Завдання 1. Знайти похідну складної функції

а) $y=\dfrac{\sqrt{2+x^2}}{\sqrt[3]{3+x^3}};\ y'(1)=?$

$$y'=\left(\frac{\sqrt{2+x^2}}{\sqrt[3]{3+x^3}}\right)'=\left\|\begin{array}{l}\left(\dfrac{u}{v}\right)'=\dfrac{u'v-uv'}{v^2}\\ \left(u^n\right)'=nu^{n-1}u'\end{array}\right\|=$$

$$=\frac{\left(\sqrt{2+x^2}\right)'\cdot\sqrt[3]{3+x^3}-\sqrt{2+x^2}\cdot\left(\sqrt[3]{3+x^3}\right)'}{\left(\sqrt[3]{3+x^3}\right)^2}=$$

$$=\left\|\begin{array}{l}\left(\sqrt{2+x^2}\right)'=\dfrac{2x}{2\sqrt{2+x^2}}=\dfrac{x}{\sqrt{2+x^2}}\\ \left(\sqrt[3]{3+x^3}\right)'=\dfrac{3x^2}{3\left(3+x^3\right)^{\frac{2}{3}}}=\dfrac{x^2}{\left(3+x^3\right)^{\frac{2}{3}}}\end{array}\right\|=$$

$$=\frac{\dfrac{x}{\sqrt{2+x^2}}\cdot\sqrt[3]{3+x^3}-\sqrt{2+x^2}\cdot\dfrac{x^2}{\left(3+x^3\right)^{\frac{2}{3}}}}{\left(\sqrt[3]{3+x^3}\right)^2}=$$

$$=\frac{x\cdot\sqrt[3]{3+x^3}\left(3+x^3\right)^{\frac{2}{3}}-\sqrt{2+x^2}\sqrt{2+x^2}\cdot x^2}{\sqrt{2+x^2}\left(\sqrt[3]{3+x^3}\right)^4}=$$

$$=\frac{x\cdot\left(3+x^3\right)-\left(2+x^2\right)\cdot x^2}{\sqrt{2+x^2}\left(\sqrt[3]{3+x^3}\right)^4}=\frac{3x+x^4-2x^2-x^4}{\sqrt{2+x^2}\left(\sqrt[3]{3+x^3}\right)^4}=\frac{3x-2x^2}{\sqrt[3]{3+x^3}\sqrt{2+x^2}\left(3+x^3\right)}.$$

$$y'(1)=\frac{3\cdot1-2\cdot1^2}{\sqrt[3]{3+1^3}\sqrt{2+1^2}\left(3+1^3\right)}=\frac{1}{4\sqrt[3]{4}\sqrt{3}}.$$

б) $y = x\sin\left(\ln x - \frac{\pi}{4}\right)$.

$$y' = \left(x\sin\left(\ln x - \frac{\pi}{4}\right)\right)' = \left\| \begin{array}{l} (uv)' = u'v + uv' \\ (\ln x)' = \frac{1}{u}\cdot u' \\ (\sin u)' = \cos u \cdot u' \end{array} \right\| =$$

$$= x'\sin\left(\ln x - \frac{\pi}{4}\right) + x\left(\sin\left(\ln x - \frac{\pi}{4}\right)\right)' = \sin\left(\ln x - \frac{\pi}{4}\right) + x\cos\left(\ln x - \frac{\pi}{4}\right)\frac{1}{x} =$$

$$= \sin\left(\ln x - \frac{\pi}{4}\right) + \cos\left(\ln x - \frac{\pi}{4}\right).$$

в) $y = 2^{\arcsin^2 3x}$.

$$y' = \left(2^{\arcsin^2 3x}\right)' = \left\| \begin{array}{l} \left(a^u\right)' = a^u \ln a \cdot u' \\ \left(u^n\right)' = nu^{n-1}u' \\ (\arcsin u)' = \frac{1}{\sqrt{1-u^2}}\cdot u' \end{array} \right\| =$$

$$= 2^{\arcsin^2 3x}\ln 2 \cdot 2\arcsin 3x \frac{1}{\sqrt{1-9x^2}} 3 = \frac{6\cdot 2^{\arcsin^2 3x}\ln 2 \cdot \arcsin 3x}{\sqrt{1-9x^2}}.$$

г) $y = 2x\operatorname{arctg}\sqrt{\sin x}$.

$$y' = \left(2x\operatorname{arctg}\sqrt{\sin x}\right)' = \left\| \begin{array}{l} (uv)' = u'v + uv' \\ (\operatorname{arctg} u)' = \frac{1}{1+u^2}\cdot u' \\ \left(u^n\right)' = nu^{n-1}u' \\ (\sin u)' = \cos u \cdot u' \end{array} \right\| = 2x'\operatorname{arctg}\sqrt{\sin x} +$$

$$+ 2x\left(\operatorname{arctg}\sqrt{\sin x}\right)' == 2\operatorname{arctg}\sqrt{\sin x} + 2x\frac{1}{1+\sin x}\cdot\frac{1}{2\sqrt{\sin x}}\cdot\cos x =$$

$$= 2\operatorname{arctg}\sqrt{\sin x} + \frac{x\cos x}{(1+\sin x)\sqrt{\sin x}}.$$

г) $y=\frac{1}{3}\ln\cos\frac{x-1}{x}.$

$$y'=\left(\frac{1}{3}\ln\cos\frac{x-1}{x}\right)'=\left\|\begin{array}{l}\left(\ln x\right)'=\frac{1}{u}\cdot u'\\ \left(\cos u\right)'=-\sin u\cdot u'\\ \left(\frac{u}{v}\right)'=\frac{u'v-uv'}{v^2}\end{array}\right\|=$$

$$=\frac{1}{3}\cdot\frac{1}{\cos\frac{x-1}{x}}\left(-\sin\frac{x-1}{x}\right)\cdot\frac{\left(x-1\right)'x-\left(x-1\right)x'}{x^2}=-\frac{1}{3}\cdot tg\frac{x-1}{x}\left(\frac{x-x+1}{x^2}\right)=$$

$$=-\frac{1}{3x^2}\cdot tg\frac{x-1}{x}$$

д) $y=\frac{2}{3}\operatorname{tg}^4\left(\frac{5\operatorname{tg}\frac{x}{2}+4}{3}\right)$

$$y'=\left(\frac{2}{3}\operatorname{tg}^4\left(\frac{5\operatorname{tg}\frac{x}{2}+4}{3}\right)\right)'=\left\|\begin{array}{l}\left(u^n\right)'=nu^{n-1}u'\\ \left(tgu\right)'=\frac{1}{\cos^2 u}\cdot u'\end{array}\right\|=$$

$$=\frac{2}{3}4\operatorname{tg}^3\left(\frac{5\operatorname{tg}\frac{x}{2}+4}{3}\right)\cdot\frac{1}{\cos^2\left(\frac{5\operatorname{tg}\frac{x}{2}+4}{3}\right)}\frac{5}{3}\cdot\frac{1}{\cos^2\frac{x}{2}}\cdot\frac{1}{2}=$$

$$=\frac{20}{9}\operatorname{tg}^3\left(\frac{5\operatorname{tg}\frac{x}{2}+4}{3}\right)\cdot\frac{1}{\cos^2\left(\frac{5\operatorname{tg}\frac{x}{2}+4}{3}\right)\cos^2\frac{x}{2}}$$

е) $y=\left(\frac{1}{2}-x\right)\arccos\sqrt{x}$

$$y'=\left(\left(\frac{1}{2}-x\right)\arccos\sqrt{x}\right)'=\left\|\begin{array}{l}(uv)'=u'v+uv'\\(\arccos u)'=\frac{1}{\sqrt{1-u^2}}\cdot u'\\(u^n)'=nu^{n-1}u'\end{array}\right\|=$$

$$=\left(\frac{1}{2}-x\right)'\arccos\sqrt{x}+\left(\frac{1}{2}-x\right)(\arccos\sqrt{x})'=$$

$$=-\arccos\sqrt{x}+\left(\frac{1}{2}-x\right)\cdot\left(\frac{1}{\sqrt{1-x}}\right)\cdot\frac{1}{2\sqrt{x}}.$$

є) $y=10^{5\log_2 3x}$

$$y'=\left(10^{5\log_2 3x}\right)'=\left\|\begin{array}{l}(a^u)'=a^u\ln a\cdot u'\\(\log_a u)'=\frac{1}{u\ln a}\cdot u'\end{array}\right\|=10^{5\log_2 3x}\cdot\ln 10\cdot 5\cdot\frac{1}{3x\ln 2}\cdot 3=$$

$$=\frac{5\cdot 10^{5\log_2 3x}\cdot\ln 10}{x\ln 2}.$$

ж) $y=\frac{3}{e^{\operatorname{arcctg}2x}}=3\cdot e^{-\operatorname{arcctg}2x}$

$$y'=\left(3\cdot e^{-\operatorname{arcctg}2x}\right)'=\left\|\begin{array}{l}(e^u)'=e^u\cdot u'\\(\operatorname{arcctg}u)'=-\frac{1}{1+u^2}\cdot u'\end{array}\right\|=3\cdot e^{-\operatorname{arcctg}2x}\cdot\frac{1}{1+4x^2}\cdot 2=$$

$$=\frac{6}{(1+4x^2)e^{\operatorname{arcctg}2x}}$$

з) $y=\sqrt{\operatorname{ctg}4x}\cdot a^{\sqrt{\operatorname{ctg}4x}}$

$$y'=\left(\sqrt{\operatorname{ctg}4x}\cdot a^{\sqrt{\operatorname{ctg}4x}}\right)'=\left\|\begin{array}{l}(uv)'=u'v+uv'\\(\operatorname{ctg}u)'=-\frac{1}{\sin^2 u}\cdot u'\\(a^u)'=a^u\ln a\cdot u'\end{array}\right\|=$$

$$= \left(\sqrt{\operatorname{ctg} 4x}\right)' \cdot a^{\sqrt{\operatorname{ctg} 4x}} + \sqrt{\operatorname{ctg} 4x} \cdot \left(a^{\sqrt{\operatorname{ctg} 4x}}\right)' =$$

$$= \frac{1}{2\sqrt{\operatorname{ctg} 4x}} \cdot \left(-\frac{1}{\sin^2 4x}\right) \cdot 4 \cdot a^{\sqrt{\operatorname{ctg} 4x}} + \sqrt{\operatorname{ctg} 4x} \cdot a^{\sqrt{\operatorname{ctg} 4x}} \cdot \ln a \cdot \left(-\frac{1}{\sin^2 4x}\right) \cdot 4 =$$

$$= -\frac{2a^{\sqrt{\operatorname{ctg} 4x}}}{\sqrt{\operatorname{ctg} 4x} \sin^2 4x} - \frac{2a^{\sqrt{\operatorname{ctg} 4x}} \cdot \ln a}{\sin^2 4x} = -\frac{2a^{\sqrt{\operatorname{ctg} 4x}}}{\sin^2 4x}\left(\frac{1}{\sqrt{\operatorname{ctg} 4x}} + \ln a\right).$$

и) $y = \sqrt[3]{x + \sqrt{2x}}$.

$$y' = \left(\sqrt[3]{x + \sqrt{2x}}\right)' = \left\| \left(u^n\right)' = nu^{n-1}u' \right\| = \frac{1}{3}\left(x + \sqrt{2x}\right)^{-\frac{2}{3}} \cdot \left(1 + \frac{1}{2\sqrt{2x}} 2\right) =$$

$$= \frac{1 + \sqrt{2x}}{3\sqrt[3]{\left(x + \sqrt{2x}\right)^2}\sqrt{2x}}.$$

і) $y = \frac{1}{15}\cos^3 x\left(3\cos^2 x - 5\right).$

$$y' = \left(\frac{1}{15}\cos^3 x\left(3\cos^2 x - 5\right)\right)' = \left\| \begin{array}{l} (uv)' = u'v + uv' \\ \left(u^n\right)' = nu^{n-1}u' \\ (\cos u)' = -\sin u \cdot u' \end{array} \right\| =$$

$$= \frac{1}{15}\left(\left(\cos^3 x\right)'\left(3\cos^2 x - 5\right) + \cos^3 x\left(3\cos^2 x - 5\right)'\right) =$$

$$= \frac{1}{15}\left(3\cos^2 x(-\sin x)\left(3\cos^2 x - 5\right) + \cos^3 x \cdot 6\cos x(-\sin x)\right) =$$

$$= -\frac{3}{15}\cos^2 x \sin x\left(\cos x\left(3\cos^2 x - 5\right) + 2\cos^2 x\right).$$

ї) $y = x^4\left(a - 2x^3\right)^2$

$$y' = \left(x^4\left(a - 2x^3\right)^2\right)' = \left\| \begin{array}{l} (uv)' = u'v + uv' \\ \left(u^n\right)' = nu^{n-1}u' \end{array} \right\| = \left(x^4\right)'\left(a - 2x^3\right)^2 + x^4\left(\left(a - 2x^3\right)^2\right)' =$$

$$= 4x^3\left(a - 2x^3\right)^2 + x^4 2\left(a - 2x^3\right)\left(-6x^2\right) = 4x^3\left(a - 2x^3\right)\left(\left(a - 2x^3\right) - 3x^3\right) =$$

$$= 4x^3\left(a - 2x^3\right)\left(a - 5x^3\right)$$

к) $y=\dfrac{10}{\sqrt[4]{\operatorname{tg}^3\dfrac{3}{x}}}=10\operatorname{tg}^{-\frac{3}{4}}\dfrac{3}{x}.$

$$y'=\left(10\operatorname{tg}^{-\frac{3}{4}}\frac{3}{x}\right)'=\left\|\begin{matrix}(uv)'=u'v+uv'\\(\operatorname{tg}u)'=\dfrac{1}{\cos^2 u}\cdot u'\end{matrix}\right\|=10\left(-\frac{3}{4}\operatorname{tg}^{-\frac{7}{4}}\frac{3}{x}\right)\cdot\frac{1}{\cos^2\dfrac{3}{x}}\cdot\left(-\frac{3}{x^2}\right)=$$

$$=\frac{45}{2}\cdot\frac{1}{\operatorname{tg}^{\frac{7}{4}}\dfrac{3}{x}\cdot\cos^2\dfrac{3}{x}\cdot x^2}.$$

л) $y=\dfrac{1}{\sqrt{6}}\arcsin\left(\sqrt{\dfrac{b}{a}x}\right)$

$$y'=\left(\frac{1}{\sqrt{6}}\arcsin\left(\sqrt{\frac{b}{a}x}\right)\right)'=\left\|(\arcsin u)'=\frac{1}{\sqrt{1-u^2}}\cdot u'\right\|=$$

$$=\frac{1}{\sqrt{6}}\cdot\frac{1}{\sqrt{1-\dfrac{b}{a}x}}\cdot\frac{1}{2\sqrt{\dfrac{b}{a}x}}\cdot\sqrt{\frac{b}{a}}=\frac{1}{2\sqrt{6}\sqrt{x}\sqrt{1-\dfrac{b}{a}x}}.$$

м) $y=\arcsin 3x\cdot\dfrac{1}{x}\cdot 5^{-x}$

$$y'=\left(\arcsin 3x\cdot\frac{1}{x}\cdot 5^{-x}\right)=\left\|\begin{matrix}(uvw)'=u'vw+uv'w+uvw'\\(\arcsin a)'=\dfrac{1}{\sqrt{1-u^2}}\cdot u'\\(u^n)'=nu^{u-1}\cdot u'\\(a^u)'=a^u\ln a\cdot u'\end{matrix}\right\|=$$

$$=\frac{1}{\sqrt{1-9x^2}}\cdot 3\cdot\frac{1}{x}\cdot 5^{-x}+\arcsin 3x\cdot\left(-\frac{1}{x^2}\right)\cdot 5^{-x}+\arcsin 3x\cdot\frac{1}{x}\cdot 5^{-x}\ln 5\,(-1)$$

Завдання 2. Знайти $\dfrac{dy}{dx}$.

Розв'язання.

$$y=(\operatorname{ctg}3x)^{3^x}$$

$$\ln y=\ln(\operatorname{ctg}3x)^{3^x}$$

$$\ln y = 3^x \ln(\operatorname{ctg} 3x)$$

$$\frac{1}{y} y' = \left(3^x\right) \ln(\operatorname{ctg} 3x) + 3^x \left(\ln(\operatorname{ctg} 3x)\right)'$$

$$\frac{1}{y} y' = 3^x \ln 3 \ln(\operatorname{ctg} 3x) + 3^x \frac{1}{\operatorname{ctg} 3x} \left(-\frac{1}{\sin^2 3x}\right) \cdot 3$$

$$y' = \left(3^x \ln 3 \ln(\operatorname{ctg} 3x) - \frac{3^{x+1}}{\cos 3x \sin 3x}\right) y$$

$$y' = \left(3^x \ln 3 \ln(\operatorname{ctg} 3x) - \frac{3^{x+1}}{\cos 3x \sin 3x}\right) (\operatorname{ctg} 3x)^{3^x}$$

$$y = \frac{\sqrt[3]{(4-x)^2} \cdot (3x^2+5)^4}{\sqrt[5]{(1+x)^2}}$$

$$\ln y = \ln\left(\frac{\sqrt[3]{(4-x)^2} \cdot (3x^2+5)^4}{\sqrt[5]{(1+x)^2}}\right)$$

$$\ln y = \ln \sqrt[3]{(4-x)^2} + \ln(3x^2+5)^4 - \ln \sqrt[5]{(1+x)^2}$$

$$\ln y = \frac{2}{3} \ln(4-x) + 4 \ln(3x^2+5) - \frac{2}{5} \ln(1+x)$$

$$\frac{1}{y} y' = \frac{2}{3} \frac{1}{4-x} (-1) + \frac{4 \cdot 6x}{3x^2+5} - \frac{2}{5(1+x)}$$

$$y' = \left(\frac{-2}{3(4-x)} + \frac{24x}{3x^2+5} - \frac{2}{5(1+x)}\right) y$$

$$y' = \left(\frac{-2}{3(4-x)} + \frac{24x}{3x^2+5} - \frac{2}{5(1+x)}\right) \frac{\sqrt[3]{(4-x)^2} \cdot (3x^2+5)^4}{\sqrt[5]{(1+x)^2}}$$

$$x^5 - 3x^3y^2 + 2y^5 - 5x^2 + 10xy - 90 = 0$$

$$5x^4 - 3(3x^2y^2 + x^3 2yy') + 10y^4 y' - 10x + 10y + 10xy' = 0$$

$$5x^4 - 9x^2y^2 - 6x^3yy' + 10y^4 y' - 10x + 10y + 10xy' = 0$$

$$y'(-6x^3y + 10y^4 + 10x) = 9x^2y^2 + 10x - 10y$$

$$y' = \frac{9x^2y^2 + 10x - 10y}{-6x^3y + 10y^4 + 10x}$$

$$\begin{cases} x = \operatorname{arctg} e^{\frac{t}{2}} \\ y = \sqrt{e^t + 1} \end{cases}$$

$$x'_t = \frac{1+t}{1-t} \cdot \frac{(1-t)'(1+t) - (1-t)(1+t)'}{(1+t)^2} = \frac{-1(1+t) - 1 + t}{(1-t)(1+t)} =$$

$$= \frac{-1-t-1+t}{1-t^2} = \frac{-2}{1-t^2}$$

$$y'_t = \frac{1}{2\sqrt{1-t^2}}(-2t) = -\frac{t}{\sqrt{1-t^2}}$$

$$\frac{dy}{dx} = \frac{-\frac{t}{\sqrt{1-t^2}}}{\frac{-2}{1-t^2}} = \frac{t(1-t^2)}{2\sqrt{1-t^2}} = \frac{t\sqrt{1-t^2}}{2}$$