can be defined as the ratio between two successive peaks of charge

$$D=\frac{q_0}{q_1}=\frac{q_0}{q_0e^{-\beta T}}=e^{-\beta T} \tag{9.54}$$

and ***logarithmic decrement*** (***log decrement***) δ, which is the natural logarithm of the decrement:

$$\delta=\ln D=\beta T. \tag{9.55}$$

The ***quality factor*** Q of the circuit is defined as

$$Q=\frac{\pi}{\delta}. \tag{9.56}$$

This magnitude shows the ratio of the energy of oscillations to the energy lost in one cycle.

The ***decay*** (or ***relaxation) time*** τ for circuit is the period of time during which the charge amplitude decreases by factor $e=2.71828$.

$$e=\frac{q_0}{q_\tau}=\frac{q_0}{q_0e^{-\beta\tau}}=e^{\beta\tau},$$

$$\tau=\frac{1}{\beta}=\frac{2L}{R}. \tag{9.57}$$

It is clear that the greater *R*, the less τ, the faster the oscillations will stop.

The relationship between *Q*-factor and relaxation time is given as

$$Q = \frac{\pi}{\beta T} = \frac{\pi\omega}{\beta 2\pi} = \frac{\omega}{2\beta} = \frac{\omega}{2}\tau \,. \qquad (9.58)$$

When we consider larger values of *R*, we find that the oscillations damp out more rapidly; in fact, there exists a critical resistance R_c value above which no oscillations occur, and a system with is said to be ***critically damped*** (Fig. 9.10, b).

If the angular frequency is $\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = 0$, the ***critical resistance*** is

$$R_c = 2\sqrt{\frac{L}{C}} \qquad (9.59)$$

When $R > R_c$ the system is said to be ***overdamped***. The oscillator is aperiodic which means that there is no periodic process, and the charge in the capacitor falls to zero exponentially and very quick (Fig. 9.10, a).

When $R < R_c$ the system is ***underdamped***. Our previous description of damped oscillations was related just the underdamped case when the amplitude decreases exponentially in time. The damped oscillation exhibited by the underdamped response is known as ***ringing***. It stems from the ability of the *L* and *C* to transfer energy back and forth between them.

9.5.3. Driven oscillations in *RLC* circuit. Resonance

Assuming that the capacitor is initially uncharged so that $I = +dq/dt$ is proportional to the increase of charge in the capacitor, the equation (9.32) can be rewritten as

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = \mathrm{E}_m \sin\omega_d t \,, \qquad (9.60)$$

where $\mathrm{E}_m \sin\omega t$ is the alternating emf with angular frequency ω_d.

Now, using the previous notation for $\omega_0^2 = 1/LC$ and $\beta = R/2L$, we get the equation

$$\ddot{q} + 2\beta\dot{q} + \omega_0^2 q = \frac{\mathrm{E}_m}{L}\sin\omega_d t \,. \qquad (9.61)$$

One possible solution to this equation is

$$q = q_m \cos(\omega_d t - \phi), \tag{9.62}$$

where the maximum charge is

$$q_m = \frac{E_m}{\omega_d \sqrt{R^2 + (\omega_d L - (1/\omega_d C))^2}}. \tag{9.63}$$

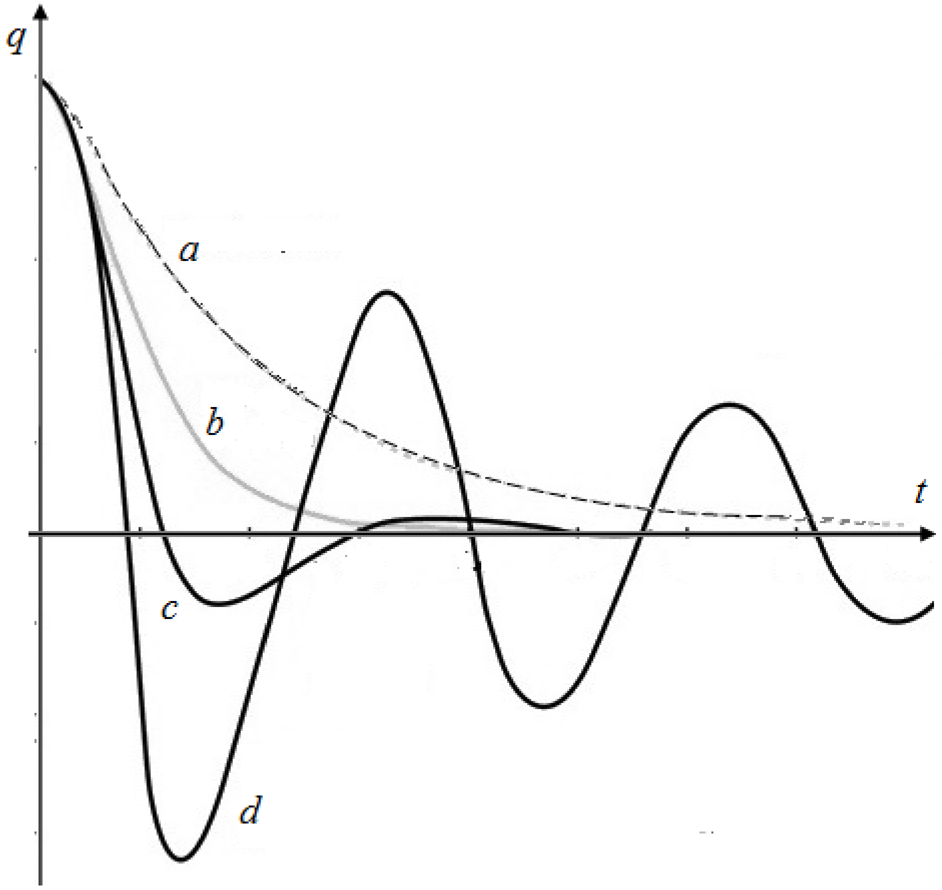

Figure 9.10 – Overdamping (a), critical dampimg (b), underdamping: one-half (c) and one-tenth (d) of critical damping

The corresponding current is

$$I = \frac{dq}{dt} = q_m \omega_d \sin(\omega_d t - \phi), \tag{9.64}$$

with the maximum current

$$I_m = q_m \omega_d = \frac{E_m}{\sqrt{R^2 + (\omega_d L - (1/\omega_d C))^2}} \tag{9.65}$$

and phase

$$\tan\phi = \frac{1}{R}\left(\omega_d L - \frac{1}{\omega_d C}\right). \tag{9.75}$$

As it can be seen from (9.65), the current amplitude I_m in *RLC* circuit is a function of the driving angular frequency ω_d of the external alternating emf. For a given resistance *R*, the amplitude is a maximum when the quantity in the parentheses is equal zero:

$$\omega_d L - \frac{1}{\omega_d C} = 0,$$

then,

$$\omega_d L = \frac{1}{\omega_d C}, \quad (9.76)$$

in other words, when the inductive reactance (9.10) is equal to the capacitive reactance (9.18)

$$X_L = X_C. \quad (9.77)$$

From (9.76), the driving angular frequency equals

$$\omega_d = \frac{1}{\sqrt{LC}}. \quad (9.78)$$

Because the natural angular frequency of the LC circuit is also equal to $\omega_0 = \frac{1}{\sqrt{LC}}$, the maximum value of the current occurs when the driving angular frequency matches the natural angular frequency $\omega_d = \omega_0$. The phenomenon at which the current reaches a maximum is called a ***resonance***, and the relative frequency (9.78) is called the ***resonant frequency***. At resonance, the impedance (9.28) becomes equal to the resistance

$$Z = \sqrt{R^2 + \left(X_L - X_C\right)^2} = R, \quad (9.79)$$

the amplitude of the current (9.65) is

$$I_m = \frac{\mathrm{E}_m}{R} \quad (9.80)$$

and the phase (9.75) is $\phi = 0$.

The qualitative behavior of the current and impedance is illustrated in Fig. 9.11.

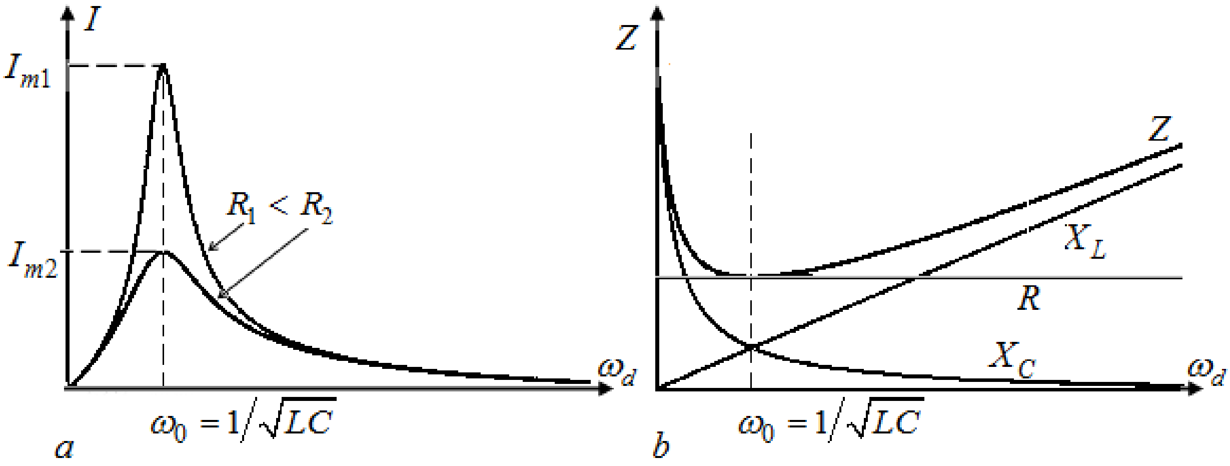

Figure 9.11 – Resonance peaks of the current vs. driven frequency (*a*), impedance vs. driven frequency (*b*)

Notice that the reactances (X_L and X_C) are dependent on the angular frequency while the resistance *R* is not. At the low frequencies, there is no inductive effect: X_L goes to zero and current is passed through the inductor, while no current is passed through the capacitor. As the frequency gets large, X_C goes to zero and current is passed through the capacitor, while no current is passed through the inductor (X_L gets large because of the quickly changing current). These properties can be used to create frequency filters: inductors are used as “low-pass” filters, and capacitors are used as “high-pass” filters. In combination, a “cross-over” circuit can be created. Additionally, since resonance in series *RLC* circuit occurs at a particular frequency, so it is used for tuning purpose as it does not allow unwanted oscillations that would otherwise cause signal distortion, noise and damage to the circuit to pass through it.

The resonance of a ***parallel RLC circuit*** (Fig. 9.12, a) is a bit more involved than the series resonance. The resonant frequency can be defined in three different ways, which converge on the same expression as the series resonant frequency if the resistance of the circuit is small.

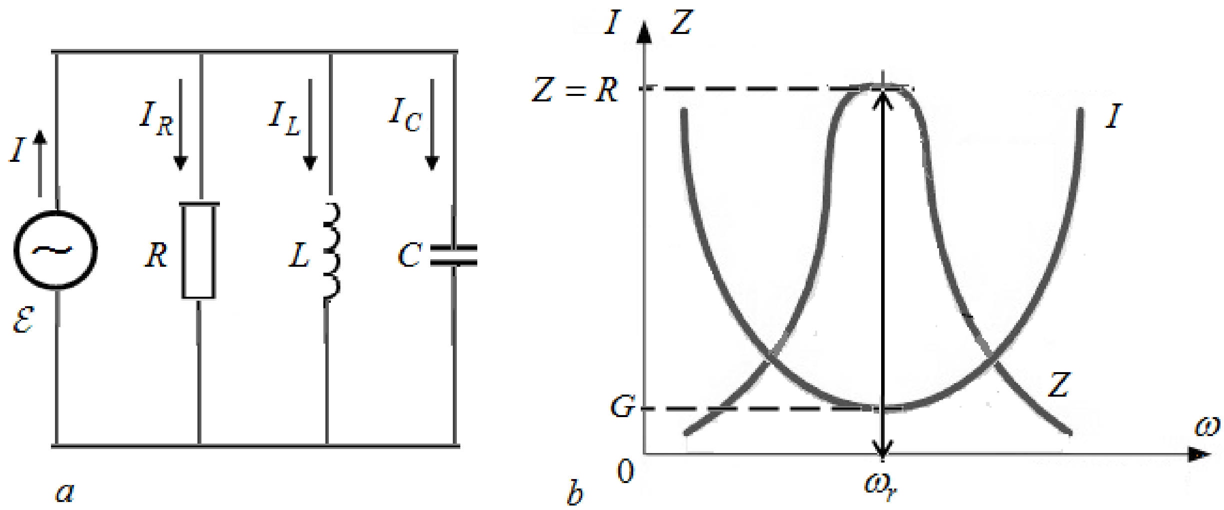

Figure 9.12 – Parallel resonance *RLC* circuit (*a*); impedance vs frequency, current vs. frequency (*b*)

This parallel *RLC* circuit is exactly opposite to series *RLC* circuit. In series *RLC* circuit, the current flowing through all the three components, i.e., the resistor, inductor and capacitor, remains the same, but in the parallel circuit, the voltage *U* across each element remains the same and the current gets divided in each component depending upon the impedance of each component. That is why parallel *RLC* circuit is said to have the dual relationship with series *RLC* circuit.

The current

$$I^2 = I_R^2 + (I_L - I_C)^2, \qquad (9.81)$$

where I is the total source current; I_R, I_L and I_C are the currents flowing through the resistor, inductor, and capacitor, respectively;

$$I = \sqrt{\left(\frac{U}{R}\right)^2 + \left(\frac{U}{X_L} - \frac{U}{X_C}\right)^2}. \qquad (9.82)$$

Admittance Y is equal to

$$Y = \frac{1}{Z} = \frac{I}{U}\sqrt{\left(\frac{1}{R}\right)^2 + \left(\frac{1}{X_L} - \frac{1}{X_C}\right)^2}. \qquad (9.83)$$

The equation (9.83) that in a parallel *RLC* circuit, each element has reciprocal of impedance, i.e., admittance; and the total admittance of the circuit can be found by simply adding each branch's admittance. The branch's admittances are ***conductance*** *G*:

$$G = \frac{1}{R}. \qquad (9.84)$$

and i***nductive*** B_L ***and capacitive*** B_C ***susceptances***, respectively:

$$B_L = \frac{1}{X_L} = \frac{1}{\omega L}, \qquad (9.85)$$

$$B_C = \frac{1}{X_C} = \omega C. \qquad (9.86)$$

Resonance occurs when $X_L = X_C$, therefore, the resonant frequency is $\omega_r = 1/\sqrt{LC}$. Notice that at resonance the parallel circuit produces the same equation as for the series resonance circuit. Therefore, it makes no difference if the inductor or capacitor are connected in parallel or series.

Unlike series *RLC* circuit, in parallel *RLC* circuit, the impedance becomes maximum and the circuit behaves like purely resistive circuit (Fig. 8.12, b).

Resonant circuits are the most important circuits used in electronics. For example, a resonant circuit, in one of many forms, allows tuning into a desired radio or television station from the vast number of signals that are around us at any time.

Fig. 9.13 shows an application of *RLC* circuit to a loudspeaker system. Low-frequency sounds are produced by the woofer (low tons), which is a speaker with a large diameter; the tweeter, a speaker with a smaller diameter, produces high-frequency sounds (high tone). In order to route signals of different frequency to the appropriate speaker, the woofer and tweeter are connected in paraIlel across the amplifier output. The capacitor in the tweeter branch blocks the low-frequency components of sound but passes the higher frequencies; the inductor in the woofer branch does the opposite.

The series or parallel resonance will generate the voltage, which is several times higher than the power source. The voltage can be applied to the equipment in the circuit, such as the capacitor, current transformer, and fuse etc, causing damage to high voltage electrical equipment insulation.

PROBLEMS

Problem 1

The oscillatory circuit consists of a coil with inductance L= 400 μH and a capacitor with a capacitance C = 0.5 μF. Initially, the capacitor was charged to a voltage of 65 V. Find the time dependence for the charge and the voltage across capacitor, and the current through inductor. Determine the current in the circuit at the instant of time when the voltage across capacitor decreased to 56.3 V. Neglect the resistance of wires in the circuit.

Solution

The initial charge of capacitor determines the initial voltage across it, which is the maximum voltage U_m = 65 V. Proceeding from this fact, the equation of the oscillations should be written as $U = U_m \cos \omega_0 t$ since it satisfies this condition, namely, the instantaneous voltage at the instant of time $t = 0$ is $U = U_m$.

The natural frequency of LC circuit is equal to

$$\omega_0 = 1/\sqrt{LC} = 1/\sqrt{400 \cdot 10^{-6} \cdot 0.5 \cdot 10^{-6}} = 7.07 \cdot 10^4 \text{ s}^{-1}.$$

The law of the voltage change across capacitor is

$$U = U_m \cos \omega_0 t = 65 \cos 7.07 \cdot 10^4 t \text{ (V)}$$

Considering that $q = CU$, the maximum charge will be

$$q_m = CU_m = 0.5 \cdot 10^{-6} \cdot 65 = 3.25 \cdot 10^{-5} \text{ (C)},$$

and the equation for the charge (since it is in phase with the voltage across the capacitor) takes on the form

$$q = q_m \cos \omega_0 t = 3.25 \cdot 10^{-5} \cos 7.07 \cdot 10^4 t \text{ (C)}.$$

The instantaneous current through the coil is determined by differentiating the dependence of the charge with respect to time:

$$I = \frac{dq}{dt} = \frac{d}{dt}\left(q_m \cos \omega_0 t\right) = -q_m \omega_0 \sin \omega_0 t .$$

According to this time dependence of current, it is possible to calculate the desired instantaneous current. It is necessary to know the instant of time. But we can facilitate the task calculating $\cos\omega_0 t$ (instead of the instant of time t), and then, using the basic trigonometric relation, we can find the corresponding magnitude of $\sin\omega_0 t$.

Since $U = U_m \cos\omega_0 t$, $\cos\omega_0 t = \dfrac{U}{U_m} = \dfrac{56.3}{65} = 0.866$.

As a result,

$$\sin\omega_0 t = \sqrt{1-\cos^2\omega_0 t} = 0.5.$$

Finally, the current at the desired time is

$$I = -q_m\omega_0 \sin\omega_0 t = -3.25\cdot 10^{-5}\cdot 7.07\cdot 10^4\cdot 0.5 = -2.3\cdot 0,5 = -1.15 \text{ A.}$$

Problem 2

RLC circuit consists of a capacitor of C = 7 μF, an inductor of C= 0.23 H and an active resistance R = 40 Ω. Initially it was charged to the charge q_0 = 0.56 μC. Find the period of oscillations, logarithmic decrement, and the equation for the time dependence of the potential difference across the capacitor.

Solution

If the circuit consists of the capacitance, inductance and active resistance, free damped oscillations take place in it. The conditional period of these oscillations is

$$T = \frac{2\pi}{\sqrt{1/LC - (R/2L)^2}} = \frac{2\pi}{\sqrt{1/\left(0.23\cdot 7\cdot 10^{-6}\right) - \left(40/2\cdot 0.23\right)^2}} = 8\cdot 10^{-3} \text{ s.}$$

To determine the logarithmic decrement of damped oscillations we have to calculate the damping coefficient

$$\beta = \frac{R}{2L} = \frac{40}{2\cdot 0.23} = 87 \text{ s}^{-1}.$$

As a result, the logarithmic decrement is equal to

$$\delta = \beta T = 87 \cdot 8 \cdot 10^{-3} = 0.7 .$$

The equation of the potential difference across the capacitor as a function of time is

$$U = U_0 e^{-\beta t} \cos(\omega t + \alpha) ,$$

where the maximum value of the voltage across the capacitor is

$$U_0 = \frac{q_0}{C} = \frac{0.56 \cdot 10^{-3}}{7 \cdot 10^{-6}} = 80 ,$$

and the frequency of damped oscillations is

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{8 \cdot 10^{-3}} = 250\pi = 785 \text{ s}^{-1}.$$

The charge and voltage across the capacitor are in phase. If the capacitor was charged at the initial instant of time, then the capacitor was at the maximum voltage. Therefore, the initial phase $\alpha = 0$. Then the equation of potential difference across capacitor is

$$U(t) = 80 e^{-0.7t} \cos(250\pi t) \text{ (V).}$$

Problem 3

Find the number of oscillations during which the charge amplitude in RLC-circuit with a logarithmic decrement of 0.0004 decreases fivefold. Write the dependences of the change in charge, voltage and current as a function of time, if the capacitor initially was charged up to 100 V. The frequency of damped oscillations is $2 \cdot 10^3$ s^{-1}, and the inductance of the coil is 0.5 mH.

Solution

The charge amplitude has been decreased by five times

$$5 = \frac{q_0}{q_t} = \frac{\cancel{q_0}}{\cancel{q_0} \cdot e^{-\beta t}} = e^{\beta t} .$$

Taking the logarithm gives

$$\beta t = \ln 5.$$

Considering that the time of the amplitude reduction consists of N oscillation periods, that is, $t = NT$, it is possible to write down

$$\ln 5 = \beta t = \beta NT = N\delta.$$

As a result, the number of oscillations is

$$N = \frac{\ln 5}{\delta} = \frac{1.609}{0.0004} = 4020.$$

This number of oscillation occurs over time t

$$t = NT = \frac{2\pi N}{\omega} = \frac{\cancel{2}\pi \cdot 4020}{\cancel{2} \cdot 10^3} = 12.6.$$

With this in mind, the damping coefficient is equal to

$$\beta = \frac{\ln 5}{t} = \frac{1.609}{12.6} = 0.128\ \text{s}^{-1}.$$

The voltage across the capacitor depends on time as

$$U = U_0 e^{-\beta t}\cos(\omega t + \alpha)$$

To write this dependence it is necessary to find the maximum voltage U_0 and initial phase α. Based on the data, the capacitor was charged to a voltage of 100 V, which means that the initial amplitude of voltage is U_0 = 100 V. If the instantaneous voltage is $U = U_0$ at $t = 0$, the initial phase $\alpha = 0$.

Angular frequency of damped oscillations is

$$\omega = 2\pi\upsilon = 2\pi \cdot 2 \cdot 10^3 = 4\pi \cdot 10^3 \text{rad/s}.$$

Finally, the voltage across capacitor depends on time as

$$U = 100 \cdot e^{-0{,}126t} \cdot \cos 4\pi \cdot 10^3 t\ \text{(V)}.$$

The charge of capacitor is changing on the time as

$$q = q_0 e^{-\beta t}\cos\left(\omega t + \alpha\right).$$

Considering that $C = \dfrac{q}{U}$, the maximum charge of the capacitor $q_0 = CU_0$.

Angular frequency of damped oscillations related to the natural angular frequency as $\omega^2 = \omega_0^2 - \beta^2$. Therefore,

$$\omega_0^2 = \omega^2 + \beta^2 = \left(4\pi \cdot 10^3\right)^2 + \left(0.128\right)^2 \approx \left(4\pi \cdot 10^3\right)^2 .$$

In other words, we can consider that $\omega = \omega_0$. Since $\omega_0^2 = \dfrac{1}{LC}$, the capacitance of capacitor is

$$C = \frac{1}{\omega_0^2 L} = \frac{1}{\omega^2 L} = \frac{1}{16\pi^2 \cdot 10^6 \cdot 0.5 \cdot 10^{-3}} = 1.3 \cdot 10^{-5} \text{ F.}$$

Then the maximum charge of capacitor is

$$q_0 = CU_0 = 1.3 \cdot 10^{-5} \cdot 100 = 1.3 \cdot 10^{-3} \text{ (C).}$$

and the time dependence of the charge on the capacitor is

$$q = 1.3 \cdot 10^{-3} \cdot e^{-0.126t} \cdot \cos 4\pi \cdot 10^3 t \quad \text{(C).}$$

To obtain the time dependence of current through the coil, it is necessary to differentiate the expression for the charge $q(t)$.

$$I = \frac{dq}{dt} = I_0 e^{-\beta t} \cos(\omega t + \alpha + \delta),$$

where $I_0 = \omega q_0$, $\cos\delta = -\beta/\omega_0$, $\sin\delta = \omega/\omega_0$.

$$I_0 = q_0 \omega = 1.3 \cdot 10^{-3} \cdot 2 \cdot 10^3 = 2.6 \text{ A},$$

Assume that $\alpha = 0$.

The phase shift can be found from

$$\tan\delta = \frac{\sin\delta}{\cos\delta} = -\frac{\omega \cdot \cancel{\omega_0}}{\cancel{\omega_0} \cdot \beta} = -\frac{\omega}{\beta} = -\frac{4\pi \cdot 10^3}{0.128} = -9.8 \cdot 10^4,$$

$$\delta = \text{arctg}\left(-9.8 \cdot 10^4\right) \approx -\frac{\pi}{2},$$

$$I = I_0 e^{-\beta t} \cos(\omega t + \alpha + \delta) = I_0 e^{-\beta t} \cos\left(\omega t - \frac{\pi}{2}\right) = -I_0 e^{-\beta t} \sin \omega t .$$

Finally, the dependence of the current through the coil on time is:

$$I = -2.6 \cdot e^{-0.128t} \cdot \sin 4\pi \cdot 10^3 t \quad \text{(A).}$$

Problem 4

The natural frequency of oscillation is υ_0 = 8 kHz, the quality factor is Q = 72. If the damped oscillations are observed in this RLC-circuit, define the change in the energy stored in circuit on time $W\left(t\right)$, and a part of energy which remains in the circuit after τ = 1 ms.

Solution

Assume that the oscillations in the RLC- circuit begin with charging the capacitor to the maximum charge q_0. In this case, the energy of the charged capacitor is equal to $W_{0C} = \frac{q_0^2}{2C}$. When $t=0$ the current in the circuit is absent, therefore, there is no magnetic field of the coil. That is, and the total energy of the circuit is equal to the energy of charged capacitor $W_0 = W_{0C} = \frac{q_0^2}{2C}$. Over time, the maximum charge of the capacitor decreases according to $q(t) = q_0 e^{-\beta \cdot t}$. Thus, the total energy is decreasing as

$$W_t = \frac{q_m^2}{2C} = \frac{\left(q_0 e^{-\beta t}\right)}{2C} = \frac{q_0^2}{2C} \cdot e^{-2\beta t} = W_0 \cdot e^{-2\beta t}$$

The energy that is remained in the circuit after the time τ = 1 ms, is equal to $W_\tau = W_0 \cdot e^{-2\beta\tau}$.

This energy is a fraction of the initial energy, therefore,

$$\frac{W_\tau}{W_0} = \frac{W_0 \cdot e^{-2\beta\tau}}{W_0} = e^{-2\beta\tau}.$$

For calculations, we have to determine the damping coefficient β. But firstly, let’s find the logarithmic decrement of damping

$$\delta = \frac{\pi}{Q} = \frac{\pi}{72} = 0.0436.$$

According to definition, $\delta = \beta T = \beta \frac{2\pi}{\sqrt{\omega_0^2 - \beta^2}}$.

Let us square this expression for finding β:

$$\delta^2 = \frac{(2\pi\beta)^2}{\omega_0^2 - \beta^2},$$

$$\delta^2\omega_0^2 - \delta^2\beta^2 = 4\pi^2\beta^2,$$

$$\delta^2\omega_0^2 = \beta^2\left(4\pi^2 + \delta^2\right),$$

$$\beta = \frac{\delta\omega_0}{\sqrt{\delta^2 + 4\pi^2}} \ .$$

Since $\delta^2 \ll 4\pi^2$, then

$$\beta = \frac{\delta\omega_0}{\sqrt{\delta^2 + 4\pi^2}} \approx \frac{\delta\omega_0}{2\pi}.$$

Then,

$$\beta = \frac{\delta\omega_0}{2\pi} = \frac{\delta 2\pi\upsilon_0}{2\pi} = \delta\upsilon_0 = 0.0436 \cdot 8 \cdot 10^3 = 349 \ \ \text{s}^{-1}.$$

Finally,

$$\frac{W_\tau}{W_0} = e^{-2\beta\tau} = e^{-2\cdot 349 \cdot 0.001} = 0.5 \ .$$

Thus, a half of the initial energy will remain in the circuit after the time interval τ = 1 ms.

Problem 5

The oscillatory circuit consists of the capacitor of capacitance C = 10 μF, the coil of inductance L= 25 mH and active resistance R= 0.1 Ω. Find for what time the stored energy decreases e^2*, and how many oscillations are necessary for this reduction. Determine the logarithmic decrement.*

Solution

The energy stored in the circuit $W = \frac{q^2}{2C}$ is proportional to q^2, therefore, the instantaneous energy is

$$W = \frac{q^2}{2C} = \frac{\left(q_0 e^{-\beta t}\cos\left(\omega t + \alpha\right)\right)^2}{2C} = \frac{q_0^2}{2C} e^{-2\beta t}\cos^2\left(\omega t + \alpha\right) = W_0 e^{-2\beta t}\cos^2\left(\omega t + \alpha\right).$$

This means that the amplitude of the energy depends on time according to $W_t = W_0 e^{-2\beta t}$. Then

$$\frac{W_0}{W_t} = \frac{W_0}{W_0 e^{-2\beta t}} = e^{2\beta t}.$$

From the data of the problem,

$$\frac{W_0}{W_t} = e^{2\beta t} = e^2,$$

As a result,

$$\beta t = 1.$$

The damping coefficient can be expressed through the parameters of the circuit as

$$\beta = \frac{R}{2L} = \frac{0{,}1}{2\cdot 25\cdot 10^{-3}} = 2\ \text{s}^{-1}.$$

Then the time interval during which the energy that was stored in the circuit decreases by a factor of e^2 is

$$t = \frac{1}{\beta} = \frac{1}{2} = 0.5\ \text{s}.$$

By the way, note that this time interval is the relaxation time.

To determine the number of oscillations for this time interval, let's find the angular frequency of damped oscillations:

$$\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = \sqrt{\frac{1}{25\cdot 10^{-3}\cdot 10\cdot 10^{-6}} - \frac{0.01}{4\cdot 625\cdot 10^{-6}}} = 2000\ \text{rad/s}$$

The conditional period of damped oscillations is

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{2000} = 3.14 \cdot 10^{-3} \text{ s.}$$

Then from $\beta t = \beta NT = 1$, the desired number of oscillations equals

$$N = \frac{1}{\beta T} = \frac{1}{2 \cdot 3{,}14 \cdot 10^{-3}} = 159 .$$

The logarithmic decrement will be

$$\delta = \beta T = 2 \cdot 3.14 \cdot 10^{-3} = 6.28 \cdot 10^{-3}.$$

Problem 6

The inductance of the LC-circuit is 0.5 mH. Find the capacity of capacitor in the circuit if the emitted wavelength λ *= 300 m.*

Solution

The angular frequency of the oscillations is $\omega_0 = \frac{1}{\sqrt{LC}}$

The wavelength of the radiated electromagnetic wave is equal to

$$\lambda = cT = c\frac{2\pi}{\omega} = c\frac{2\pi}{\omega_0} = 2\pi c\sqrt{LC} .$$

Therefore, the capacitance is expressed as

$$C = \frac{\lambda^2}{4\pi^2 c^2 L} = \frac{9 \cdot 10^4}{4 \cdot \pi^2 \cdot 9 \cdot 10^{16} \cdot 0.5 \cdot 10^{-3}} = 5 \cdot 10^{-11} \text{ F = 50 pF.}$$

Problem 7

Find the capacitance of capacitor if the wavelength of electromagnetic radiation emitted by this circuit is λ *= 300 m, the rate of the current change in the coil is* $\Delta I/\Delta t$ *= 4 A/s, and the motional emf arising in the circuit is* ε_s *= 0.04 V.*

Solution

The emf of self-induction that arises in the oscillatory circuit according to Faraday's law is

$$\varepsilon_s = -L\frac{dI}{dt}.$$

The inductance of the circuit is

$$L = \frac{\varepsilon_s}{\Delta I/\Delta t} = \frac{0.04}{4} = 0.01 \text{ H}.$$

If the period of undamped electromagnetic oscillations is $T_0 = 2\pi\sqrt{LC}$, and the speed of the electromagnetic waves propagation in vacuum (air) is $c = 3\cdot 10^8$ m/s, the wavelength that the LC-circuit emits will be

$$\lambda = cT = c2\pi\sqrt{LC}.$$

Hence, the capacitance of the capacitor is determined as

$$C = \frac{\lambda^2}{4\pi^2 c^2 L} = \frac{300^2}{4\pi^2\cdot\left(3\cdot 10^8\right)^2\cdot 0.01} = 2.53\cdot 10^{-12} \text{ F} = 2.53 \text{ pF}.$$