NATIONAL TECHNICAL UNIVERSITY
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STUDY GUIDE

<u>“RELATIVITY”</u>

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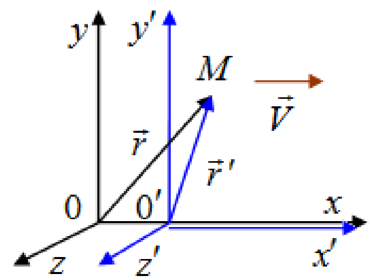
RELATIVITY

1. Galileo transformation

There are two frames of references (K and K'), and K' is moving relatively K at velocity $\vec{V}$. The relationship between coordinates, velocities and accelerations of the object M in these frames are the following:



$$\begin{cases} x' = x - Vt \\ y' = y \\ z' = z \\ t' = t \end{cases} \quad \Rightarrow \quad \vec{r}' = \vec{r} - \vec{V}t$$

$$\vec{v}' = \frac{d\vec{r}'}{dt'} = \frac{d\vec{r}'}{dt} = \frac{d}{dt}\left(\vec{r} - \vec{V}t\right) = \frac{d\vec{r}}{dt} - \vec{V} = \vec{v} - \vec{V}.$$

$$\vec{v}' = \vec{v} - \vec{V} \; .$$

This is the classical (Galilean) addition law for velocities.

$$\vec{a}' = \frac{d\vec{v}'}{dt'} = \frac{d\vec{v}'}{dt} = \frac{d}{dt}\left(\vec{v} - \vec{V}\right) = \frac{d\vec{v}}{dt} = \vec{a}.$$

The acceleration is the same in both systems. The equality of accelerations in two inertial frames of reference expressed the fact that the acceleration is Galilean invariant.

Forces depend on the mutual arrangement $\left(\vec{r}_{12}\right)$ and the velocity $\left(\vec{u}_{12}\right)$ of relative motion of the points

$$\left.\begin{aligned} \vec{r}_{12}' &= \vec{r}_1' - \vec{r}_2' = \left(\vec{r}_1 - \vec{V}t\right) - \left(\vec{r}_2 - \vec{V}t\right) = \vec{r}_1 - \vec{V}t - \vec{r}_2 + \vec{V}t = \vec{r}_{12} \\ \vec{u}_{12}' &= \vec{v}_1' - \vec{v}_2' = \left(\vec{v}_1 - \vec{V}\right) - \left(\vec{v}_2 - \vec{V}\right) = \vec{v}_1 - \vec{V} - \vec{v}_2 + \vec{V} = \vec{u}_{12} \end{aligned}\right\} \Rightarrow \vec{F}' = \vec{F} \; .$$

Consequently, the force is Galilean invariant as well.

Moreover, it is adopted, that the equation of Newton's 2 law $\vec{F} = m\vec{a}$ is also Galileo invariance. So, *mechanical (Galileo's) principle* of relativity is the following: *all mechanical phenomena occur identically under the same conditions*.

2. Speed of light. The Michelson-Morley experiment. Einstein's postulates

According to Galilean addition law for velocities, the speed of light is different in the frame of reference at rest in moving frame. The most famous experiment designed to detect these small changes in the speed of light was first performed in 1881 by Albert A. Michelson (1852–1931) and later repeated under various conditions by Michelson and Edward W. Morley (1838–1923). They examined the speed of light along and opposite the Earth rotation, but no changes of the light speed was ever observed.

In 1905 Albert Einstein (1879-1955) proposed a theory that explained the result of the Michelson–Morley experiment and completely altered our notions of space and time. He based his special theory of relativity on two postulates:

1. ***The principle of relativity*:** All the laws of physics are the same in all inertial frames.

2. ***The constancy of the speed of light*:** The speed of light in a vacuum has the same value, $c = 2.99792458 \cdot 10^8 \approx 3 \cdot 10^8$ m/s, in all inertial reference frames, regardless of the velocity of the observer or the velocity of the source emitting the light.

3. Lorentz transformation

The Galilean transformation is not valid when the velocity of an object approaches the speed of light. This transformation, known as the Lorentz transformation, was derived by Dutch physicist Hendrik A. Lorentz (1853–1928) in 1890. The Lorentz coordinate transformation is a set of formulas that relates the space and time coordinates of two inertial observers moving with a relative speed V. If $\beta = \frac{V}{c}$ and $\frac{1}{\sqrt{1-\beta^2}}$ - the *Lorentz factor,* Lorenz transformation is

$$\begin{cases} x' = \dfrac{x - Vt}{\sqrt{1-\beta^2}} \\ y' = y \\ z' = z \\ t' = \dfrac{t - \dfrac{V}{c^2}x}{\sqrt{1-\beta^2}} \end{cases}$$

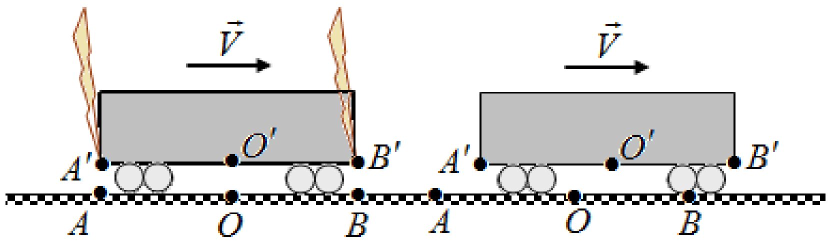
In the Lorentz transformation, time t' (in moving frame) depends on both t and x (time and coordinate in frame at rest). This is unlike the case of the Galilean transformation, in which $t = t'$. When $V \ll c$, the Lorentz transformation should reduce to the Galilean transformation.

4. Consequences of special relativity

We restrict our discussion to the concepts of length, time, and simultaneity, which are quite different in relativistic mechanics and Newtonian mechanics. For example, we will find that the distance between two points and the time interval between two events depend on the frame of reference in which they are measured. That is, there is no such thing as absolute length or absolute time in relativity. Furthermore, events at different locations that occur simultaneously in one frame are not simultaneous in another frame moving uniformly past the first.

(a) ***Simultaneity and the relativity of time.*** Two events that are simultaneous in one frame are in general not simultaneous in a second frame moving with respect to the first. That is, simultaneity is not an absolute concept, but one that depends on the state of motion of the observer.

Einstein devised the following thought experiment to illustrate this point. A boxcar moves with uniform velocity $\vec{V}$, and two lightning bolts strike the ends of the boxcar, leaving marks on the boxcar and ground. The marks left on the boxcar are labeled A' and B'; those on the ground are labeled A and B. An observer at O' moving with the boxcar is midway between A' and B', and a ground observer at O is midway between A and B. The events recorded by the observers are the light signals from the lightning bolts. The two light signals reach the observer at O at the same time. This observer realizes that the light signals have traveled at the same speed over equal distances. Thus, observer O concludes that the events at A and B occurred simultaneously. Now consider the same events as viewed by the observer on the boxcar at O'. By the time the light has reached observer O,



observer O' has moved forward. Thus, the light signal from B' has already swept past O', but the light from A' has not yet reached O'. According to Einstein, *observer O' must find that light travels at the same speed as that measured by observer O*. Therefore, observer O' concludes that the lightning struck the front of the boxcar before it struck the back. This thought experiment clearly demonstrates that the two events, which appear to O to be simultaneous, do not appear to O' to be simultaneous. Although the two observers reach different conclusions, both are correct in their own reference frame because the concept of simultaneity is not absolute. Both observers are correct, because the principle of relativity states that *there is no preferred inertial frame of reference*. This, in fact, is the central point of relativity: any uniformly moving frame of reference can be used to describe events and do physics.

(b) ***Time dilation.*** Observers in different inertial frames always measure different time intervals between a pair of events

$$\tau = \frac{\tau_0}{\sqrt{1-\beta^2}},$$

where the proper time τ_0 is the time interval between two events measured by an observer moving along with the clock, and τ is the time interval measured by the observer moving with respect to the clock. As it is seen from this formula, a moving clock runs slower.

Time dilation is a very real phenomenon that has been verified by various experiments. For example, muons are unstable elementary particles that have a charge equal to that of an electron and a mass 207 times that of the electron. Muons are naturally produced by the collision of cosmic radiation with atoms at a height of several thousand meters above the surface of the Earth. Muons have a lifetime of only 2.2 μs when measured in a reference frame at rest with respect to them. If we take 2.2 μs (proper time) as the average lifetime of a muon and assume that its speed is close to the speed of light, we would find that these particles could travel a distance of about 650 m before they decayed. Hence, they could not reach the Earth from the upper atmosphere where they are produced. However, experiments show that a large number of muons do reach the Earth. The phenomenon of time dilation explains this effect. Relative to an observer on Earth, the muons have a lifetime equal to $\tau = \frac{\tau_0}{\sqrt{1-\beta^2}} = \frac{2.2 \cdot 10^{-6}}{\sqrt{1-0.99^2}} = 16$ μs, if muon's speed is $v = 0.99c$. Hence, the average distance traveled as measured by an observer on Earth is about $4700\,\text{m}$.

Twins Paradox. There are two identical 20-year-old twins. The twins carry with them identical clocks that have been synchronized. The first brother sets out on a journey to planet X. His spaceship is capable of a speed of 0.500*c* relative to the inertial frame of his twin brother. After reaching planet X, the brother returns to Earth at the same high speed. On his return, the astronaut discovered that his twin brother on the Earth *has aged more than he* and is now 60 years of age. The astronaut, on the other hand, has aged by only 34.6 years. But which twin is the traveler and which twin would really be the younger of the two? If motion is relative, the twins are in a symmetric situation and either's point of view is equally valid. From the point of view of astronaut, it is he who is at rest while twin on the Earth is on a high-speed space journey. This leads to the paradox: Which twin will have developed the signs of excess aging?

To resolve this apparent paradox, recall that special relativity deals with inertial frames of reference moving with respect to one another at uniform speed.

However, the trip situation is not symmetric. The space traveler must experience acceleration during his journey. As a result, his state of motion is not always uniform, and consequently he is not in an inertial frame. Therefore, there is no paradox; there is an incorrect application of special relativity.

(c) ***Length contraction.*** The measured distance between two points depends on the frame of reference. The proper length l_0 of an object is defined as the length of the object measured by someone who is at rest with respect to the object. The proper length is defined similarly to proper time, i.e. it is measured by an observer who is at rest with respect to the object. The length l of an object measured by someone in a reference frame that is moving relative to the object is always less than the proper length.

$$l = l_0\sqrt{1-\beta^2} .$$

Note that the length contraction takes place only along the direction of motion.

(d) Relativistic addition law for velocities

$$v' = \frac{v - V}{1 - \frac{Vv}{c^2}} .$$

5. Relativistic dynamics

Relativistic momentum

$$\vec{p} = \frac{m_0\vec{v}}{\sqrt{1-\beta^2}} ,$$

where m_0 is the rest mass. Note that at $v << c$, it reduces to $p = m_0 v$.

The magnitude of linear momentum is

$$p = \frac{mv}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \frac{m\beta c}{\sqrt{1-\beta^2}}$$

Relativistic mass $m = \frac{m_0}{\sqrt{1-\beta^2}}$

Newton's 2 Law $\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}\left(\frac{m_0\vec{v}}{\sqrt{1-\beta^2}}\right)$,

Total energy $E = mc^2 = \frac{m_0c^2}{\sqrt{1-\beta^2}}$.

Rest energy $E_0 = m_0c^2$.

Kinetic energy

$$E_k = E - E_0 = mc^2 - m_0c^2 = \frac{m_0c^2}{\sqrt{1-\beta^2}} - m_0c^2 = m_0c^2\left(\frac{1}{\sqrt{1-\beta^2}} - 1\right).$$

Relationship between the energy and linear momentum

$\vec{p} = \frac{E}{c^2}\vec{v}$,

$E^2 - p^2c^2 = m_0^2c^4$,

$E = c\sqrt{p^2 + m_0^2c^2}$.

PROBLEMS

Problem 1

How long would a meter stick appear if it were thrown like a spear at 99.5% the speed of light?

Solution

The proper length l_0 is the distance between two points measured by an observer who is *at rest* relative to both of the points. The proper length of the meter stick is $l_0 = 1$m, the Lorenz factor is $\beta = \frac{v}{c} = \frac{0.995c}{c} = 0.995$, therefore, the observed length of the moving stick is

$$l = l_0\sqrt{1-\beta^2} = l_0\sqrt{1-\left(\frac{v}{c}\right)^2} = 1\cdot\sqrt{1-0.995^2} = 0.0999 \text{ m.}$$

Problem 2

Find the speed of the moving body if its length in the direction of motion is twice shorter than its length at rest.

Solution

The relationship between the length l of the moving object and its proper length l_0 is given by

$$l = l_0\sqrt{1-\beta^2}.$$

Since $l = l_0/2$,

$$l_0/2 = l_0\sqrt{1-\beta^2}.$$

Then the Lorentz factor is

$$\beta = \sqrt{3}/2 = 0.866,$$

and the speed of the moving body is

$$v = \beta c = 0.866c = 2.6\cdot 10^8 \text{ m/s.}$$

Problem 3

The average lifetime of an elementary particle π-meson in its own frame of reference (i.e., the proper lifetime) is $2.6\cdot10^{-8}$ s. If π-meson moves at v =0.98c, what is its mean lifetime as measured by an observer on the Earth?

Solution

The π-meson's lifetime in its own frame of reference is the proper time interval, $\tau_0 = 2.6\cdot10^{-8}$ s. An earthbound observer measures a longer dilated time interval τ. The Lorentz factor is

$$\beta = \frac{v}{c} = 0.98.$$

The life time measured by the observer on the Earth is

$$\tau = \frac{\tau_0}{\sqrt{1-\beta^2}} = \frac{2.6\cdot10^{-8}}{\sqrt{1-0.98^2}} = 1.31\cdot10^{-7}\ \text{s}.$$

Problem 4

Find the mass of an electron moving at a speed of 0.999c.

Solution

Applying the equation for the mass of the object moving at relativistic velocity and taking into account the rest mass of electron is $m_0 = 9.1\cdot10^{-31}$ kg, we find that electron mass increases from its rest mass to the magnitude

$$m = \frac{m_0}{\sqrt{1-\beta^2}} = \frac{9.1\cdot10^{-31}}{\sqrt{1-0.999^2}} = 2.04\cdot10^{-29}\ \text{kg}.$$

Problem 5

Find the speed at which the relativistic mass is thrice greater than its rest mass?

Solution

When the object is moving at the velocity which is closed to the speed of light its mass is

$$m = \frac{m_0}{\sqrt{1-\beta^2}},$$

where $\beta = \frac{v}{c}$, and m_0 is the rest mass.

Since $m = 3m_0$, then $3m_0 = \frac{m_0}{\sqrt{1-\beta^2}}$, and

$$\sqrt{1-\beta^2} = \frac{1}{3},$$

$$\beta = 0.94.$$

The speed of the object is

$$v = \beta c = 2.82 \cdot 10^8 \text{ m/s}.$$

Problem 6

A certain chemical reaction requires 48.2 kJ of energy input for it to go. What is the increase in mass of the product over the reactants?

Solution

The relation of the mass and energy is

$$E = mc^2.$$

Hence, the energy increase gives rise to the increase in mass $\Delta E = \Delta mc^2$. From there,

$$\Delta m = \frac{\Delta E}{c^2} = \frac{4.82 \cdot 10^4}{\left(3 \cdot 10^8\right)^2} = 5.36 \cdot 10^{-13} \text{kg}.$$

Problem 7

A particle of rest mass m_0 *travels at a speed* v_1*=0.2c. At what speed* v_2 *will its momentum be doubled?*

Solution

If the particle travels at the speed $v_1 = 0.2c$, its linear momentum is

$$p_1 = mv_1 = \frac{m_0 v_1}{\sqrt{1-\left(\frac{v_1}{c}\right)^2}} = \frac{m_0 \cdot 0.2c}{\sqrt{1-(0.2)^2}} = 0.2m_0 c.$$

Under the condition of the problem, $p_2 = 2p_1$, therefore,

$$p_2 = 2p_1 = 2 \cdot 0.2m_0 c = 0.4m_0 c.$$

$$p_2 = mv_2 = \frac{m_0 v_2}{\sqrt{1-\left(\frac{v_2}{c}\right)^2}}.$$

Since $\frac{v_2}{c} = \beta_2$, the second speed is $v_2 = \beta_2 c$, and

$$\frac{m_0 \beta_2 c}{\sqrt{1-\beta_2^2}} = 0.4m_0 c.$$

$$\frac{\beta_2}{\sqrt{1-\beta_2^2}} = 0.4$$

$$\frac{\beta_2^2}{1-\beta_2^2} = (0.4)^2,$$

$$\beta_2 = 0.37.$$

The speed at which the momentum of the particle is doubled is

$v_2 = \beta_2 c = 0.37c = 0.37 \cdot 3 \cdot 10^8 = 1.11 \cdot 10^8$ m/s.

Problem 8

An electron is accelerated from rest through a potential difference of 10^6 *V. Find the electron's rest energy, kinetic energy, final total energy, and final speed.*

Solution

The rest energy of electron is

$E_0 = m_0c^2 = 9.1\cdot 10^{-31}\cdot\left(3\cdot 10^8\right)^2 = 8.2\cdot 10^{-14}$ J.

We can express this result in units of MeV, using the conversion 1 eV $=1.6\cdot 10^{-19}$ J.

$$E_0 = 8.2\cdot 10^{-14}\text{ J} = \frac{8.2\cdot 10^{-14}}{1.6\cdot 10^{-19}} = 5.125\cdot 10^5\text{eV} = 0.5125\cdot 10^6\text{eV} = 0.51\text{ MeV}$$

Electron gains the kinetic energy due to the work of electric field qU. Since the charge of electron is e, the work of electric field is eU.

$KE = eU = 1.6\cdot 10^{-19}\cdot 10^6 = 1.6\cdot 10^{-13}\text{ J} = 10^6$ eV $=1$ MeV.

The electron's final total energy is

$E = KE + E_0 = 1 + 0.51 = 1.51\text{MeV}.$

Finally we relate energy to speed

$$E = mc^2 = \frac{m_0c^2}{\sqrt{1-\beta^2}} = \frac{E_0}{\sqrt{1-\beta^2}},$$

$$\sqrt{1-\beta^2} = \frac{E_0}{E} = \frac{0.51}{1.51} = 0.338.$$

Solving for β, we find

$\beta = 0.941,$

or

$v = \beta c = 0.941c = 0.941\cdot 3\cdot 10^8 = 2.82\cdot 10^8$ m/s.

Problem 9

Determine the accelerating potential difference passing through which the electron have been accelerated to the velocity equaled to 95% of the light speed.

Solution

The kinetic energy of the electron was gained due to the work of electric field: $KE = W$, or $KE = eU$. Since the particle is moving with relativistic velocity its kinetic energy is

$$KE = m_0c^2\left(\frac{1}{\sqrt{1-\beta^2}} - 1\right).$$

Therefore,

$$U = \frac{m_0c^2}{e}\left(\frac{1}{\sqrt{1-\beta^2}} - 1\right) = \frac{9.1\cdot10^{-31}\cdot9\cdot10^{16}}{1.6\cdot10^{-19}}\left(\frac{1}{\sqrt{1-0.95^2}} - 1\right) = 1.13\cdot10^{6}\text{ V}.$$

Problem 10

The energy of the particle is 10 times greater than its rest energy. Find the velocity of the particle.

Solution

The total energy and rest energy are, respectively,

$$E = mc^2 = \frac{m_0c^2}{\sqrt{1-\beta^2}},$$

$$E_0 = m_0c^2$$

Since $E = 10E_0$, we obtain

$$mc^2 = 10m_0c^2,$$

$$\frac{m_0c^2}{\sqrt{1-\beta^2}} = 10m_0c^2.$$

$\beta = 0.995$,

$v = \beta c = 0.995c = 2.98 \cdot 10^8$ m/s.

Problem 11

Find the velocity of the particle if its kinetic energy is equal to its rest energy.

Solution

Kinetic energy of the particle is equal its rest energy $KE = E_0$, therefore,

$$m_0 c^2 \left(\frac{1}{\sqrt{1-\beta^2}} - 1 \right) = m_0 c^2 ,$$

$$\frac{1}{\sqrt{1-\beta^2}} = 2 ,$$

$$\beta = \frac{\sqrt{3}}{2} = 0.866 .$$

Finally the speed of the particle is

$v = \beta c = 0.866 \cdot 3 \cdot 10^8 = 2.56 \cdot 10^8$ m/s.

Problem 12

The linear accelerator accelerates the electrons from rest through a potential difference of 50 GV. Determine the speed of these electrons.

Solution

Electrons gained their kinetic energy in the accelerating electric field:

$KE = qU$,

where $KE = m_0 c^2 \left(\frac{1}{\sqrt{1-\beta^2}} - 1 \right)$, $\beta = v/c$, and $q = e$.

$$m_0 c^2 \left(\frac{1}{\sqrt{1-\beta^2}} - 1 \right) = eU ,$$

$$\frac{1}{\sqrt{1-\beta^2}}-1=\frac{eU}{m_0c^2},$$

$$\frac{1}{\sqrt{1-\beta^2}}=\frac{eU}{m_0c^2}+1.$$

The usage of the units of energy electron-Volts allows simplifying the calculations. 1 electron-Volt is equal to the energy of electron passed through the potential difference 1 V, therefore, 1 eV= $1.6\cdot 10^{-19}$ J. Consequently, the energy of the electron accelerated by potential difference of 50 GV = $5\cdot 10^{10}$ V is $5\cdot 10^{10}$ eV. The rest energy of electron is $m_0c^2=0.512\,\text{MeV}=5.12\cdot 10^5$ eV. Then the ratio

$$\frac{eU}{m_0c^2}=\frac{50\cdot 10^9}{0.512\cdot 10^6}=97656.25.$$

$$\frac{1}{\sqrt{1-\beta^2}}=97656.25+1=97657.25,$$

$$\beta=0.999.$$

The electron velocity is $0.999c=2.997\cdot 10^8$ m/s.

Problem 13

The mass of the proton after passing through the accelerating potential difference becomes equal the mass of α-particle with the kinetic energy $KE=10^9$ eV. Find this potential difference.

Solution

The total energy of the α-particle consists of the kinetic energy and the rest energy

$E_\alpha=KE_\alpha+E_{0\alpha}$, therefore, $KE_\alpha=E_\alpha-E_{0\alpha}$, or

$$KE_\alpha=m_\alpha c^2-m_{0\alpha}c^2,$$

$$m_\alpha = \frac{KE_\alpha + m_{0\alpha}c^2}{c^2}.$$

The kinetic energy and the mass of the proton are, respectively,

$$KE_p = m_p c^2 - m_{0p}c^2,$$

$$m_p = \frac{KE_p + m_{0p}c^2}{c^2}.$$

According to the given data, $m_p = m_\alpha$, therefore,

$$\frac{KE_\alpha + m_{0\alpha}c^2}{c^2} = \frac{KE_p + m_{0p}c^2}{c^2},$$

$$KE_\alpha + m_{0\alpha}c^2 = KE_p + m_{0p}c^2.$$

Taking into account that $m_{0\alpha} = 4m_{0p}$, we obtain

$$KE_p = KE_\alpha + m_{0\alpha}c^2 - m_{0p}c^2 = KE_\alpha + 4m_{0p}c^2 - m_{0p}c^2 = KE_\alpha + 3m_{0p}c^2.$$

This kinetic energy was gained by proton in the electric field

$$KE_p = eU,$$

$$KE_\alpha + 3m_{0p}c^2 = eU,$$

Finally, the accelerating potential difference is

$$U = \frac{KE_\alpha + 3m_{0p}c^2}{e} = \frac{10^9 \cdot 1.6 \cdot 10^{-19} + 3 \cdot 1.66 \cdot 10^{-27} \cdot 9 \cdot 10^{16}}{1.6 \cdot 10^{-19}} = 3.8 \cdot 10^9\,\text{V}.$$

Problem 14

An electron travels at 0.69 c in a circle at right angles to a uniform magnetic field of strength 2 T. Find the radius of the circle and compare it to the radius calculated without considering mass dilation.

Solution

The force on the charged particle in magnetic field is $\vec{F} = q\left[\vec{v}, \vec{B}\right]$. The equation of motion is

$$ma = \frac{mv^2}{R} = qvB\sin\alpha \ .$$

If the charged particle is electron ($q = e$), the motion is perpendicular to the magnetic field ($\alpha = 90^0, \sin 90^0 = 1$), and the acceleration of the electron is centripetal acceleration ($a = \frac{v^2}{R}$), then

$$r = \frac{m_0 v}{eB} = \frac{m_0 v}{eB} .$$

$$r = \frac{9.1 \cdot 10^{-31} \cdot 0.69 \cdot 3 \cdot 10^8}{1.6 \cdot 10^{-19} \cdot 2} = 5.9 \cdot 10^{-4} \text{m}.$$

When the mass is dilated, $m = \frac{m_0}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$, so

$$r = \frac{mv}{eB} = \frac{m_0 v}{eB\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

$$r = \frac{9.1 \cdot 10^{-31} \cdot 0.69 \cdot 3 \cdot 10^8}{1.6 \cdot 10^{-19} \cdot 2 \cdot \sqrt{1 - \left(\frac{0.69 \cdot c}{c}\right)^2}} = 8.1 \cdot 10^{-4} \text{m}.$$

Problem 15

Cosmic rays collide with atoms or molecules in the upper atmosphere. If a proton moving at 0.7c makes a head-on collision with a nitrogen atom, initially at rest, and the proton recoils at 0.63c, what is the speed of the nitrogen atom after the collision? (The mass of a nitrogen atom is about 14 times the mass of a proton)

Solution

According to the law of conservation of the linear momentum

$$\vec{p}_p = \vec{p}\,'_p + \vec{p}\,'_N .$$

If we chose the positive direction of x-axis and take into account that proton after collision moved in the direction opposite to its initial direction, we can write for magnitudes of momenta

$$p_p = -p'_p + p'_N .$$

Since the particle move at the relativistic velocities we deal with relativistic momenta. Therefore,

$$p_p = m_p v_p = \frac{m_{p0}\beta_1 c}{\sqrt{1-\beta_1^2}} = m_{p0}c \cdot \frac{\beta_1}{\sqrt{1-\beta_1^2}} = m_{p0}c \cdot \frac{0.7}{\sqrt{1-0.7^2}} = 0.9802 \cdot m_{p0}c .$$

$$p'_p = m'_p v'_p = \frac{m_{p0}\beta_2 c}{\sqrt{1-\beta_2^2}} = m_{p0}c \cdot \frac{\beta_2}{\sqrt{1-\beta_2^2}} = m_{p0}c \cdot \frac{0.63}{\sqrt{1-0.63^2}} = 0.8112 \cdot m_{p0}c$$

$$p'_N = p_p + p'_N = 0.9802 \cdot m_{p0}c + 0.8112 \cdot m_{p0}c = 1.7914 \cdot m_{p0}c$$

$$p'_N = m_N v_N = \frac{m_{N0} \cdot \beta c}{\sqrt{1-\beta^2}} = \frac{14 m_{p0} \cdot \beta c}{\sqrt{1-\beta^2}}$$

$$\frac{14 m_{p0} \cdot \beta c}{\sqrt{1-\beta^2}} = 1.7914 \cdot m_{p0}c ,$$

$$\frac{\beta}{\sqrt{1-\beta^2}} = \frac{1.7914}{14} = 0.128 .$$

$$\beta = 0.127 .$$

The velocity of nitrogen atom is $v = 3.84 \cdot 10^7$ m/s