NATIONAL TECHNICAL UNIVERSITY
“KHARKIV POLYTECHICAL INSTITUTE”

DEPARTMENT OF PHYSICS

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STUDY GUIDE

“QUANTUM PHYSICS”

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Chapter 2. QUANTUM OPTICS

1. BLACK BODY RADIATION

1.1. Thermal radiation and its characteristics

Thermal radiation is the electromagnetic radiation emitted by heated bodies due to their internal energy and depending only on the temperature and the optic properties of these bodies.

Thermal radiation takes place at any temperature: depending on the temperature the bodies change the spectral composition of the emitted radiation. The color of the heated bodies seemed to depend only on their temperature and not on the material, quality of the surface, etc.

Thermal radiation is the only one which is in thermal equilibrium with the substance. *Thermal equilibrium* means that the system is in the state when the amount of emitted and absorbed power does not depend on time but reaches a steady state. Therefore, this radiation is the equilibrium radiation.

The intensity of thermal radiation is characterized by the energy flux, emitted by a unit area of the body surface along all directions - *radiant emittance* R – the total energy radiated per square meter per second at a temperature T or the rate at which radiation is emitted from a unit area.

Radiant emittance (*radiant exitance*) is the function of the temperature and the frequency. It is shown by indices ω and T - $R_{\omega T}$.

If the unit area emitted the energy flux $dR_{\omega T}$ in the frequency range $d\omega$, then,

$$dR_{\omega T} = r_{\omega T} d\omega$$

where $r_{\omega T}$ is the *emissive (radiating) power* or spectral *density of radiant emittance.* As the radiation consists of the waves of different frequencies ω, therefore,

$$R_{\omega T} = \int dR_{\omega T} = \int_0^\infty r_{\omega T} d\omega .$$

$$[R] = \frac{\text{J}}{\text{s} \cdot \text{m}^2} = \frac{\text{W}}{\text{m}^2} .$$

The *radiating power* $r_{\omega T}$ depends on the temperature and characterizes the power of radiation emitted per unit area per unit frequency.

$$\left[r_{\omega T} \right] = \frac{\text{J}}{\text{m}^2} .$$

The *absorption coefficient (absorptivity or absorption power)* $a_{\omega T}$ is the fraction of the incident power absorbed per unit area per unit frequency by the heated object or it is a ratio of the absorbed and incident energy fluxes.

$$a_{\omega T} = \frac{\left(d\Phi_\omega \right)_{absorbted}}{\left(d\Phi_\omega \right)_{incident}} .$$

$a_{\omega T}$ is the function of frequency and temperature.

A *blackbody* is an ideal system that absorbs all radiation incidents on it. Mathematically, a black body is defined to have $a_{\omega T} \equiv 1$ at any temperature.

A grey body $a_{\omega T} < 1$.

1.2. Kirchhoff's law of radiation

Kirchhoff's law of radiation: the ratio of the emissive power and the absorption coefficient is independent on the nature of the body. It is a universal function (the same for all bodies) that depends only on ω, the light frequency, and T, the absolute temperature of the body, and is called *black body emissive power* $r^*_{\omega T}$:

$$\left(\frac{r_{\omega T}}{a_{\omega T}} \right)_1 = \left(\frac{r_{\omega T}}{a_{\omega T}} \right)_2 = \ldots = \left(\frac{r_{\omega T}}{a_{\omega T}} \right)_{black} = r^*_{\omega T} = f\left(\omega, T \right)$$

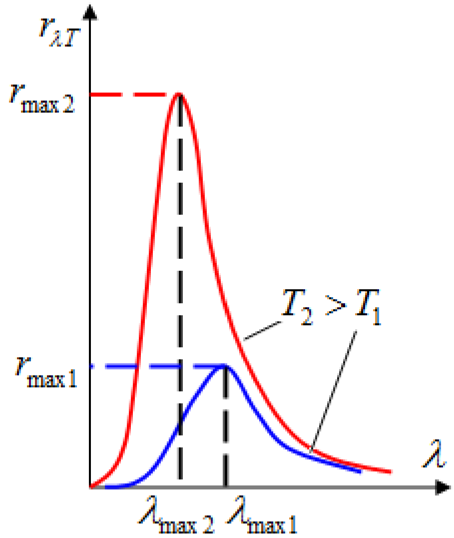
1.3. Stefan-Boltzmann Law

Austrian physicist Josef Stefan (1835–1893) in 1879 found experimentally that the total power per unit area emitted at all frequencies by a hot solid black body, $R^*_{\omega T}$, was proportional to the fourth power of its absolute temperature T. 5 years later Ludwig Boltzmann derived this law from a combination of thermodynamics and Maxwell's equations. Therefore, *Stefan-Boltzmann law* is

$$R^*_{\omega T} = \int_0^{\omega} r^*_{\omega T} d\omega = \sigma T^4,$$

where $\sigma = 5.67 \cdot 10^{-8}$ W·m^{-2}·K^{-4} is the Stefan-Boltzmann constant.

1.4. Wien's displacement law



The peak of the wavelength distribution shifts to shorter wavelengths as the temperature increases. This behaviour is described by the following relationship, called *Wien's displacement law*:

$$\lambda_m = \frac{b}{T},$$

where $b = 2.898 \cdot 10^{-3}$ m·K $= \approx 2.9 \cdot 10^{-3}$ m·K is the Wien's constant, λ_m is the *wavelength corresponding to the blackbody's maximum intensity* (*spectral density of radiant emittance*), and T is the absolute temperature of the surface of the object emitting the radiation.

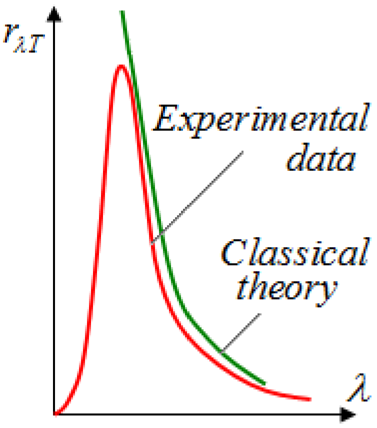
1.5. Rayleigh-Jeans Law

Attempts to use classical ideas to explain the shapes of the curves $r_{\omega T}(\lambda)$ failed. Rayleigh and Jeans suggested the expression based on classical physics

$$r^*_{\omega T} = f(\omega, T) = \frac{\omega^2}{4\pi^2 c^2} kT,$$

where c is the speed of light and k is the Boltzmann's constant.

An experimental plot (red curve) of the blackbody radiation spectrum, together with the theoretical picture of what this curve should look like based on classical theories (green curve) are shown at the figure. At long wavelengths, classical theory is in good agreement with the experimental data. At short wavelengths, however, major disagreement exists between classical theory and experiment. As λ approaches zero, classical theory erroneously predicts that the intensity should go to infinity, when the experimental data shows it should approach zero. This mismatch of theory and experiment was so disconcerting that scientists called it the ultraviolet catastrophe. (This "catastrophe" – infinite energy – occurs as the wavelength approaches zero in the violet short wavelength range)



In 1900 Planck developed a theory for blackbody radiation that was in complete agreement with experiments at all wavelengths. Planck made assumption that atoms oscillating in any frequency could only have energy that is some multiple of a small constant h times the frequency, that is, possible energies of atomic oscillators were given as

$$\varepsilon = nh\upsilon,$$

where $n = 0, 1, 2, \ldots$, and $h = 6.63 \cdot 10^{-34}$ J·s is Planck's constant.

The electromagnetic radiation is emitted, propagates and is absorbed in form of discreet portions – *quanta*. The energy of quant is proportional to the frequency

$$\varepsilon = h\upsilon = h\frac{c}{\lambda} = h\frac{\omega}{2\pi} = \hbar\omega,$$

where $\hbar = \frac{h}{2\pi} = 1.05 \cdot 10^{-34}$ J·s is Planck's constant.

Planck gives the expression for the spectral density of radiant emittance (*Planck's law*)

$$r^*_{\omega T} = f\left(\omega, T\right) = \frac{\hbar\omega^3}{4\pi^2 c^2} \cdot \frac{1}{e^{\frac{\hbar\omega}{kT}} - 1}.$$

Since $\lambda = cT = \frac{c}{\nu} = \frac{2\pi c}{\omega}$, the emissive power is the function of the wavelength and the temperature, i.e. $r_{\lambda T}$. In these terms *Planck's law* for radiation can be rewritten in the form:

$$r^*_{\lambda T} = f\left(\lambda, T\right) = \frac{2\pi hc^2}{\lambda^5} \cdot \frac{1}{e^{\frac{hc}{\lambda kT}} - 1}$$

The *maximum magnitude of emissive power* may be defined as

$$(r^*_{\lambda T})_{\max} = CT^5,$$

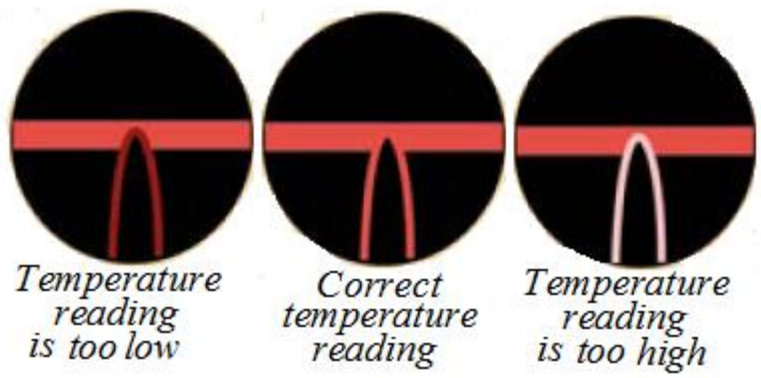
where $C = 1.3 \cdot 10^{-5} \frac{\mathrm{W}}{\mathrm{m}^3 \cdot \mathrm{K}^5}$.

Planck's law for radiation is in good correlation with experimental data. The Stefan-Boltzmann law and Wien’s displacement law may be obtained from it.

1.6. Practical applications

(a) ***Pyrometry*** (“*pyro*” - Greek “fire”) is the set of noncontact methods of temperature measurements. The appropriate devices are called *pyrometers* (or *pyroscopes*). There two types of pyrometers:

1. *Optical pyrometers.* They work on the basic principle of using the human eye to match the brightness of the hot object to the brightness of a calibrated lamp filament inside the instrument. Optical pyrometers are of *brightness* and *colour* types. Example, pyrometer with “disappearing” filament.



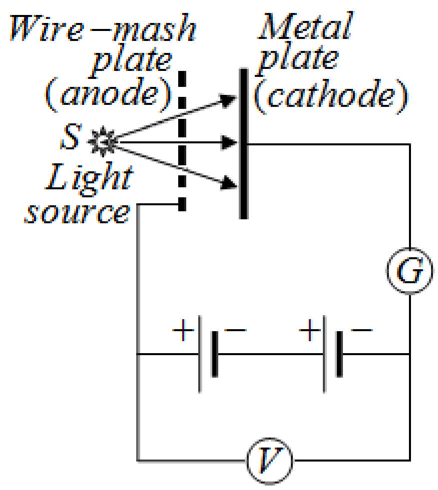
2. *Radiation pyrometer* determines the temperature of an object from the radiation (infrared and, if present, visible light) given off by the object. The radiation is directed at a heat-sensitive element such as a thermocouple, a device that produces an electric current when part of it is heated. The hotter the object, the more current is generated by the thermocouple. The current operates a dial that indicates temperature.

(b) ***Filament lamps***

Hot object glow is used for making light sources. The first filament lamps were invented by A. Lodygin (1873) and the electric arc lamps – by P. Yablochkov (1876). The main characteristic of the filament lamps is light efficiency. For the light efficiency increase: (i) the filament is made of wolfram which has high melting point and high selectivity of its thermal radiation; and (ii) glass balloons are filled by inert gas for prevention of filament evaporation. The efficiency of modern filament lamps is ~ 5%.

2. PHOTOELECTRIC EFFECT

2.1. Stoletov's experiment



For the first time the phenomenon latter referred to as photoelectric effect was observed by Hertz, who showed that metals under ultraviolet light emit charges, which were later identified to be electrons by Thomson.

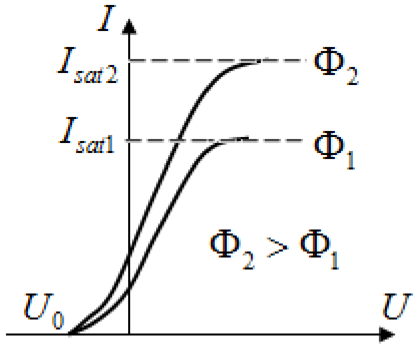
In 1888-1889 A. Stoletov carefully investigated the phenomenon of electron emission by solid and liquid objects under the influence of EM radiation – *electron photoemission* or *photoelectric effect*.

Stoletov' experiments showed:

1. The saturation photocurrent I_{sat} is proportional to the incident light flux Φ (*Stoletov's law*):

$$I_{sat} = \gamma \cdot \Phi ,$$

where γ is the *sensitivity* of illuminated surface that depends on the nature and the quality of the surface, and on the wavelength of incident light.



2. Maximum initial velocity of electrons and their kinetic energy do not depend on the intensity of the light, but increase with light frequency.

3. No electrons are emitted if the incident light frequency falls below some *cutoff (or threshold) frequency* ω_0 (or if the incident light wavelength is greater than *cutoff (or threshold)* wavelength λ_0), which is characteristic of the material being illuminated.

4. There is *no time lag* between the start of illumination and the start of the photocurrent.

2.2. Einstein's law for photoelectric effect

Einstein based on Planck's idea about quantization to electromagnetic waves successfully explained (1905; Nobel Prize, 1921) the nature of photoelectric effect which could not be explained using classical concepts.

When light illuminates a piece of metal, the light will kick off electrons from the metal's surface and these electrons can be detected as a change in the electric charge of the metal or as an electric current. Hence the name: "photo" for light and "electric" for the current. At photoelectric effect the photon (quantum of light) gives all its energy ε to the single electron of the illuminated metal. According to Einstein, the maximum kinetic energy for these liberated photoelectrons is

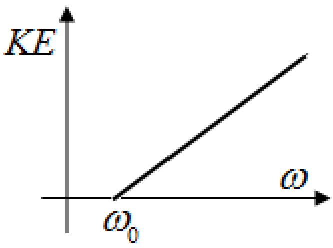
$KE = \varepsilon - A$,

where A is called the *work function* of the metal.

Einstein's equation for photoelectric effect is

$\varepsilon = A + KE$,

where $\varepsilon = h\upsilon = \dfrac{hc}{\lambda} = \hbar\omega$ is the energy of the photons of the incident light; KE is kinetic energy which is $KE = \dfrac{mv^2}{2}$ for classical case ($v \ll c$) and $KE = mc^2\left(1 - \dfrac{1}{\sqrt{1-\beta^2}}\right)$ relativistic case ($v \sim c$); $\beta = v/c$; and the *work function* A.



The work function, which represents the minimum energy with which an electron is bound in the metal, therefore, this is minimum amount of energy which is necessary to start photoelectric emission. If the amount of energy of incident radiation is less than the work function of metal, no photoelectrons are emitted. Work function is the property of material. Different materials have different values of work function which is usually is on the order of a few electron-Volts.

$A = h\upsilon_0 = \dfrac{hc}{\lambda_0} = \hbar\omega_0$,

where $\upsilon_0 = \dfrac{A}{h}$, $\omega_0 = \dfrac{A}{\hbar} = \dfrac{2\pi \cdot A}{h}$ and $\lambda_0 = \dfrac{hc}{A}$ are the cutoff (threshold) frequency, angular frequency and wavelength.

The negative potential at which the photoelectric current becomes zero is called stopping potential or cutoff potential. Stopping potential U_s is that value of retarding potential difference between two plates which is just sufficient to halt the most energetic photoelectrons emitted.

$h\upsilon = A + eU_s$

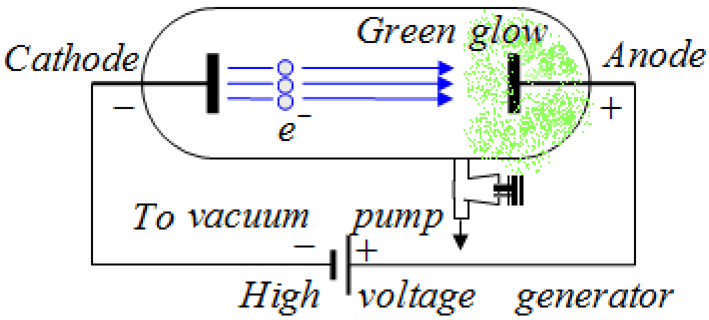
The photoelectric effect has many practical applications which include the photocell and solar cells. A photocell is usually a vacuum tube with two electrodes. One is a photosensitive cathode which emits electrons when exposed to light and the other is an anode which is maintained at a positive voltage with respect to the cathode. Thus when light shines on the cathode, electrons are attracted to the anode and an electron current flows in the tube from cathode to anode. The current can be used to operate a relay, which might turn a motor on to open a door or ring a bell in an alarm system. The system can be made to be responsive to light, as described above, or sensitive to the removal of light as when a beam of light incident on the cathode is interrupted, causing the current to stop. Photocells are also useful as exposure meters for cameras in which case the current in the tube would be measured directly on a sensitive meter.

Solar cells, usually made of specially prepared silicon, act like a battery when exposed to light. Individual solar cells produce voltages of about 0.6 volts but higher voltages and large currents can be obtained by appropriately connecting many solar cells together. Electricity from solar cells is still quite expensive but they are very useful for providing small amounts of electricity in remote locations where other sources are not available. It is likely however that as the cost of producing solar cells is reduced they will begin to be used to produce large amounts of electricity for commercial use.

2.3. Discovery of electron

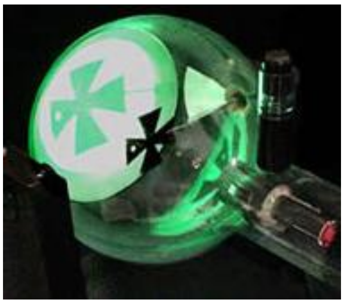

Crookes tube

In 1885, Sir William Crookes carried out a series of investigations into the behavior of metals heated in a vacuum. He used discharge tube. It was the long glass tube having two metal plates, sealed at its two ends as electrodes. It has a side tube through which air can be pumped out by using a vacuum pump, so that experiments can be performed at low pressure. When the pressure of air in the discharge tube is reduced to 10^{-3} mm of mercury and a high voltage is applied to the electrodes, the emission of light by air stops. But the phenomenon of fluorescence in which the walls of the discharge tube at the end opposite to the cathode begin to glow with a greenish light is observed. The experiment of Crookes and others showed that a heated cathode produced a stream of radiation, which could cause gases at low pressure to glow and which made other substances emit light too. The radiation emitted from the cathode was given the name *'cathode rays'*.



Experimentally observed properties of cathode rays:

1. They travel in straight lines: when an opaque object like a metal Maltese cross is placed in the path of cathode rays in a discharge tube; a shadow of the metal cross is formed at the end opposite to the cathode.



2. Cathode rays produce mechanical effect. They can rotate a high paddle wheel placed in their path.

3. The cathode rays are deflected in electric and magnetic fields.

4. The cathode ray particles being negatively charged.

5. The nature of cathode rays does not depend on the nature of gas taken in the discharge tube or material of the cathode.

6. The ratio of the charge to mass (e/m ratio) of cathode ray particles obtained from different gases was found to be exactly the same.

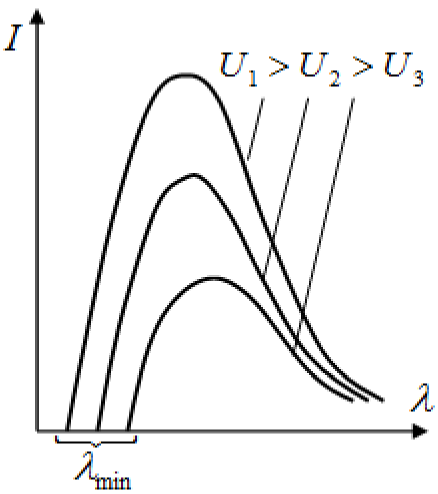
Eventually, J.J.Thomson (1856-1940), working in Cavendish Laboratory (University of Cambridge), showed that cathode rays consists of *electrons* (Nobel Prize, 1906).

2.4. X-rays

X-rays were discovered in 1895 by Wilhelm Conrad Roentgen (Röntgen) (1845-1923; Nobel Prize, 1901) and identified as electromagnetic waves in 1912 by Max von Laue (1879-1960; Nobel Prize, 1914).

Wilhelm Roentgen experimented with Crookes' tube. He had covered his tube with black cardboard and darkened the room. Suddenly he noticed a weak light shimmering on a little bench nearby and discovered that the source of the mysterious light was a little barium platinocyanide screen lying on the bench. The cathode rays was able to travel only several centimeters in air but the little screen was a meter away from the tube hence cathode rays couldn't be the cause of this light. Roentgen supposed that this fluorescence was coursed by unknown radiation, X-rays.

He found that the X-rays traveled in straight lines, and, unlike the cathode rays, were not deflected by electric and magnetic fields. He found they passed through flesh almost unimpeded, but bone cast a shadow. By having his wife place her hand between the point source of X-rays on the Crookes' tube and some unexposed film in a box, then developing the film, he took a picture of the bones of her hand. Roentgen announced his finding complete with the bone picture on January 1, 1896. Already in 1896 several hospitals had X-ray facilities, and X-ray photographs were ruled as acceptable evidence in courts in France, England and the USA. And nevertheless that the dangers became apparent and several persons suffered severe skin damage, by 1903 lead-impregnated rubber shielding devices were being used.



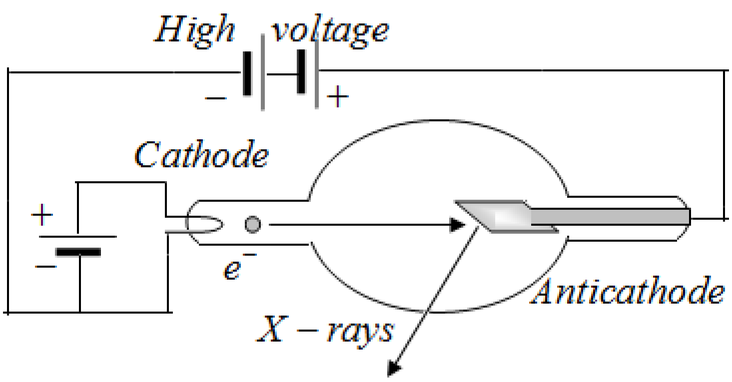
The further investigations showed the following properties of X-rays.

1. Since X-rays were undeflected by a magnetic field, there were not charged particles.

2. X-rays were ionizing radiation: as they passed through air, ions were created.

3. X-rays are electromagnetic radiation with the wavelength $\lambda \leq 10^{-8}$ m which can diffract on the periodic structures of crystals.

The explanation of the X-rays origin is following. Electrons are accelerated in an electric field through the potential difference of several thousand Volts and gain a kinetic energy $KE = eU$ that can be converted into electromagnetic radiation (photons) when the electrons hit and slow down in metal target. The emitted radiation gives continuous spectrum. At that, there is the minimum wavelength $\lambda_{\min}$ (or maximum frequency $\omega_{\max}$) that is independent of target material and limited by energy conservation: electron's total kinetic energy is converted into a single X-ray photon.



$$eU = h\upsilon_{\max} = \frac{hc}{\lambda_{\min}}.$$

Short-wavelength cutoff of X-rays: $\lambda_{\min} = \dfrac{hc}{eU}$,

or in terms of frequency: $\omega_{\max} = \dfrac{eU}{\hbar}$, $\upsilon_{\max} = \dfrac{eU}{h}$.

Applications of X-rays:

1. X-rays find vast application in medical science. They are used to detect fractures, tumors and presence of foreign matter in human body.

2. X-rays are used to locate the cracks, blow holes and other defects in metals, especially, in different pieces, component parts of machines, etc.

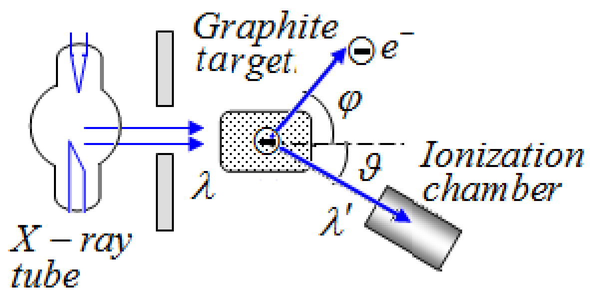
3. By taking X-ray diffraction of substances it is possible to get valuable information about their molecular and crystal structure.

2.5. The Compton Effect

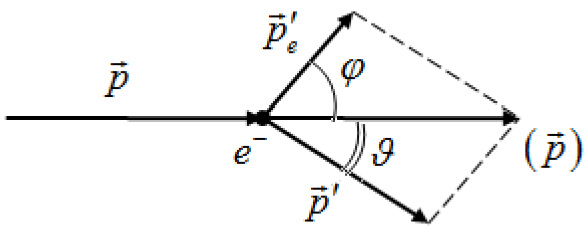
In the experiments conducted by Arthur H. Compton in 1923, X-ray beam of wavelength λ was directed towards a block of graphite (paraffin or other light substance). The wavelength of scattered radiation and the energy of emitted electron are determined as a function of the angle relative to the incident beam. The wavelength of the scattered X-rays is longer as the scattering angle becomes larger. The relationship between incident λ and scattered λ' radiation wavelengths is found to be

$$\Delta\lambda = \lambda' - \lambda = \lambda_c(1 - \cos\vartheta) = 2\lambda_c \sin^2(\vartheta/2),$$

where $\Delta\lambda$ is *Compton's shift*, $\lambda_c = \dfrac{h}{m_0 c} = 2.43 \cdot 10^{-12}$ m = 2.43 pm is the *Compton wavelength of electron* and ϑ is scattering angle.



Compton assumed that the incident X-rays collide against the electron in the graphite as a "particle" with the energy $\varepsilon = h\upsilon = h\dfrac{c}{\lambda}$ and linear momentum $p = \dfrac{h}{\lambda} = \dfrac{h\upsilon}{c}$. The energy and the momentum of the X-rays scattered in the scattering angle ϑ are $\varepsilon' = h\upsilon' = h\dfrac{c}{\lambda'}$ and $p' = \dfrac{h}{\lambda'} = \dfrac{h\upsilon'}{c}$, respectively. The electron of



mass m_0 recoils with the energy ε'_e and the momentum $\vec{p}'_e$. The energies and momenta in Compton effect are conserved.

$$\varepsilon + m_0 c^2 = \varepsilon' + m_0 c^2 + \varepsilon'_e, \qquad \varepsilon = \varepsilon' + \varepsilon'_e,$$

$$\vec{p} = \vec{p}' + \vec{p}'_e.$$

The Compton effect is explained by treating light (electromagnetic radiation) as particles that can transfer energy and momentum, and the energy isn't completely absorbed in the interaction with material particles (electrons). This effect provides solid evidence for the particle nature of electromagnetic radiation.

2.6. Matter waves

In 1924, Louis de Broglie proposed that that light was not the only phenomenon to exhibit wave particle duality. He said that material particles like electrons and atoms, which are recognized as being particles, can exhibit wave behavior. To make this statement, he had to define a meaningful wavelength for a material particle; the quintessential wave properties of diffraction and interference require that the wavelength be comparable to the slit width or slit spacing. To define a wavelength, de Broglie used the photon to make an analogy: if the photon has a momentum related to its wavelength through the equation $p = h/\lambda$, then perhaps material particles with momentum p have a wavelength that satisfies the same relation:

$$\lambda = \frac{h}{p},$$

where the linear momentum is $p = mv$ for the particles moving with velocities $v \ll c$ or $p = \frac{m\beta c}{\sqrt{1-\beta^2}}$ $(\beta = v/c)$ for relativistic particles moving with $v \approx c$.

This wavelength is called *de Broglie wavelength* of the particles.

The de Broglie relation is the mathematical expression for the wave nature of the particles. It gives the wavelength of the wave associated with a particle with momentum *p*.

The de Broglie wavelength may be determined for any particle moving at different speeds. For example, the ball of mass m = 1 kg travelling at the velocity v = 1 m/s has the de Broglie wavelength

$$\lambda = \frac{h}{mv} = \frac{6.62 \cdot 10^{-34}}{1 \cdot 1} = 6.62 \cdot 10^{-34} \text{ m.}$$

This magnitude is extremely small in comparison with the dimension of any physical system. Thus it is impossible to either prove or disprove the validity of de Broglie's postulate by investigating the motion of a macroscopic particle.

For electron of mass $m = 9.1 \cdot 10^{-31}$ kg moving at the velocity $v = 10^5$ m/s de Broglie wavelength is $\lambda = \dfrac{h}{mv} = \dfrac{6.62 \cdot 10^{-34}}{9.1 \cdot 10^{-31} \cdot 10^5} = 7.2 \cdot 10^{-9}$ m.

This very small wavelength is of the order of the atom size, and, consequently, of the inter-atomic spacing in a crystal. Due to this fact it is possible to observe the diffraction effects at the transmission of the electron beam through the crystal. A practical device that relies on the wave characteristics of electrons is the *electron microscope*.

2.7. The wave-particle duality

Phenomena such as the photoelectric effect and the Compton effect offer evidence that when light (or other forms of electromagnetic radiation) and matter interact, the light behaves as if it were composed of particles having energy $\varepsilon = hc/\lambda$ and momentum $p = h/\lambda$. In other contexts, however, light acts like a wave, exhibiting interference and diffraction effects.

Overcoming the opposition of wave and quantum properties of light is one of the most significant achievements of the XX century physics. The properties of continuity inherent to the electromagnetic field of light wave do not exclude the properties of discreteness inherent to light quanta – photons. This is a dialectic unity of two opposites (*wave-particle duality).*

BLACK BODY RADIATION

Problem 1

The radius of our Sun is 6.96·10^8 m, and its total power output is 3.77·10^{26} W. (a) Assuming that the Sun's surface emits as a black body, calculate its surface temperature. (b) Using the result of part (a), find λ_m *for the Sun.*

Solution

(a) The radiant emittance R is the magnitude equaled to the total energy radiated per square meter per second at a temperature T or the rate at which radiation is emitted from a unit area. The radiant emittance of a black body is

$$R^* = \frac{E}{S \cdot t}$$

Taking into account that the energy radiated per the time unit is the radiating power $\left(N = E/t\right)$, we obtain

$$R^* = \frac{N}{S}.$$

According to Stefan-Boltzmann Law, the radiant emittance of the black body is directly proportional to the fourth power of the object's temperature in Kelvin:

$$R^* = \sigma T^4,$$

where $\sigma = 5.67 \cdot 10^{-8}$ W·m^{-2}·K^{-4} is the Stefan-Boltzmann constant.

Combining two expressions for the radiant emittance, we obtain

$$\frac{N}{S} = \sigma T^4,$$

Since the surface area of the Sun is $S = 4\pi r^2$, the temperature is given by

$$T = \sqrt[4]{\frac{N}{\sigma S}} = \sqrt[4]{\frac{N}{\sigma 4\pi r^2}} = \sqrt[4]{\frac{3.77 \cdot 10^{26}}{5.67 \cdot 10^{-8} \cdot 4\pi \cdot \left(6.96 \cdot 10^8\right)^2}} = 5749\,\text{K}.$$

(b) Wien's displacement law determines the dependence of the wavelength λ_m corresponding to the blackbody's maximum spectral density of radiant emittance on the absolute temperature T of the object surface emitting the radiation:

$$\lambda_m = \frac{b}{T},$$

where $b = 2.898 \cdot 10^{-3}$ m·K $\approx 2.9 \cdot 10^{-3}$ m·K is the Wien's constant.

$$\lambda_m = \frac{2.898 \cdot 10^{-3}}{5749} = 5.04 \cdot 10^{-7} \text{ m.}$$

Problem 2

The Earth has an average temperature of 288K. What is the Earth's wavelength of the maximum emission (maximum spectral density of radiant emittance)?

Solution

Use Wien's displacement Law $\lambda_m = \frac{b}{T}$, we have

$$\lambda_m = \frac{2.898 \cdot 10^{-3}}{288} = 1 \cdot 10^{-5} \text{ m} = 10 \text{ μm.}$$

Maximum emission falls at the infrared thermal radiation.

Problem 3

The power of the radiation of the black body N = 10 kW. Find the area of the radiating surface if the maximum spectral density of radiant emittance falls at the wavelength λ_m = 700 nm.

Solution

The power of the radiation of the black body is

$$N = R^* \cdot S = \sigma T^4 S$$

The temperature may be determined using Wien's displacement Law $\lambda_m = \frac{b}{T}$.

Then the area of the radiating surface

$$S = \frac{N}{\sigma T^4} = \frac{N}{\sigma} \cdot \left(\frac{\lambda_m}{b}\right)^4 = \frac{10^4}{5.67 \cdot 10^{-8}} \cdot \left(\frac{0.7 \cdot 10^{-6}}{2.9 \cdot 10^{-3}}\right)^4 = 6 \cdot 10^{-4} \text{ m}^2.$$

Problem 4

The black body is at the temperature T_1 *= 2900 K. As a result of cooling, the wavelength of maximum emission was changed by* $\Delta\lambda_m$ *= 9* μ*m. Find the final temperature* T_2.

Solution

The wavelengths of maximum emission for two temperatures according to the Wien's Law $\lambda_m = \frac{b}{T}$ are

$$\Delta\lambda_m = \lambda_{m2} - \lambda_{m1} = \frac{b}{T_2} - \frac{b}{T_1},$$

where $b = 2.898 \cdot 10^{-3}$ m·K.

The required temperature is

$$T_2 = \frac{bT_1}{b + \Delta\lambda_m \cdot T_1} = \frac{2.97 \cdot 10^{-3} \cdot 2900}{2.97 \cdot 10^{-3} + 9 \cdot 10^{-6} \cdot 2900} = 296\,\text{K}.$$

Problem 5

The temperature of the black body was changed from 1000 K to 3000 K. Find the changes in (a) the radiant emittance R_2^* / R_1^**, (b) the spectral density of radiant emittance* $\Delta\lambda_m$*, and (c) maximum spectral density of radiant emittance* $r_{\lambda 2} / r_{\lambda 1}$.

Solution

Using the Stefan-Boltzmann Law gives

$$\frac{R_2^*}{R_1^*} = \frac{\cancel{\sigma}T_2^4}{\cancel{\sigma}T_1^4} = \left(\frac{3000}{1000}\right)^4 = 81.$$

The change in the spectral density of radiant emittance is

$$\lambda_{m1} - \lambda_{m2} = \frac{b}{T_1} - \frac{b}{T_2} = b\left(\frac{1}{T_1} - \frac{1}{T_2}\right) = 2.97 \cdot 10^{-3}\left(\frac{1}{1000} - \frac{1}{3000}\right) = 1.97 \cdot 10^{-6} \text{ m.}$$

The change in maximum spectral density of radiant emittance

$$\frac{(r_\lambda)_{\max 2}}{(r_\lambda)_{\max 1}} = \frac{\cancel{C}T_2^5}{\cancel{C}T_1^5} = \left(\frac{3000}{1000}\right)^5 = 243.$$

Problem 6

What power has to be transferred to the black metal sphere of radius r = 2 cm to keep it at the temperature 27 K higher than the room temperature. The room temperature is 293 K. Assume that the sphere is the black body; and the heat is lost only by radiation.

Solution

Power of the radiation that is emitted by the sphere at the room temperature T is

$$N = R^* \cdot S = \sigma T^4 S,$$

and power of radiation at higher temperature $T_1 = T + \Delta T$ is

$$N_1 = R_1^* \cdot S = \sigma T_1^4 S = \sigma\left(T + \Delta T\right)^4 \cdot S.$$

The difference between these two powers is compensated by external heat

$$\Delta N = N_1 - N = S\left(R_1^* - R\right) = 4\pi r^2 \sigma\left(T_1^4 - T^4\right) =$$

$$= 4\pi \cdot 4 \cdot 10^{-4} \cdot 5.67 \cdot 10^{-8}\left(320^4 - 293^4\right) = 0.89 \text{ W.}$$

Problem 7

The current I =4.55 A flows across the heating element of the electric oven under the voltage U =220 V. 20% of the power input is dispersed by the walls, and the rest part is the thermal radiation from the hole of diameter d = 10 cm.

Find the cutoff temperature T_1 of the operating zone assuming it the 'grey" body with the coefficient $k = 0.8$. *Ambient temperature is* T_0 *=295 K.*

Solution

The power input is

$N_0 = I \cdot U = 4{,}55 \cdot 220 = 1000\,\mathrm{W}.$

Since 20% is dispersed by the walls, 80% is radiated, therefore, the efficiency factor $\eta = 0.8$.

Radiated power is

$N = \eta \cdot N_0 = 0.8 \cdot 1000 = 800\,\mathrm{W}.$

The radiating power of the oven at room temperature is

$$N_0 = R_0 \cdot S = k \cdot R_0^* \cdot S = k \cdot \sigma T_0^4 \cdot \frac{\pi d^2}{4}.$$

The radiating power at the temperature T is

$$N_1 = R_1 \cdot S = k \cdot R_1^* \cdot S = k \cdot \sigma T_1^4 \cdot \frac{\pi d^2}{4}.$$

The difference in powers is compensated by the power N:

$N = N_1 - N_0,$

$$\eta \cdot N_0 = k \cdot \sigma T_1^4 \cdot \frac{\pi d^2}{4} - k \cdot \sigma T_0^4 \cdot \frac{\pi d^2}{4} = k \cdot \sigma \cdot \frac{\pi d^2}{4}\left(T_1^4 - T_0^4\right).$$

Required temperature is

$$T_1 = \sqrt[4]{\frac{4\eta N}{k\sigma\pi d^2} + T_0^4} = \sqrt[4]{\frac{4 \cdot 0.8 \cdot 10^3}{0.8 \cdot 5.67 \cdot 10^{-8} \cdot \pi \cdot 10^{-2}} + 295^4} = 1225\,\mathrm{K}.$$

PHOTONS. PHOTOELECTRIC EFFECT

Problem 1

Find the energy and the linear momentum of the photon which wavelength is λ = 1.24 pm.

Solution

The energy of the photon is given by $\varepsilon = \dfrac{hc}{\lambda}$, where $h = 6.63 \cdot 10^{-34}$ J·s is the Planck's constant and $c = 3 \cdot 10^{8}$ m/s is the speed of light.

$$\varepsilon = \frac{hc}{\lambda} = \frac{6.63 \cdot 10^{-34} \cdot 3 \cdot 10^{8}}{1.24 \cdot 10^{-12}} = 1.6 \cdot 10^{-13}\ \text{J}.$$

The linear momentum is

$$p = \frac{h}{\lambda} = \frac{6.63 \cdot 10^{-34}}{1.24 \cdot 10^{-12}} = 5.34 \cdot 10^{-22}\ \text{kg·m/s}.$$

Problem 2

Find the speed of electron if its kinetic energy is equal to the energy of photon with wavelength (a) λ = 520 nm; (b) λ =1.2 pm.

Solution

(a) The photon's energy is $\varepsilon = \dfrac{hc}{\lambda}$. The kinetic energy of the charged particle electron is $KE = \dfrac{mv^2}{2}$.

By condition $KE = \varepsilon$, therefore,

$$\frac{mv^2}{2} = \frac{hc}{\lambda}.$$

If $h = 6.63 \cdot 10^{-34}$ J·s, $c = 3 \cdot 10^{8}$ m/s and mass of electron $m = 9.1 \cdot 10^{-31}$ kg, the speed of electron is

$$v=\sqrt{\frac{2hc}{\lambda m}}=\sqrt{\frac{2\cdot 6.63\cdot 10^{-34}\cdot 3\cdot 10^{8}}{520\cdot 10^{-9}\cdot 9.1\cdot 10^{-31}}}=9.16\cdot 10^{5}\text{ m/s.}$$

(b) If we use the same expression for this case when the wavelength of photon is $\lambda=1.2\cdot 10^{-12}$ m, we obtain the speed of electron

$$v=\sqrt{\frac{2hc}{\lambda m}}=\sqrt{\frac{2\cdot 6.63\cdot 10^{-34}\cdot 3\cdot 10^{8}}{1{,}2\cdot 10^{-12}\cdot 9.1\cdot 10^{-31}}}=6\cdot 10^{8}\text{ m/s.}$$

In is impermissible and impossible result because the speed of any material object must be less than the speed of light. This result was obtained due to the usage of classical expression for the kinetic energy. Let’s use the relativistic formula $E_{кин}=m_0c^2\left(\frac{1}{\sqrt{1-\beta^2}}-1\right)$, where $\beta=v/c$. Then

$$m_0c^2\left(\frac{1}{\sqrt{1-\beta^2}}-1\right)=\frac{hc}{\lambda},$$

$$\beta=\sqrt{1-\left(\frac{\lambda m_0 c}{h+\lambda m_0 c}\right)^2}=\sqrt{1-\left(\frac{1.2\cdot 10^{-12}\cdot 9.1\cdot 10^{-31}\cdot 3\cdot 10^{8}}{6.63\cdot 10^{-34}+1.2\cdot 10^{-12}\cdot 9.1\cdot 10^{-31}\cdot 3\cdot 10^{8}}\right)^2}=0.94.$$

The speed of electron is

$$v=\beta c=0.94c=2.8\cdot 10^{8}\text{ m/s.}$$

Problem 3

Find the speed of electron if its linear momentum is equal to the momentum of photon with wavelength (a) λ = 520 nm; (b) λ =1.2 pm.

Solution

(a) The linear momentum of electron is $p_e=mv$ and the momentum of photon is $p=h/\lambda$.

By the data, $mv=\frac{h}{\lambda}$. Therefore, the speed of electron is

$$v = \frac{h}{\lambda m} = \frac{6.63 \cdot 10^{-34}}{520 \cdot 10^{-9} \cdot 9.1 \cdot 10^{-31}} = 1.4 \cdot 10^{3} \text{ m/s.}$$

(b) The same expression used for the second wavelength gives
$$v = \frac{h}{\lambda m} = \frac{6.63 \cdot 10^{-34}}{1.2 \cdot 10^{-12} \cdot 9.1 \cdot 10^{-31}} = 6.06 \cdot 10^{8} \text{ m/s.}$$

This impermissible result connected with the usage of classical formula. The relativistic expression $p_e = \frac{m_0 v}{\sqrt{1-\beta^2}} = \frac{m_0 \beta c}{\sqrt{1-\beta^2}}$ gives

$$\frac{m_0 \beta c}{\sqrt{1-\beta^2}} = \frac{h}{\lambda},$$

$$\beta = \frac{h}{\sqrt{h^2 + (m_0 c \lambda)^2}} = \frac{6,63 \cdot 10^{-34}}{\sqrt{\left(6.63 \cdot 10^{-34}\right)^2 + \left(9.1 \cdot 10^{-31} \cdot 3 \cdot 10^{8} \cdot 1.2 \cdot 10^{-12}\right)^2}} = 0.9.$$

The speed of electron is

$$v = 0.9c = 2.68 \cdot 10^{8} \text{ m/s.}$$

Problem 4

The energy of photon is equal to the kinetic energy of electron that had the initial velocity of 10^6 m/s and accelerated through the potential difference of 4 V. Find the wavelength of photon.

Solution

The change in the kinetic energy of the electron is due the work of electric field:

$$KE_2 - KE_1 = W,$$

$$\frac{mv^2}{2} - \frac{mv_0^2}{2} = eU.$$

The energy of photon is equal to the energy of accelerated electron, therefore,

$$\varepsilon = KE_2,$$

Taking into account that $\varepsilon = \dfrac{hc}{\lambda}$ and $KE_2 = \dfrac{mv^2}{2} = \dfrac{mv_0^2}{2} + eU$,

$$\frac{hc}{\lambda} = \frac{mv_0^2}{2} + eU ,$$

$$\lambda = \frac{hc}{\left(mv_0^2/2\right) + eU},$$

$$\lambda = \frac{6.63\cdot 10^{-34}\cdot 3\cdot 10^{8}}{\left(9.1\cdot 10^{31}\cdot 10^{12}/2\right) + 1.6\cdot 10^{-19}\cdot 4} = 1.8\cdot 10^{-7} \text{ m.}$$

Problem 5

The threshold wavelength for the metal is $\lambda_0 = 275$ *nm. Find the minimum energy of photons that will cause the photoelectric effect, and the work function* A*. Determine the maximum speed* v_m *and maximum kinetic energy of photoelectrons as they are emitted. The wavelength of the light used for illumination are (a)* λ *= 180 nm; (b)* λ *=1.2 pm.*

Solution

Use Einstein's photoelectric effect equation to determine the energy of the incident radiation

$$\varepsilon = A + KE_m .$$

Minimum energy that will cause the photoelectric effect is $\varepsilon_{\min} = A$. The work function $A = \dfrac{hc}{\lambda_0}$, therefore,

$$\varepsilon_{\min} = A = \frac{hc}{\lambda_0} = \frac{6.63\cdot 10^{-34}\cdot 3\cdot 10^{8}}{275\cdot 10^{-9}} = 7.2\cdot 10^{-18}\text{J = 4.5 eV.}$$

The kinetic energy and the velocity of the photoelectrons we will find for two wavelengths. For λ = 180 nm we obtain

$$KE_m = \varepsilon - A = \frac{hc}{\lambda} - A = \frac{6.63\cdot 10^{-34}\cdot 3\cdot 10^{8}}{180\cdot 10^{-9}} = 1.1\cdot 10^{-18}\text{J = 6.9 eV.}$$

Taking into account that the mass of electron is $m = 9.1 \cdot 10^{-31}$, its velocity from $KE_m = \frac{mv^2}{2}$ is

$$v_m = \sqrt{\frac{2 \cdot KE_m}{m}} = \sqrt{\frac{2 \cdot 1.1 \cdot 10^{-18}}{9.1 \cdot 10^{-31}}} = 1.55 \cdot 10^6 \text{ m/s.}$$

For the second wavelength λ =1.2 pm =$1.2 \cdot 10^{-12}$ m the energy of the photon is

$$\varepsilon = \frac{hc}{\lambda} = \frac{6.63 \cdot 10^{-34} 3 \cdot 10^8}{1.2 \cdot 10^{-12}} = 1.7 \cdot 10^{-13} \text{ J} = 1.03 \cdot 10^6 \text{ eV.}$$

This magnitude significantly larger than the energy for the first wavelength, therefore, we will use relativistic expression for the kinetic energy for calculation. Moreover, the energy of photon is much greater than work function, hence, $\varepsilon = KE_m = m_0 c^2 \left(\frac{1}{\sqrt{1-\beta^2}} - 1 \right)$,

$$\beta = \sqrt{1 - \left(\frac{m_0 c^2}{KE + m_0 c^2} \right)^2} = \sqrt{1 - \left(\frac{9.1 \cdot 10^{-31} \cdot 9 \cdot 10^{16}}{1.7 \cdot 10^{-13} + 9.1 \cdot 10^{-31} \cdot 9 \cdot 10^{16}} \right)^2} = 0.95,$$

$$v_m = \beta c = 0.95c = 2.83 \cdot 10^8 \text{ m/s.}$$

Problem 6

Find the frequency of incident light which causes the emission of photoelectrons that are stopped by potential difference $U_s = 3V$. The threshold frequency is $\upsilon_0 = 6 \cdot 10^4$ Hz. Find the work function for this metal.

Solution

The threshold (or cutoff) frequency is the smallest frequency of photon that will still result in the photoelectric effect occurring. Therefore, the work function is

$$A = \varepsilon_{\min} = h\upsilon_0 = 6.63 \cdot 10^{-34} \cdot 6 \cdot 10^{14} = 3.98 \cdot 10^{-19} \text{ J} = 6.23 \text{ eV.}$$

Using Einstein's equation

$\varepsilon = A + KE_m$,

and taking into account that $KE_m = eU_s$, and the energy of incident photon is $\varepsilon = h\upsilon$, we can find the desired frequency from

$h\upsilon = A + eU_s$,

$$\upsilon = \frac{A + eU_s}{h} = \frac{3.98 \cdot 10^{-19} + 1.6 \cdot 10^{-19} \cdot 3}{6.63 \cdot 10^{-34}} = 1.32 \cdot 10^{15} \text{ Hz.}$$

Problem 7

When light of wavelength λ = 360 nm is incident, the stopping potential is found to be U_s = 0.8 V. (a) What is the maximum kinetic energy of the emitted electrons? (b) What is the work function? (c) What is the longest wavelength that will eject any electrons from the metal?

Solution

The maximum kinetic energy of electrons must be the energy that corresponds the voltage U_s = 0.8 V. From the definition of the electron-Volt, the energy is 0.8 eV. Converting this value to Joules, we obtain $KE_m = 0.8 \cdot 1.6 \cdot 10^{-19} = 1.28 \cdot 10^{-19}$ J.

The work function is the energy lost by electron as it leaves the metal. Light of the wavelength λ = 360 nm gives each electron an energy

$$\varepsilon = \frac{hc}{\lambda} = \frac{6.63 \cdot 10^{-34} \cdot 3 \cdot 10^{8}}{360 \cdot 10^{-9}} = 5.53 \cdot 10^{-19} \text{ J = 3.45 eV.}$$

The maximum kinetic energy of electron is $KE_m = 0.8 \cdot 1.6 \cdot 10^{-19} = 1.28 \cdot 10^{-19}$ J. Therefore, using Einstein's photoelectric effect equation $\varepsilon = A + KE_m$ gives the work function as

$A = \varepsilon - KE_m = 3.45 - 0.8 = 2.65$ eV= $4.25 \cdot 10^{-19}$ J.

In order to eject electron, the photon energy must be at least equal to the work function. Taking into account that the photon's energy is $\varepsilon = \dfrac{hc}{\lambda}$,

$$A = \varepsilon = \frac{hc}{\lambda_0}.$$

So, the longest wavelength that will eject any electrons from the metal is

$$\lambda_0 = \frac{hc}{A} = \frac{6.63 \cdot 10^{-34} \cdot 3 \cdot 10^{8}}{4.25 \cdot 10^{-19}} = 468 \cdot 10^{-9}\,\text{m} = 468\ \text{nm}.$$

Problem 8

The photons with the energy ε = 4.9 eV produce photoelectrons with work function A = 4.5 eV. Find the maximum linear momentum $p_{\max}$ that is transferred to the metal surface by each emitting electron.

Solution

From Einstein's photoelectric effect equation $\varepsilon = A + KE_{max}$, the kinetic energy of the emitting photoelectrons is

$$KE_m = \varepsilon - A = 4.9 - 4.5 = 0.4\ \text{eV} = 6.4 \cdot 10^{-20}\,\text{J}.$$

Using classical relationship between the linear momentum and the kinetic energy $E = \frac{mv^2}{2} = \frac{(mv)^2}{2m} = \frac{p^2}{2m}$, we have

$$p_m = \sqrt{2m \cdot KE} = \sqrt{2 \cdot 1.6 \cdot 10^{-19} 6.4 \cdot 10^{-20}} = 3.4 \cdot 10^{-25}\ \text{kg·m/s}.$$

Problem 9

Ultraviolet radiation, which is part of the solar spectrum, causes a photoelectric effect in certain materials. If the kinetic energy of the photoelectrons from an aluminum sample is $5.6 \cdot 10^{-19}$ J and the work function of aluminum is 4.1 eV, what is the frequency of the photons that produce the photoelectrons?

Solution

Einstein's photoelectric effect equation $\varepsilon = A + KE_m$ gives

$h\upsilon = A + KE_m$. The frequency will be

$$\upsilon = \frac{A + KE_m}{h} = \frac{4.1 \cdot 1.6 \cdot 10^{-19} + 5.6 \cdot 10^{-19}}{6.63 \cdot 10^{-34}} = 1.83 \cdot 10^{15} \text{ Hz.}$$

Problem 10

Light with a frequency of 7.5 ·10^14 Hz is able to eject electrons from the metal surface of a photocell that has a threshold frequency of 5.2 ·10^14 Hz. What stopping potential is needed to stop the emitted photoelectrons?

Solution

From the Einstein relation we have that the kinetic energy of photoelectrons is $KE_m = \varepsilon - A$. The stopping potential for electrons given off by the metal is just the voltage required to stop the most energetic electrons, $KE_m = eU_s$, where $e = 1.6 \cdot 10^{-19}$ C is the charge of electron

$$U_s = \frac{KE_m}{e} = \frac{\varepsilon - A}{e}.$$

Taking into account that $\varepsilon = h\upsilon$ and $A = h\upsilon_0$, the stopping potential is

$$U_s = \frac{h\upsilon - h\upsilon_0}{e} = \frac{h(\upsilon - \upsilon_0)}{e} = \frac{6.63 \cdot 10^{-34} (7.5 - 5.2) \cdot 10^4}{1.6 \cdot 10^{-19}} = 0.95 \text{ V.}$$

Problem 11

When the wavelength of the incident light was diminished from λ_1=600 nm to λ_2=400 nm, the maximum speed of photoelectrons was changed twice. Find the cutoff wavelength λ_0 for this metal.

Solution

Einstein's photoelectric effect equations for two wavelengths are

$$\begin{cases} \varepsilon_1 = A + KE_{max1}, \\ \varepsilon_2 = A + KE_{max2}. \end{cases}$$

where $\varepsilon_1 = \frac{hc}{\lambda_1}$ and $\varepsilon_2 = \frac{hc}{\lambda_2}$ ate the energies of the photons, $KE_{max1} = \frac{mv_{max1}^2}{2}$ and $KE_{m2} = \frac{mv_{m2}^2}{2}$ are the kinetic energies of electrons.

Relying on the given data that $\lambda_1 > \lambda_2$, we have $\varepsilon_2 > \varepsilon_1$, therefore, $KE_{m2} > KE_{m1}$, finally, $v_{m2} > v_{m1}$ and $v_{m2} = 2v_{m1}$.

$$\begin{cases} \frac{hc}{\lambda_1} = A + \frac{mv_{m1}^2}{2}, \\ \frac{hc}{\lambda_2} = A + \frac{mv_{m2}^2}{2}, \end{cases} \qquad \begin{cases} \frac{hc}{\lambda_1} = A + \frac{mv_{m1}^2}{2}, \\ \frac{hc}{\lambda_2} = A + \frac{m4v_{m1}^2}{2}. \end{cases}$$

Multiply both sides of the first equation by 4 and subtract the second equation from the first one:

$$\frac{4hc}{\lambda_1} - \frac{hc}{\lambda_2} = 4A - A + \frac{m4v_{m1}^2}{2} - \frac{m4v_{m1}^2}{2} = 3A,$$

$$A = \frac{hc}{3}\left(\frac{4}{\lambda_1} - \frac{1}{\lambda_2}\right).$$

Due to $A = \frac{hc}{\lambda_0}$, we have

$$\frac{hc}{\lambda_0} = \frac{hc}{3}\left(\frac{4}{\lambda_1} - \frac{1}{\lambda_2}\right) = \frac{hc}{3}\left(\frac{4\lambda_2 - \lambda_1}{\lambda_1 \cdot \lambda_2}\right).$$

The cutoff wavelength is

$$\lambda_0 = \frac{3 \cdot \lambda_1 \cdot \lambda_2}{4\lambda_2 - \lambda_1} = \frac{3 \cdot 600 \cdot 10^{-9} \cdot 400 \cdot 10^{-9}}{4 \cdot 400 \cdot 10^{-9} - 600 \cdot 10^{-9}} = 7.2 \cdot 10^{-7}\ \text{m} = 720\ \text{nm}.$$

Problem 12

When metal is illuminated with radiation with a wavelength λ_1*= 491nm the stopping potential is* U_{s1}*=0.7 V. When the light with* λ_2 *is used the stopping potential is* U_{s2} *=1.43 V. Determine the wavelength* λ_2 *and the work function for the metal.*

Solution

Equations for two wavelengths based on Einstein's relation are

$$\begin{cases} \dfrac{hc}{\lambda_1} = A + KE_{m1}, \\ \dfrac{hc}{\lambda_2} = A + KE_{m2}. \end{cases}$$

The stopping potential is determined by the kinetic energy of the electrons, so

$$\begin{cases} \dfrac{hc}{\lambda_1} = A + eU_{s1}, \\ \dfrac{hc}{\lambda_2} = A + eU_{s2}, \end{cases}$$

Subtraction the first equation from the second equation gives

$$\frac{hc}{\lambda_2} - \frac{hc}{\lambda_1} = e\left(U_{s2} - U_{s1}\right),$$

$$\lambda_2 = \frac{hc}{\dfrac{hc}{\lambda_1} + e\left(U_{s2} - U_{s1}\right)} = \frac{\lambda_1 hc}{hc + \lambda_1 e\left(U_{s2} - U_{s1}\right)} = \frac{\lambda_1}{1 + \dfrac{\lambda_1 e}{hc}\left(U_{s2} - U_{s1}\right)},$$

$$\lambda_2 = \frac{\lambda_1}{1 + \dfrac{\lambda_1 e}{hc}\left(U_{s2} - U_{s1}\right)} = \frac{4.91 \cdot 10^{-7} \lambda_1}{1 + \dfrac{4.91 \cdot 10^{-7} \cdot 1.6 \cdot 10^{=19}}{6.63 \cdot 10^{-34} \cdot 3 \cdot 10^{8}}\left(1.43 - 0.7\right)} = 3.81 \cdot 10^{-7}$$

The work function from the first equation is $A = \dfrac{hc}{\lambda_1} - eU_{s1}$**.** Substitution of given data gives

$$A = \frac{6.63 \cdot 10^{-34} \cdot 3 \cdot 10^{8}}{4.91 \cdot 10^{-7}} - 1.6 \cdot 10^{-19} \cdot 0.7 = 2.93 \text{ J} = 1.83 \text{ eV}.$$

X-RAYS

Problem 1

Calculate the minimum-wavelength X-ray that can be produced when a target is struck by an electron that has been accelerated through a potential difference of (a) 15 kV and (b) 100 kV.

Solution

We have to find the short-wavelength cutoff of X-rays which is equal to $\lambda_{min} = \frac{hc}{eU}$.

When the accelerating potential is 15 kV the wavelength is

$$\lambda_{min} = \frac{6.63 \cdot 10^{-34} \cdot 3 \cdot 10^{8}}{1.6 \cdot 10^{-19} \cdot 15000} = 8.23 \cdot 10^{-11} \text{m}.$$

For the second voltage:

$$\lambda_{min} = \frac{6.63 \cdot 10^{-34} \cdot 3 \cdot 10^{8}}{1.6 \cdot 10^{-19} \cdot 10^{5}} = 1.24 \cdot 10^{-11} \text{m}.$$

Problem 2

The extremes of the x-ray portion of the electromagnetic spectrum range from approximately $1 \cdot 10^{-8}$ m to $1 \cdot 10^{-13}$ m. Find the minimum accelerating voltages required to produce wavelengths at these two extremes.

Solution

Using the expression for the short-wavelength cutoff of X-rays gives

$$U_1 = \frac{hc}{e\lambda_{\min 1}} = \frac{6.63 \cdot 10^{-34} \cdot 3 \cdot 10^{8}}{1.6 \cdot 10^{-19} \cdot 10^{-8}} = 124 \text{ V}.$$

$$U_2 = \frac{hc}{e\lambda_{\min 2}} = \frac{6.63 \cdot 10^{-34} \cdot 3 \cdot 10^{8}}{1.6 \cdot 10^{-19} \cdot 10^{-13}} = 1.24 \cdot 10^{7} \text{ V} = 12.4 \text{ MV}.$$

Problem 3

Find the short-wavelength cutoff of X-rays if the speed of electrons hitting the electrode of the tube is $v = 0.85c$, *where* $c = 3 \cdot 10^8$ *m/s is the speed of light.*

Solution

The relation for the short-wavelength cutoff of X-rays is

$$\lambda_{min} = \frac{hc}{eU},$$

where U is the accelerating voltage.

The work W of electric field changes the speed of electrons moving from cathode of X-ray tube to its anode. As a result the kinetic energy of electrons is changed:

$$W = \Delta KE = KE_2 - KE_1$$

An initial energy is zero $KE_1 = 0$, therefore,

$$eU = KE_2$$

Taking into account that given speed is closed to the speed of light, the relativistic expression for kinetic energy has to be used:

$$eU = m_0 c^2 \left(\frac{1}{\sqrt{1-\beta^2}} - 1 \right).$$

The short-wavelength cutoff of X-rays is

$$\lambda_{min} = \frac{hc}{eU} = \frac{h\cancel{c}}{m_0 c^{\cancel{2}} \left(\frac{1}{\sqrt{1-\beta^2}} - 1 \right)} = \frac{6.63 \cdot 10^{-34}}{9.1 \cdot 10^{-31} \cdot 3 \cdot 10^{8} \left(\frac{1}{\sqrt{1-(0.85)^2}} - 1 \right)} = 2.7 \cdot 10^{-12} \text{ m}.$$

Problem 4

When the accelerating voltage across the X-tube was increased twice $(n=2)$, *the short-wavelength cutoff of X-rays* λ_{min} *was changed by* $\Delta\lambda$ *= 50 pm. Find* λ_{min1}.

Solution

At the increase of the voltage across the tube the short-wavelength cutoff of X-rays λ_{min} decreases.

$$\begin{cases} \lambda_{min1} = \dfrac{hc}{eU_1}, \\ \lambda_{min2} = \lambda_{min1} - \Delta\lambda = \dfrac{hc}{eU_2}. \end{cases}$$

Dividing the second equation by the first equation and using of the given data gives

$$\frac{\lambda_{min2}}{\lambda_{min1}} = \frac{\lambda_{min1} - \Delta\lambda}{\lambda_{min1}} = \frac{U_1}{U_2} = \frac{U_1}{nU_1} = \frac{1}{n}.$$

Finally, the initial value of the short-wavelength is

$$\lambda_{min1} = \frac{n}{n-1}\Delta\lambda = \frac{2\cdot 50\cdot 10^{-12}}{2-1} = 10^{-10} \text{ m.}$$

COMPTON EFFECT

Problem 1

X-rays of wavelength λ *= 0.2 nm are scattered from a block of material. The scattered X-rays are observed at an angle of 45° to the incident beam. (a) Calculate the wavelength of the X-rays scattered at this angle. (b) Compute the fractional change in the energy of a photon in the collision.*

Solution

(a) The change of the wavelength during the Compton effect (Compton shift) is

$$\Delta\lambda = \frac{h}{m_0 c}\left(1-\cos\vartheta\right) = \frac{6.63\cdot 10^{-34}}{9.1\cdot 10^{-31}\cdot 3\cdot 10^{8}}\left(1-\cos 45°\right) = 7.11\cdot 10^{-13} \text{ m} = 0.711 \text{ pm}.$$

Since λ = 0.2 nm = 200 pm,

$\lambda' = \lambda + \Delta\lambda = 200 + 0.7 = 2.5\,\text{pm}.$

(b) The change of energy is $\dfrac{\Delta\varepsilon}{\varepsilon} = \dfrac{\varepsilon' - \varepsilon}{\varepsilon} = \dfrac{\left(hc/\lambda'\right) - \left(hc/\lambda\right)}{hc/\lambda}$,

$$\frac{\Delta\varepsilon}{\varepsilon} = \frac{\left(1/\lambda'\right) - \left(1/\lambda\right)}{1/\lambda} = \frac{\lambda - \lambda'}{\lambda'} = -\frac{\Delta\lambda}{\lambda'} = -\frac{7.11\cdot 10^{-13}}{0.2\cdot 10^{-9}} = -3.54\cdot 10^{-3}$$

Problem 2

Find the wavelength of the incident X-ray radiation if after the Compton scattering the rays with the wavelength $\lambda' = 25{,}4$ are deflected at $\vartheta = 60°$*relative to the direction of the incident rays. Find: (a) the energies and linear momenta of incident (*ε*,* p*) and scattered (*ε'*,* p'*) X-rays; (b) energy (*ε_e'*), linear momentum (*p_e'*) and velocity* $\left(v\right)$ *of recoiled electron; (c) the scattering angle* ϑ*, the angle of recoiled electron* φ*, and the spreading angle* α.

Solution

The expression for the Compton shift

$$\Delta\lambda = \lambda' - \lambda = \lambda_c\left(1-\cos\vartheta\right)$$

allow to find the wavelength of scattered photon

$$\lambda = \lambda' - \lambda_c(1-\cos\vartheta) = 25.4 - 2.43(1-\cos 60°) = 24.19\,\text{m}.$$

The energies of the incident ε and the scattered ε' photons are

$$\varepsilon = \frac{hc}{\lambda} = \frac{6.63\cdot 10^{-34}\cdot 3\cdot 10^{8}}{24.19\cdot 10^{-12}} = 8.22\cdot 10^{-15}\,\text{J},$$

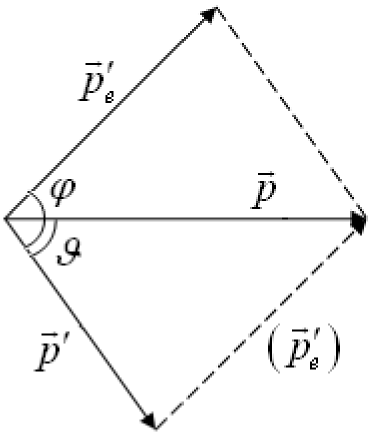
$$\varepsilon' = \frac{hc}{\lambda} = \frac{6.63\cdot10^{-34}\cdot3\cdot10^{8}}{25.4\cdot10^{-12}} = 7.82\cdot10^{-15}\,\text{J}.$$

The energy of the recoiled electron may be found using the law of conservation of energy

$$\varepsilon = \varepsilon' + \varepsilon_e',$$

$$\varepsilon_e' = \varepsilon - \varepsilon' = 8.22\cdot10^{-15} - 7.82\cdot10^{-15} = 4\cdot10^{-16}\ \text{J}.$$

The linear momenta of the incident p and scattered p' photons are



$$p = \frac{h}{\lambda} = \frac{6.63\cdot10^{-34}}{24.19\cdot10^{-12}} = 2.74\cdot10^{-23}\ \text{kg·m/s},$$

$$p' = \frac{h}{\lambda'} = \frac{6.62\cdot10^{-34}}{25.4\cdot10^{-12}} = 2.6\cdot10^{-23}\ \text{kg·m/s}.$$

From the law of conservation of linear momentum $\vec{p} = \vec{p}_e + \vec{p}'$ and the figure that illustrated these momenta the linear momentum of recoiled electron may be found by the help of the cosine law:

$$p_e = \sqrt{p^2 + p'^2 - 2pp'\cos\vartheta} =$$

$$= \sqrt{\left(2.74\cdot10^{-23}\right)^2 + \left(2.6\cdot10^{-23}\right)^2 - 2\cdot2.74\cdot10^{-23}\cdot2.6\cdot10^{-23}\cos60^\circ} =$$

$$= 2.67\cdot10^{-23}\ \text{kg·m/s}.$$

Now let’s find the speed of electron. Usage of the classical expression for momentum $p = mv$, we obtain

$$v = \frac{p}{m} = \frac{2.67\cdot10^{-23}}{9.1\cdot10^{-31}} = 2.93\cdot10^{7}\ \text{m/s}.$$

As seen the speed is closed to the speed of light, therefore, we have to use relativistic relation for momentum $p_e = \dfrac{m_0 v}{\sqrt{1-\beta^2}} = \dfrac{m_0\beta c}{\sqrt{1-\beta^2}}$ $(\beta = v/c)$. It gives

$$\beta = \frac{p_e}{\sqrt{p_e^{\,2} + (m_0 c)^2}} = \frac{2.67 \cdot 10^{-23}}{\sqrt{\left(2.67 \cdot 10^{-23}\right)^2 + \left(9.1 \cdot 10^{-31} \cdot 3 \cdot 10^8\right)^2}} =$$

$$= \frac{2.67 \cdot 10^{-23}}{\sqrt{7.13 \cdot 10^{-46} + 7.46 \cdot 10^{-44}}} = 0.097.$$

$$v = \beta c = 0.097c = 2.9 \cdot 10^7 \text{ m/s}.$$

The desired angles may be determined from sine law

$$\frac{p_e}{\sin\vartheta} = \frac{p'}{\sin\varphi},$$

$$\sin\varphi = \frac{p'}{p_e}\sin\vartheta = \frac{2.6 \cdot 10^{-23}}{2.67 \cdot 10^{-23}}\sin 60° = 0.843,$$

$$\varphi = 57.5°.$$

The spreading angle is $\alpha = \vartheta + \varphi = 60° + 57.5° = 117.5°$.

Problem 3

In a Compton collision with an electron, the energy of the scattered photon is equaled to the half of the energy of incident photon. If the scattered X-rays are detected at 90° relative to the incident X-rays, determine the Compton shift $\Delta\lambda$ at this angle, the energy ε and linear momentum p of the incident photon, the energy ε' and linear momentum p' of the scattered photon, and the energy ε_e', linear momentum p_e' and speed v of the recoiling electron.

Solution

The increase in a photon's wavelength (the Compton shift) when it is scattered through an angle $\vartheta = 90°$ by an electron is given by

$$\Delta\lambda = \lambda' - \lambda = \lambda_c\left(1 - \cos\vartheta\right) = \lambda_c\left(1 - \cos\frac{\pi}{2}\right) = \lambda_c,$$

$$\lambda' - \lambda = \lambda_c.$$

From $\varepsilon = \dfrac{hc}{\lambda}$, the wavelengths of the incident and scattered photons are $\lambda = \dfrac{hc}{\varepsilon}$ and $\lambda' = \dfrac{hc}{\varepsilon'}$, respectively. Taking into account that $\lambda_c = \dfrac{h}{m_0 c}$, we can write the equation $\lambda' - \lambda = \lambda_c$ in the following form

$$\frac{hc}{\varepsilon'} - \frac{hc}{\varepsilon} = \frac{h}{m_0 c}.$$

Let's multiply and divide the right part of the equation by speed of light c

$$\frac{hc}{\varepsilon'} - \frac{hc}{\varepsilon} = \frac{hc}{m_0 c^2}.$$

Reducing the fractions by a factor hc, we obtain

$$\frac{1}{\varepsilon'} - \frac{1}{\varepsilon} = \frac{1}{m_0 c^2}.$$

Since $m_0 c^2 = \varepsilon_0 = 0.512\,\text{MeV} = 8.18 \cdot 10^{-14}\,\text{J}$ is the electron rest energy, the relationship between energies takes on form

$$\frac{1}{\varepsilon'} - \frac{1}{\varepsilon} = \frac{1}{\varepsilon_0}.$$

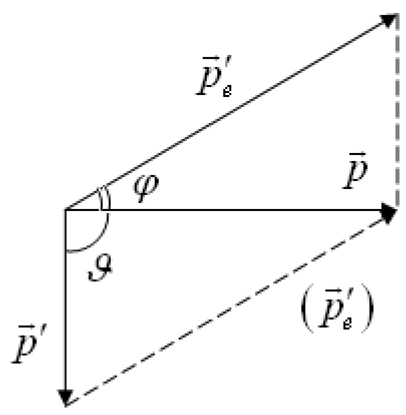
According to the data of the problem, $\varepsilon' = \varepsilon/2$. Then

$$\frac{2}{\varepsilon} - \frac{1}{\varepsilon} = \frac{1}{\varepsilon_0}.$$

As a result, $\varepsilon = \varepsilon_0$ and $\varepsilon' = \varepsilon'_e = \varepsilon_0/2$.

Therefore, the energy of incident photon is $\varepsilon = 0.512\,\text{MeV} = 8.18 \cdot 10^{-14}\,\text{J}$, and the energy of the scattered photon is $\varepsilon' = 0.256$ MeV $= 409 \cdot 10^{-14}$ J.

The linear momentums of the incident and scattered photons are



$$p = \frac{\varepsilon}{c} = \frac{8.18 \cdot 10^{-14}}{3 \cdot 10^{8}} = 2.72 \cdot 10^{-22}\,\frac{\text{kg} \cdot \text{m}}{\text{s}},$$

$$p' = \frac{\varepsilon'}{c} = \frac{4.09 \cdot 10^{-14}}{3 \cdot 10^{8}} = 1.36 \cdot 10^{-22}\,\frac{\text{kg} \cdot \text{m}}{\text{s}}.$$

From the law of linear momentum conservation $\vec{p} = \vec{p}'_e + \vec{p}'$, the linear momentum of the recoiling electron using the Pythagorean Theorem is

$$p'_e = \sqrt{(p)^2 + (p')^2} = \sqrt{\left(1.36\cdot 10^{-22}\right)^2 + \left(2.72\cdot 10^{-22}\right)^2} = 3\cdot 10^{-22}\,\frac{\mathrm{kg\cdot m}}{\mathrm{s}}$$

The speed of electron can be found by means of the relativistic formula for momentum $p_e = \dfrac{m_0\beta c}{\sqrt{1-\beta^2}}$:

$$\beta = \frac{p_e}{\sqrt{p_e{}^2 + (m_0c)^2}} = \frac{3\cdot 10^{-22}}{\sqrt{\left(3\cdot 10^{-22}\right)^2 + \left(9.1\cdot 10^{-31}\cdot 3\cdot 10^{8}\right)^2}} = 0.74.$$

The speed of electron is

$v = \beta c = 0.74c = 2.2\cdot 10^8$ m/s.

Problem 4

In a Compton collision with an electron, a photon with energy $\varepsilon = 1.536$ MeV is backward scattered through an angle $\vartheta = 180°$. Find the energies of the scattered photon ε and recoiling electron ε'_e, the linear momentums of the incident p and scattered p' photons, and the linear momentum p'_e and the speed v of the recoiling electron.

Solution

The Compton shift when the photon is scattered through the angle $\vartheta = 180°$ is

$\lambda' - \lambda = \lambda_c\left(1 - \cos\vartheta\right) = \lambda_c\left(1 - \cos\pi\right) = 2\lambda_c$.

Using that $\lambda = \dfrac{hc}{\varepsilon}$ and the Compton wavelength of the electron is $\lambda_c = \dfrac{h}{m_0c}$, let's bring the equation $\lambda' - \lambda = 2\lambda_c$ to a form

$$\frac{hc}{\varepsilon'} - \frac{hc}{\varepsilon} = \frac{2h}{m_0c}.$$

Multiply and divide the right part of the equation by speed of light c.

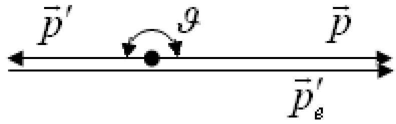
$$\frac{hc}{\varepsilon'} - \frac{hc}{\varepsilon} = \frac{2hc}{m_0c^2},$$

$$\frac{1}{\varepsilon'}-\frac{1}{\varepsilon}=\frac{2}{m_0c^2}\,.$$

Taking into account that $m_0c^2=\varepsilon_0$ = 0.512 MeV, we obtain that ε = 1.536 MeV = $3\varepsilon_0$. Therefore,

$$\frac{1}{\varepsilon'}-\frac{1}{3\varepsilon_0}=\frac{2}{\varepsilon_0}\,,$$

$$\frac{1}{\varepsilon'}=\frac{1}{3\varepsilon_0}+\frac{2}{\varepsilon_0}=\frac{7}{3\varepsilon_0}\,.$$



$$\varepsilon'=\frac{3}{7}\varepsilon_0=\frac{3\cdot 0.512}{7}=0.219\,\text{MeV} = 3.52\cdot10^{-14}\ \text{J}.$$

The law of energy conservation for Compton Effect is $\varepsilon=\varepsilon'+\varepsilon'_e$. From here it follows that

$$\varepsilon'_e=\varepsilon-\varepsilon'=3\varepsilon_0-0.219\varepsilon_0=2.781\varepsilon_0=2.781\cdot0.512\,\text{MeV} = 1.43\ \text{MeV} = 2.28\cdot10^{-13}\ \text{J}.$$

The linear momenta of the incident and scattered photons are

$$p=\frac{\varepsilon}{c}=\frac{3\cdot\varepsilon_0}{c}=\frac{3\cdot m_0c^{\cancel{2}}}{\cancel{c}}=3\cdot m_0c=3\cdot2.731\cdot10^{-22}=8.193\cdot10^{-22}\,\frac{\text{kg}\cdot\text{m}}{\text{s}}\,,$$

$$p'=\frac{\varepsilon'}{c}=\frac{2.781\cdot\varepsilon_0}{c}=2.781\frac{m_0c^{\cancel{2}}}{\cancel{c}}=2.781\cdot2.731\cdot10^{-22}=7.59\cdot10^{-22}\,\frac{\text{kg}\cdot\text{m}}{\text{s}}\,.$$

From the law of linear momentum conservation $\vec{p}=\vec{p}'_e+\vec{p}'$, the magnitude of the linear momentum of electron is

$$p'_e=p+p'=2.72\cdot10^{-22}+7.59\cdot10^{-22}=1.03\cdot10^{-21}\,\frac{\text{kg}\cdot\text{m}}{s}\,.$$

Then speed of electron can be found using $p_e=\dfrac{m_0\beta c}{\sqrt{1-\beta^2}}$ as

$$\beta=\frac{p_e}{\sqrt{p_e{}^2+(m_0c)^2}}=\frac{1.03\cdot10^{-21}}{\sqrt{\left(1.03\cdot10^{-21}\right)^2+\left(9.1\cdot10^{-31}\cdot3\cdot10^{8}\right)^2}}=0.967\,,$$

$$v=\beta c=0.967c=2.9\cdot10^{8}\ \text{m/s}.$$

Problem 5

In a Compton effect, the wavelengths of the photons scattered through the angles ϑ_1 = 60° and ϑ_2 = 120°, differ twofold. Find the wavelength of the incident X- ray radiation.

Solution

The Compton shifts are

$$\begin{cases} \lambda_1' - \lambda = \lambda_c\left(1-\cos\vartheta_1\right) = \lambda_c\left(1-\cos 60^\circ\right) = \dfrac{\lambda_c}{2}, \\ \lambda_2' - \lambda = \lambda_c\left(1-\cos\vartheta_2\right) = \lambda_c\left(1-\cos 120^\circ\right) = \dfrac{3\lambda_c}{2}. \end{cases}$$

The second wavelength λ_2' of the scattered radiation is $\lambda_2' = 2\lambda_1'$. Then

$$\begin{cases} \lambda_1' - \lambda = \dfrac{\lambda_c}{2}, \\ 2\lambda_1' - \lambda = \dfrac{3\lambda_c}{2}. \end{cases}$$

Express the wavelength $\lambda_1' = \lambda + \dfrac{\lambda_c}{2}$ from the first equation and substitute it to the second equation

$$2\left(\lambda + \frac{\lambda_c}{2}\right) - \lambda = \frac{3\lambda_c}{2}.$$

Taking into account that Compton wavelength is $\lambda_c = 2.43\cdot 10^{-12}\,\text{m}$ = 2.43 pm, we obtain

$$\lambda = 0.5\lambda_c = 0.5\cdot 2.43 = 1.215\,\text{pm}.$$

Speed of light in vacuum	c = 2,998·10^8 m/s = □ $3\cdot 10^8$ m/s
Plank's constant	$h = 2\pi\hbar = 6.62\cdot 10^{-34}\,\mathrm{J\cdot s}$ $\hbar = h/2\pi = 1.055\cdot 10^{-34}\,\mathrm{J\cdot s}$
Stefan-Boltzmann constant	$\sigma = 5.67\cdot 10^{-8}\,\mathrm{W\cdot m^{-2}\cdot K^{-4}}$
Wien's displacement constant	$b = 2.898\cdot 10^{-3}\,\mathrm{m\cdot K}$
Constant	$C = 1.3\cdot 10^{-5}\,\mathrm{W\cdot m^{-3}\cdot K^{-5}}$
Compton wavelength for the electron	$\lambda_c = h/m_0c = 2.43\cdot 10^{-12}$ m =2.43 pm
Electron rest mass	$m_{0e} = 9.1\cdot 10^{-31}$ kg
Proton rest mass	$m_{0p} = 1.67\cdot 10^{-27}$ kg
α-particle rest mass	$m_{0\alpha} = 6.64\cdot 10^{-27}$ kg
Elementary charge (proton/electron)	$e = \pm 1.6\cdot 10^{-19}$ C
α-particle charge	$q_\alpha = 2e = 3.2\cdot 10^{-19}$ C
Electric constant (vacuum permittivity)	ε_0 = 8.85·10^{-12} F/m $1/4\pi\varepsilon_0$ = 9·10^9 m/ F
1 electron-Volt	1eV = $1.6\cdot 10^{-19}$ J
Rest energy of electron	$\varepsilon_{0e} = m_{0e}c^2 = 8.187\cdot 10^{-14}$ J = = $5.12\cdot 10^5$ eV = 0.512 MeV
Rest energy of proton	$\varepsilon_{0p} = m_{0p}c^2 = 1.49\cdot 10^{-10}$ J = =9.315·10^8 eV = 0.93 GeV
Rest energy of α-particle	$\varepsilon_{0\alpha} = m_{0\alpha}c^2 = 5.97\cdot 10^{-10}$ J = =$3.72\cdot 10^9$ eV = 3.72 GeV
	$m_{0e}c = 2.73\cdot 10^{-22}$ kg·m/s $(m_{0e}c)^2 = 7.46\cdot 10^{-44}$ kg²·m²/s²
	$hc = 1.986\cdot 10^{-25}$ J·m
	$(hc)^2 = 3.95\cdot 10^{-50}$ (J·m)²